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THE P-CLASS AND Q-CLASS FUNCTIONS ON SYMMETRIC CONES

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Dedicated to Prof. Juan Enrique Martínez Legaz on the occasion of his 70th birthday

Abstract. In this paper, we investigate the functions of P-class and of Q-class associated with symmetric cones. We provide the characterizations for P-class functions on symmetric cones, and discuss the relationship between P-class functions and monotone functions in the setting of symmetric cones. In addition, we also discuss the sufficient conditions for Q-class functions being monotone in the setting of symmetric cones.

Keywords. Symmetric cone; P-class function; Q-class function; SC-convex; SC-monotone.

1. Introduction and Preliminaries

The families of P-class and Q-class functions attract much attention in analysis, not only because they include convex functions and nonnegative monotone functions as special cases, but also they induce some important inequalities; see, e.g., [7, 8, 9, 16, 17, 18, 20] and the references therein. Their official definitions are stated as below. For an interval J in $\mathbb{R}$, a function $f : J \to \mathbb{R}$ is said to be of P-class on J or is a P-class function on J if

$$f(\eta a + (1-\eta)b) \le f(a) + f(b),$$

for all $a, b \in J$ and all $\eta \in [0,1]$, and f is of Q-class on or is a Q-class function on J if

$$f(\eta a + (1-\eta)b) \le \frac{f(a)}{\eta} + \frac{f(b)}{1-\eta}.$$

for all $a, b \in J$ and all $\eta \in (0,1)$. The Q-class function was introduced by Godunova and Levin [13] in 1985 and coincides with the so-called Schur function [17]. These two families of functions were generalized to operator P-class functions [1] and operator Q-class functions [10], respectively.

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In this paper, we investigate the P-class functions associated with symmetric cones (SC P-class functions for short) and the Q-class functions associated with symmetric cones (SC Q-class functions for short). We shall present some relation between the SC P-class functions and SC-monotone functions; relation between SC Q-class functions and SC-monotone functions.

To proceed, we recall some basic definitions and properties about the Euclidean Jordan algebra and the associated symmetric cone. A *Euclidean Jordan algebra* [12] is a finite dimensional inner product space $(\mathbb{V}, \langle\cdot,\cdot\rangle)$ ($\mathbb{V}$ for short) over the field of real numbers $\mathbb{R}$ equipped with a bilinear map $(x,y) \mapsto x\circ y : \mathbb{V}\times\mathbb{V}\to\mathbb{V}$, which satisfies the following conditions:

(i) $x\circ y = y\circ x$ for all $x,y\in\mathbb{V}$;
(ii) $x\circ(x^2\circ y) = x^2\circ(x\circ y)$ for all $x,y\in\mathbb{V}$;
(iii) $\langle x\circ y, z\rangle = \langle x, y\circ z\rangle$ for all $x,y,z\in\mathbb{V}$,

where $x^2 := x\circ x$, and $x\circ y$ is called the *Jordan product* of x and y. If a Jordan product only satisfies the conditions (i) and (ii) in the above definition, the algebra $\mathbb{V}$ is said to be a *Jordan algebra*. Moreover, if there is an (unique) element $e\in\mathbb{V}$ such that $x\circ e = x$ for all $x\in\mathbb{V}$, the element e is called the *identity element* in $\mathbb{V}$. Note that a Jordan algebra does not necessarily have an identity element. Throughout this paper, we assume that $\mathbb{V}$ is a Euclidean Jordan algebra with an identity element e.

In a given Euclidean Jordan algebra $\mathbb{V}$, the set of squares $\mathscr{K} := \{x^2 : x\in\mathbb{V}\}$ is a *symmetric cone* [12, Theorem III.2.1]. This means that $\mathscr{K}$ is a self-dual closed convex cone and, for any two elements $x,y\in\operatorname{int}(\mathscr{K})$, there exists an invertible linear transformation $\Gamma:\mathbb{V}\longrightarrow\mathbb{V}$ such that $\Gamma(x)=y$ and $\Gamma(\mathscr{K})=\mathscr{K}$. We introduce the second-order cone in $\mathbb{R}^n$, an important example of symmetric cones, which is defined as follows:

$$\mathscr{K}^n := \left\{x=(x_0,\bar{x})\in\mathbb{R}\times\mathbb{R}^{n-1} \mid x_0\geq\|\bar{x}\|\right\},$$

and the corresponding Jordan product of x and y in $\mathbb{R}^n$ with $x=(x_0,\bar{x}), y=(y_0,\bar{y})\in\mathbb{R}\times\mathbb{R}^{n-1}$ is given by

$$x\circ y := \begin{bmatrix} x^T y \\ x_0\bar{y}+y_0\bar{x} \end{bmatrix}.$$

This algebra is called the Jordan spin algebra and is denoted by $\mathbb{J}^n$. We note that $e=(1,0)\in\mathbb{R}\times\mathbb{R}^{n-1}$ acts as the Jordan identity.

For any given $x\in\mathbb{V}$, we denote $m(x)$ the *degree* of the minimal polynomial of x, that is,

$$m(x) := \left\{k>0 \,|\, \{e,x,\cdots,x^k\} \text{ is linearly dependent}\right\}.$$

Since $m(x)\leq\dim(\mathbb{V})$, where $\dim(\mathbb{V})$ is the dimension of $\mathbb{V}$, the *rank* of $\mathbb{V}$ is well-defined by $r := \max\{m(x)\,|\,x\in\mathbb{V}\}$. In Euclidean Jordan algebra $\mathbb{V}$, an element $e^{(i)}\in\mathbb{V}$ is an *idempotent* if $(e^{(i)})^2 = e^{(i)}$, and it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. The idempotents $e^{(i)}$ and $e^{(j)}$ are said to be *orthogonal* if $e^{(i)}\circ e^{(j)}=0$. In addition, we say that a finite set $\{e^{(1)},e^{(2)},\cdots,e^{(r)}\}$ of primitive idempotents in $\mathbb{V}$ is a *Jordan frame* if

$$e^{(i)}\circ e^{(j)} = 0 \ \text{ for } \ i\neq j, \ \text{ and } \ \sum_{i=1}^{r} e^{(i)} = e.$$

Note that $\langle e^{(i)}, e^{(j)}\rangle = \langle e^{(i)}\circ e^{(j)}, e\rangle$ whenever $i\neq j$.

With the above, there have the spectral decomposition of an element x in $\mathbb{V}$.

Theorem 1.1. (The Spectral Decomposition Theorem) [12, Theorem III.1.2] *Let $\mathbb{V}$ be a Euclidean Jordan algebra. Then there is a number r such that, for every $x \in \mathbb{V}$, there exists a Jordan frame $\{e^{(1)}, \cdots, e^{(r)}\}$ and real numbers $\lambda_1(x), \cdots, \lambda_r(x)$ with*

$$x = \lambda_1(x)e^{(1)} + \cdots + \lambda_r(x)e^{(r)}.$$

Here, the numbers $\lambda_i(x)$ $(i = 1, \cdots, r)$ are the spectral values of x, the expression $\lambda_1(x)e^{(1)} + \cdots + \lambda_r(x)e^{(r)}$ is the spectral decomposition of x. Moreover, $\operatorname{tr} x := \sum_{i=1}^{r} \lambda_i(x)$ *is called the trace of x, and* $\det(x) = \lambda_1(x)\lambda_2(x)\cdots\lambda_r(x)$.

Suppose that $f : J \subseteq \mathbb{R} \to \mathbb{R}$ is a real-valued function. Let $\mathbb{V}_J$ be a subset in $\mathbb{V}$ such that all $x \in \mathbb{V}_J$ have the spectral values in J. Then, by the spectral decomposition $\sum_{j=1}^{r} \lambda_j(x)e^{(j)}$ of $x \in \mathbb{V}_J$, it is natural to define a vector valued function [2, 14] $f^{\mathrm{SC}} : \mathbb{V}_J \to \mathbb{V}$ by

$$f^{\mathrm{SC}}(x) := f(\lambda_1(x))e^{(1)} + f(\lambda_2(x))e^{(2)} + \cdots + f(\lambda_r(x))e^{(r)}. \tag{1.1}$$

This function is also called the Löwner's operator. Sun and Sun [21] studied many differential properties of Löwner's operator and spectral functions in Euclidean Jordan algebras.

Given a Euclidean Jordan algebra $\mathbb{V}$ with $\dim(\mathbb{V}) = n > 1$, from Proposition III 4.4-4.5 and Theorem V.3.7 in [12], we know that any Euclidean Jordan algebra $\mathbb{V}$ and its corresponding symmetric cone $\mathscr{K}$ are, in a unique way, a direct sum of simple Euclidean Jordan algebras and the constituent symmetric cones therein, respectively, i.e.,

$$\mathbb{V} = \mathbb{V}_1 \times \cdots \times \mathbb{V}_m \quad \text{and} \quad \mathscr{K} = \mathscr{K}^1 \times \cdots \times \mathscr{K}^m,$$

where every $\mathbb{V}_i$ is a simple Euclidean Jordan algebra (that cannot be a direct sum of two Euclidean Jordan algebras) with the corresponding symmetric cone $\mathscr{K}^i$ for $i = 1, \cdots, m$, and $n = \sum_{i=1}^{m} n_i$ (n_i is the dimension of $\mathbb{V}_i$). Therefore, for any $x = (x_1, \cdots, x_m)^T$ and $y = (y_1, \cdots, y_m)^T \in \mathbb{V}$ with $x_i, y_i \in \mathbb{V}_i$, we have

$$x \circ y = (x_1 \circ y_1, \cdots, x_m \circ y_m)^T \in \mathbb{V} \quad \text{and} \quad \langle x, y\rangle = \langle x_1, y_1\rangle + \cdots + \langle x_m, y_m\rangle.$$

For simplicity, we focus on the single symmetric cone $\mathscr{K}$ because all the analysis can be carried over to the setting of Cartesian product. The classification theorem [12, Chapter V] says that every simple Euclidean Jordan algebra is isomorphic to one of the following:

(i): The Jordan spin algebra $\mathbb{J}^n$.
(ii): The algebra $\mathbb{S}^n$ of $n \times n$ real symmetric matrices.
(iii): The algebra $\mathbb{H}^n$ of all $n \times n$ complex Hermitian matrices.
(iv): The algebra $\mathbb{Q}^n$ of all $n \times n$ quaternion Hermitian matrices.
(v): The algebra $\mathbb{O}^3$ of all 3×3 octonion Hermitian matrices.

From [12, Theorem III.2.1], we know that the set of all squares $\mathscr{K} := \{x \circ x \,|\, x \in \mathbb{V}\}$ in $\mathbb{V}$ is a symmetric cone, i.e., a self-dual homogeneous closed convex cone. So, there is a natural partial order in $\mathbb{V}$. We write $x \succeq_{\mathscr{K}} y$ if $x - y \in \mathscr{K}$, and $x \succ_{\mathscr{K}} y$ if $x - y \in \operatorname{int}\mathscr{K}$.

Now we introduce the concepts of SC-monotone and SC-convex functions.

Definition 1.1. Let $(\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$ be a simple Euclidean Jordan algebra of rank r. For any given $f : J \subseteq \mathbb{R} \to \mathbb{R}$, let $f^{\mathrm{SC}} : \mathbb{V}_J \to \mathbb{V}$ be defined as in (1.1). Then,

(a): f is said to be SC-monotone of order r if, for any $x, y \in \mathbb{V}_J$, it holds that

$$x \preceq_{\mathscr{K}} y \implies f^{\mathrm{SC}}(x) \preceq_{\mathscr{K}} f^{\mathrm{SC}}(y).$$

(b): f is said to be SC-convex of order r if, for any $x, y \in \mathbb{V}_J$ and $\eta \in (0,1)$, it holds that

$$f^{\mathrm{SC}}(\eta x + (1-\eta)y) \preceq_{\mathscr{K}} \eta f^{\mathrm{SC}}(x) + (1-\eta) f^{\mathrm{SC}}(y).$$

We call f *SC-monotone* (*SC-convex*) if it is SC-monotone (SC-convex) of all orders.

When $\mathbb{V}$ is the algebra $\mathscr{S}^n$ of $n \times n$ real symmetric matrices, Definition 1.1 represents the concepts of matrix monotone and matrix convex functions of order n; when $\mathbb{V}$ is the Jordan spin algebra, it gives the concepts of SOC-monotone and SOC-convex functions [5, 6]. The concepts of SC-monotone functions and operator monotone functions share "matrix monotone functions" in common, see Figure 1.

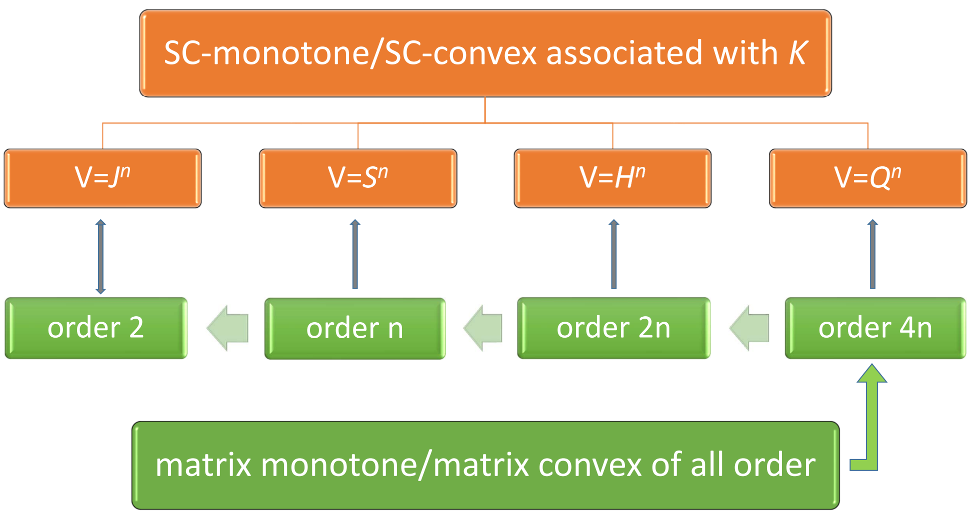

FIGURE 1. Relationship between SC-monotone/convex and operator monotone/convex.

2. THE P-CLASS FUNCTIONS ON SYMMETRIC CONES

Following the definitions of operator P-class function [1], we introduce the definition of SC P-class functions as below.

Definition 2.1. Let $f : J \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function. For all x, y with spectral in J, we say that f is of SC P-class on J if

$$f^{\mathrm{SC}}(\eta x + (1-\eta)y) \preceq_{\mathscr{K}} f^{\mathrm{SC}}(x) + f^{\mathrm{SC}}(y), \tag{2.1}$$

for all $\eta \in [0,1]$.

Example 2.1. **(a):** Every nonnegative SC-convex function is of SC P-class.
(b): Every non-zero SC P-class function has nonnegative values.

The following lemma summarizes the properties needed in subsequent analysis. We omit the proofs since they can be found in [12].

Lemma 2.1. *Let $\mathbb{V}$ be a Euclidean Jordan algebra and $\mathscr{K}$ be the associated symmetric cone. Then, the followings hold.*

(a): *For any $u \in \mathscr{K}$, there exists $v \in \mathscr{K}$ such that $v^2 = u$.*
(b): *$w \succeq_{\mathscr{K}} 0$ if and only if $\langle w, u\rangle \geq 0$ for all $u \succeq_{\mathscr{K}} 0$.*

Theorem 2.1. *If f is an SC P-class function on $(0,+\infty)$ such that $\lim_{t\to\infty} f(t) = 0$, then $-f$ is SC-monotone.*

Proof. Let $0 \prec_{\mathscr{K}} x \preceq_{\mathscr{K}} y$. Fixing $v \succ_{\mathscr{K}} 0$, we put $z = y - x + v$ and assume that all the spectral values of z are contained in $[\alpha, \beta]$ for some $0 < \alpha < \beta$. For each $\varepsilon > 0$, there exists $M > 0$ such that

$$f(t) \leq \varepsilon \ \text{ for all } \ t \geq M$$

since $\lim_{t\to\infty} f(t) = 0$. In addition, it follows from $\lim_{\eta\to 1^-} \frac{\eta}{1-\eta} = +\infty$ that there exists $\delta > 0$ such that

$$\frac{\eta}{1-\eta} \geq \frac{M}{\alpha} \ \text{ for all } \ \eta \in (1-\delta, 1).$$

Hence for all $i = 1, 2, \ldots, r$ and $\eta \in (1-\delta, 1)$,

$$\lambda_i\left(\frac{\eta}{1-\eta} z\right) \geq \frac{M}{\alpha}\lambda_i(z) \geq M,$$

which implies

$$f^{\mathrm{SC}}\left(\frac{\eta}{1-\eta} z\right) = \sum_{j=1}^{r} f\left(\lambda_j\left(\frac{\eta}{1-\eta} z\right)\right) e^{(j)} \preceq_{\mathscr{K}} \sum_{j=1}^{r} \varepsilon e^{(j)} = \varepsilon e.$$

Thus, for all $u \in \mathbb{V}$ and $\eta \in (1-\delta, 1)$,

$$\left\langle f^{\mathrm{SC}}\left(\frac{\eta}{1-\eta} z\right), u^2 \right\rangle \leq \langle \varepsilon e, u^2\rangle = \varepsilon \|u\|^2.$$

Since $\eta(y+v) = \eta x + (1-\eta)\left(\frac{\eta}{1-\eta} z\right)$ and f is of SC P-class function, we have

$$f^{\mathrm{SC}}(\eta(y+v)) \preceq_{\mathscr{K}} f^{\mathrm{SC}}(x) + f^{\mathrm{SC}}\left(\frac{\eta}{1-\eta} z\right)$$

for all $\eta \in (1-\delta, 1)$. Hence

$$\begin{aligned}\left\langle f^{\mathrm{SC}}(\eta(y+v)), u^2 \right\rangle &\leq \left\langle f^{\mathrm{SC}}(x), u^2 \right\rangle + \left\langle f^{\mathrm{SC}}\left(\frac{\eta}{1-\eta} z\right), u^2 \right\rangle \\ &\leq \left\langle f^{\mathrm{SC}}(x), u^2 \right\rangle + \varepsilon \|u\|^2,\end{aligned}$$

for all $u \in \mathbb{V}$. As $\eta \to 1^-$ and $\varepsilon \to 0^+$, we obtain

$$\left\langle f^{\mathrm{SC}}(y+v), u^2 \right\rangle \leq \left\langle f^{\mathrm{SC}}(x), u^2 \right\rangle$$

for all $u \in \mathbb{V}$. Then, taking $v \to 0$ (alternatively, $v = \alpha e$ and letting $\alpha \downarrow 0$), we conclude that $f^{\mathrm{SC}}(y) \preceq_{\mathscr{K}} f^{\mathrm{SC}}(x)$ by Lemma 2.1(b). $\square$

Corollary 2.1. *If $f : (0,+\infty) \to (0,+\infty)$ is an SC-convex function such that $\lim_{t\to\infty} f(t) = 0$, then $-f$ is SC-monotone.*

Let $f : J \to \mathbb{R}$ be an arbitrary function. For two fixed elements $x, y \in \mathbb{V}_J$, we define the function $f_{x,y} : [0,1] \to \mathbb{R}$ by

$$f_{x,y}(\eta) = f^{\mathrm{SC}}(\eta x + (1-\eta) y). \tag{2.2}$$

Via this function $f_{x,y}$, we provide characterization of SC P-class functions as below.

Proposition 2.1. *Let $f_{x,y}$ be defined as in (2.2). Then, the following statements are equivalent:*

(a): *f is an SC P-class function on J.*

(b): *For every $x, y \in \mathbb{V}_J$, the function $f_{x,y}$ satisfies*

$$f_{x,y}(t\eta_1 + (1-t)\eta_2) \preceq_{\mathscr{K}} f_{x,y}(\eta_1) + f_{x,y}(\eta_2).$$

for all $\eta_1, \eta_2 \in [0,1]$ and $t \in [0,1]$.

Proof. (a) $\Rightarrow$ (b): For any $\eta_1, \eta_2 \in [0,1]$ and $t \in [0,1]$,

$$\begin{aligned}
f_{x,y}(t\eta_1 + (1-t)\eta_2) &= f^{\mathrm{SC}}((t\eta_1 + (1-t)\eta_2)x + (1 - t\eta_1 - (1-t)\eta_2)y) \\
&= f^{\mathrm{SC}}(t[\eta_1 x + (1-\eta_1)y] + (1-t)[\eta_2 x + (1-\eta_2)y]) \\
&\preceq_{\mathscr{K}} f^{\mathrm{SC}}(\eta_1 x + (1-\eta_1)y) + f^{\mathrm{SC}}(\eta_2 x + (1-\eta_2)y) \\
&= f_{x,y}(\eta_1) + f_{x,y}(\eta_2),
\end{aligned}$$

which shows the assertion.

(b) $\Rightarrow$ (a): For any $x, y \in \mathbb{V}_J$ and $\eta \in [0,1]$,

$$\begin{aligned}
f^{\mathrm{SC}}(\eta x + (1-\eta)y) &= f_{x,y}(\eta) \\
&= f_{x,y}(\eta \cdot 1 + (1-\eta) \cdot 0) \\
&\leq f_{x,y}(1) + f_{x,y}(0) \\
&= f^{\mathrm{SC}}(x) + f^{\mathrm{SC}}(y),
\end{aligned}$$

that is, f is an SC P-class function. $\square$

We may also define a map $f_\eta : \mathbb{V}_J \times \mathbb{V}_J \to \mathbb{V}_J$ by

$$f_\eta(x,y) = f^{\mathrm{SC}}(\eta x + (1-\eta)y) \tag{2.3}$$

for fixed $\eta \in [0,1]$. Again, we have a characterization of SC P-class functions.

Proposition 2.2. *Let f_η be defined as in (2.3). Then, the followings assertions hold:*

(a): *If f is an SC P-class function on J, then f_η satisfies*

$$f_\eta(\xi(x,y) + (1-\xi)(u,v)) \preceq_{\mathscr{K}} f_\eta(x,y) + f_\eta(u,v) \tag{2.4}$$

for all $(x,y), (u,v) \in \mathbb{V}_J \times \mathbb{V}_J$ and $\xi \in [0,1]$.

(b): *If $\mathbb{V}_J$ is an convex cone in $\mathbb{V}$ and f_η satisfies inequality (2.4) for all $(x,y), (u,v) \in \mathbb{V}_J \times \mathbb{V}_J$ and $\xi \in [0,1]$, then f is an SC P-class function on J.*

Proof. (i) Fix $\eta \in [0,1]$ and let $(x,y),(u,v) \in \mathbb{V}_J \times \mathbb{V}_J$. Then, for all $\xi \in [0,1]$,

$$\begin{array}{rcl} f_\eta\left(\xi(x,y)+(1-\xi)(u,v)\right) & = & f_\eta(\xi x+(1-\xi)u, \xi y+(1-\xi)v) \\ & = & f^{\mathrm{SC}}(\eta(\xi x+(1-\xi)u)+(1-\eta)(\xi y+(1-\xi)v)) \\ & = & f^{\mathrm{SC}}(\xi(\eta x+(1-\eta)y)+(1-\xi)(\eta u+(1-\eta)v)) \\ & \preceq_{\mathscr{K}} & f^{\mathrm{SC}}(\eta x+(1-\eta)y)+f^{\mathrm{SC}}(\eta u+(1-\eta)v) \\ & = & f_\eta(x,y)+f_\eta(u,v), \end{array}$$

which shows the desired inequality.

(ii) Let $x,y \in \mathbb{V}_J$ and $\eta \in (0,1)$. If $\mathbb{V}_J$ is a convex cone in $\mathbb{V}$, that is, $\mathbb{V}_J+\mathbb{V}_J \subseteq \mathbb{V}_J$ and $\alpha\mathbb{V}_J \subseteq \mathbb{V}_J$, then $\eta^{-1}x, (1-\eta)^{-1}y \in \mathbb{V}_J$ and $(\eta^{-1}x,0),(0,(1-\eta)^{-1}y) \in \mathbb{V}_J \times \mathbb{V}_J$. Hence, we have

$$\begin{array}{rcl} f^{\mathrm{SC}}(\eta x+(1-\eta)y) & = & f_\eta(x,y) \\ & = & f_\eta(\eta(\eta^{-1}x,0)+(1-\eta)(0,(1-\eta)^{-1}y)) \\ & \preceq_{\mathscr{K}} & f_\eta(\eta^{-1}x,0)+f_\eta(0,(1-\eta)^{-1}y) \\ & = & f^{\mathrm{SC}}(x)+f^{\mathrm{SC}}(y). \end{array}$$

Moreover, for each $x \in \mathbb{V}_J$, $f^{\mathrm{SC}}(x) = f_{\frac{1}{2}}(x,x) \succeq_{\mathscr{K}} 0$, the last inequality holds for $\eta = 0,1$ as well. Therefore, f^{SC} is an SC P-class function. □

Proposition 2.3. *The following statements are equivalent:*

(a): *f is an SC P-class function on J.*

(b): *For every $x,y \in \mathbb{V}_J$ and $t > 0$ such that $(1+t)x-ty \in \mathbb{V}_J$, the following inequality holds*

$$f^{\mathrm{SC}}((1+t)x-ty) \succeq_{\mathscr{K}} f^{\mathrm{SC}}(x)-f^{\mathrm{SC}}(y).$$

Proof. (a) $\Rightarrow$ (b): Note that

$$x = \frac{1}{1+t}[(1+t)x-ty]+\frac{t}{1+t}y,$$

we have

$$f^{\mathrm{SC}}(x) \preceq_{\mathscr{K}} f^{\mathrm{SC}}((1+t)x-ty)+f^{\mathrm{SC}}(y),$$

which says $f^{\mathrm{SC}}((1+t)x-ty) \succeq_{\mathscr{K}} f^{\mathrm{SC}}(x)-f^{\mathrm{SC}}(y)$.

(b) $\Rightarrow$ (a): For any arbitrary $x \in \mathbb{V}_J$. According to the assumption, we have

$$f^{\mathrm{SC}}(2x-x) \succ_{\mathscr{K}} f^{\mathrm{SC}}(x)-f^{\mathrm{SC}}(x) = 0.$$

which also says that f is nonnegative. Suppose that $x,y \in \mathbb{V}_J$ and $\eta \in [0,1]$. It is obvious that (2.1) holds for $\eta = 0,1$ since f is nonnegative. For $0 \le \eta \le 1$, we set $t = \frac{1-\eta}{\eta}$ and $z = \eta x+(1-\eta)y$. We notice that $z \in \mathbb{V}_J$, $t > 0$, and $(1+t)z-ty = x \in \mathbb{V}_J$. This implies

$$f^{\mathrm{SC}}(x) = f^{\mathrm{SC}}((1+t)z-ty) \succeq_{\mathscr{K}} f^{\mathrm{SC}}(z)-f^{\mathrm{SC}}(y),$$

which is equivalent to

$$f^{\mathrm{SC}}(\eta x+(1-\eta)y) = f^{\mathrm{SC}}(z) \preceq_{\mathscr{K}} f^{\mathrm{SC}}(x)+f^{\mathrm{SC}}(y).$$

Thus, f is an SC P-class function. $\square$

Theorem 2.2. *Let $f : J \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous P-class function. Then, for any $x \in \mathbb{V}_J$ and unit vector $v \in \mathbb{R}^n$, there holds*

$$f(\langle x \circ v, v \rangle) \leq \operatorname{tr}\left(f^{\mathrm{SC}}(x)\right).$$

Proof. By the Spectral decomposition, we write $x = \sum_{i=1}^r \lambda_i(x) e^{(i)}$ for some positive integer r. Then, we have

$$\begin{aligned} f(\langle x \circ v, v \rangle) &= f\left(\left\langle \left(\sum_{i=1}^r \lambda_i(x) e^{(i)}\right) \circ v, v \right\rangle\right) \\ &= f\left(\left\langle \left(\sum_{i=1}^r \lambda_i(x) e^{(i)}\right), v^2 \right\rangle\right) \\ &= f\left(\sum_{i=1}^r \lambda_i \left\langle e^{(i)}, v^2 \right\rangle\right), \end{aligned}$$

where the second equality holds by the condition (iii) of Euclidean Jordan algebra. We note that

$$\left\langle e^{(i)}, v^2 \right\rangle \geq 0 \quad \text{for all } i = 1, 2, \ldots, r$$

since $e^{(i)}, v^2 \in \mathscr{K}$, and

$$\sum_{i=1}^r \left\langle e^{(i)}, v^2 \right\rangle = \left\langle \sum_{i=1}^r e^{(i)}, v^2 \right\rangle = \langle e, v^2 \rangle = \|v\|^2 = 1.$$

Because f is of P-class, we have

$$f(\langle x \circ v, v \rangle) = f\left(\sum_{i=1}^r \lambda_i \left\langle e^{(i)}, v^2 \right\rangle\right) \leq \sum_{i=1}^r f(\lambda_i) = \operatorname{tr}\left(f^{\mathrm{SC}}(x)\right).$$

This proves the assertion. $\square$

3. Symmetric cone Q-class functions

As mentioned in Section 1, the Q-class function was introduced by Godunova and Levin [13] in 1985. Fujii, Kian and Moslehian [10] further extended it to operator Q-class function and established some inequalities for these functions. Moreover, they also discussed the sufficient conditions for operator Q-class function being operator monotone functions. In this section, we investigate the symmetric cone Q-class function.

Definition 3.1. Let $f : J \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function. For all x, y with spectral in J, we say that f is of SC Q-class on J if

$$f^{\mathrm{SC}}(\eta x + (1-\eta) y) \preceq_{\mathscr{K}} \frac{f^{\mathrm{SC}}(x)}{\eta} + \frac{f^{\mathrm{SC}}(y)}{1-\eta}. \tag{3.1}$$

for all $\eta \in (0, 1)$.

It is obvious that every nonnegative SC P-class function is of SC Q-class. In addition, every non-zero SC Q-class function has nonnegative values; see the following proposition.

Proposition 3.1. *Let $f : J \to \mathbb{R}$ be a continuous function. If f is an SC Q-class function, then $f^{\mathrm{SC}}(x) \succeq_{\mathscr{K}} 0$ for all $x \in \mathbb{V}_J$. Furthermore, f is nonnegative on J.*

Proof. Let $x, y \in J$ and $\eta \in (0,1)$. Since f is an SC Q-class function, we have

$$f^{\mathrm{SC}}(\eta x + (1-\eta) y) \preceq_{\mathscr{K}} \frac{f^{\mathrm{SC}}(x)}{\eta} + \frac{f^{\mathrm{SC}}(y)}{1-\eta}.$$

Multiplying both side by $\eta(1-\eta)$, we obtain

$$\eta(1-\eta) f^{\mathrm{SC}}(\eta x + (1-\eta) y) \preceq_{\mathscr{K}} (1-\eta) f^{\mathrm{SC}}(x) + \eta f^{\mathrm{SC}}(y).$$

Letting $\eta \to 0^+$, we conclude $f^{\mathrm{SC}}(x) \succeq_{\mathscr{K}} 0$. Furthermore, f is nonnegative on J. $\square$

Theorem 3.1. *Let $\alpha > 0$, $\beta > 1$ and $f : (\alpha, +\infty) \to \mathbb{R}$ be a continuous function with $f(t) \leq t^{\beta}$. If $t \mapsto f(t^{-1})$ is an SC Q-class function on $(0, 1/\alpha)$, then f is SC-monotone.*

Proof. Let $0 \prec_{\mathscr{K}} \alpha e \prec_{\mathscr{K}} x$, $0 \prec_{\mathscr{K}} y$, and $0 \preceq_{\mathscr{K}} v$. We note that $\lambda_i(x) > \alpha$ for all $i = 1, 2, \ldots, r$. By the SC-monotonicity of $-t^{-1}$, we have

$$\begin{aligned} z &:= (x+y+v)^{-1} \prec_{\mathscr{K}} x^{-1} \prec_{\mathscr{K}} \frac{1}{\alpha} e, \\ w &:= x^{-1} - (x+y+v)^{-1} \prec_{\mathscr{K}} x^{-1} \prec_{\mathscr{K}} \frac{1}{\alpha} e. \end{aligned}$$

Since $f(t^{-1})$ is an SC Q-class function on $(0, 1/\alpha)$ and for all $\eta \in (0,1)$

$$\eta(z+w) = \eta z + (1-\eta) \frac{\eta}{1-\eta} w,$$

we can obtain

$$\begin{aligned} f^{\mathrm{SC}}\left((\eta(z+w))^{-1}\right) &\preceq_{\mathscr{K}} \frac{f^{\mathrm{SC}}(z^{-1})}{\eta} + \frac{1}{1-\eta} f^{\mathrm{SC}}\left(\left(\frac{\eta}{1-\eta} w\right)^{-1}\right) \\ &\preceq_{\mathscr{K}} \frac{f^{\mathrm{SC}}(z^{-1})}{\eta} + \frac{1}{1-\eta}\left(\frac{1-\eta}{\eta}\right)^{\beta} w^{-\beta} \\ &= \frac{f^{\mathrm{SC}}(z^{-1})}{\eta} + \frac{(1-\eta)^{\beta-1}}{\eta^{\beta}} w^{-\beta}. \end{aligned}$$

Letting $\eta \to 1$, we have

$$f^{\mathrm{SC}}(x) = f^{\mathrm{SC}}\left((z+w)^{-1}\right) \prec_{\mathscr{K}} f^{\mathrm{SC}}\left(z^{-1}\right) = f^{\mathrm{SC}}(x+y+v).$$

Then, taking $v \to 0$, we conclude that $f^{\mathrm{SC}}(x) \preceq_{\mathscr{K}} f^{\mathrm{SC}}(x+y)$ for all $y \succeq_{\mathscr{K}} 0$, that is, f is SC-monotone on $(\alpha, +\infty)$. $\square$

Example 3.1. The following functions satisfy the conditions of Theorem 3.1

(a): In the setting of second-order cone, we consider the function $f(t) = t^r$ on $(1,+\infty)$. We first note that $t^r \leq t^2$ for all $t \in (1,+\infty)$. It is known from [6, Proposition 5.2(a)] that $t \mapsto f(t^{-1})$ is nonnegative SOC-convex on $(0,1)$, which implies $f(t^{-1})$ is an SC Q-class function in the case of second-order cone.

(b): For the function $f(t) = \ln t$ on $(1,+\infty)$. It is easy to verify that $\ln t \leq t \leq t^\beta$ on $(1,+\infty)$. From [4, Example 3.2(i)], we know that the function $f(t^{-1}) = -\ln t$ is nonnegative SC-convex on $(0,1)$, and hence is of SC Q-class on $(0,1)$.

Theorem 3.2. *Let $\alpha > 0$, $\beta > 1$ and $f : (0,1/\alpha) \to \mathbb{R}$ be a continuous function with $f(t) \leq t^{-\beta}$. If f is an SC Q-class function on $(0,1/\alpha)$, then $-f$ is SC-monotone.*

Proof. It follows from Theorem 3.1 that $t \mapsto f(t^{-1})$ is SC-monotone on $(\alpha,+\infty)$. Then, for any $0 \prec_{\mathscr{K}} x \preceq_{\mathscr{K}} y \prec_{\mathscr{K}} \frac{1}{\alpha}e$, we have $\alpha e \prec_{\mathscr{K}} y^{-1} \preceq_{\mathscr{K}} x^{-1}$. Hence $f^{\mathrm{SC}}(y) \preceq_{\mathscr{K}} f^{\mathrm{SC}}(x)$. □

Corollary 3.1. *Let $f : (0,+\infty) \to \mathbb{R}$ be a continuous function with $f(t) \leq t^{-\beta}$ for some $\beta > 1$. If f is an SC Q-class function on $(0,+\infty)$, then $-f$ is SC-monotone.*

Theorem 3.3. *If f is an SC Q-class function on $(0,+\infty)$ such that with $tf(t) \leq f(t^{-1})$ and $\lim_{t\to 0^+} f(t) = 0$, then $-f$ is SC-monotone.*

Proof. Let $0 \prec_{\mathscr{K}} x \preceq_{\mathscr{K}} y$ and $0 \prec_{\mathscr{K}} v$. We note that, for any $\eta \in (0,1)$

$$\eta(y+v) = \eta x + (1-\eta)\frac{\eta}{1-\eta}(y-x+v),$$

which together with f being of SC Q-class implies that

$$\begin{aligned}
f^{\mathrm{SC}}(\eta(y+v)) &\preceq_{\mathscr{K}} \frac{f^{\mathrm{SC}}(x)}{\eta} + \frac{f^{\mathrm{SC}}\left(\frac{\eta}{1-\eta}(y-x+v)\right)}{(1-\eta)} \\
&\preceq_{\mathscr{K}} \frac{f^{\mathrm{SC}}(x)}{\eta} + \frac{1}{1-\eta}\frac{1-\eta}{\eta}(y-x+v)^{-1} f^{\mathrm{SC}}\left(\frac{1-\eta}{\eta}(y-x+v)^{-1}\right) \\
&= \frac{f^{\mathrm{SC}}(x)}{\eta} + \frac{1}{\eta}(y-x+v)^{-1} f^{\mathrm{SC}}\left(\frac{1-\eta}{\eta}(y-x+v)^{-1}\right).
\end{aligned}$$

Using the same techniques as in Section 2, that is, letting $\eta \to 1$ and $v \to 0$, we conclude $f^{\mathrm{SC}}(y) \preceq_{\mathscr{K}} f^{\mathrm{SC}}(x)$. □

4. Concluding Remarks

In this paper, we built up the concepts of P-class and Q-class functions to the setting of symmetric cone. It is worth to note that the symmetric cone monotonicity and symmetric cone convexity are more general than the operator monotonicity and operator convexity. Indeed, for the important type of symmetric cones, the second-order cone (or called the Lorentz cone), we know that the SOC monotonicity is equivalent to the matrix monotonicity of order 2, and SOC convexity is equivalent to the matrix convexity of order 2 (see [6, 15]). In view of this, we derived some more general results than the ones in [1, 10, 19]. For further study, we shall investigate how to use these results to design algorithm on symmetric cone programming.

Besides, for real-valued functions, the P-class functions can be regarded as the generalizations from either nonnegative convex functions or nonnegative monotone functions, see [1].

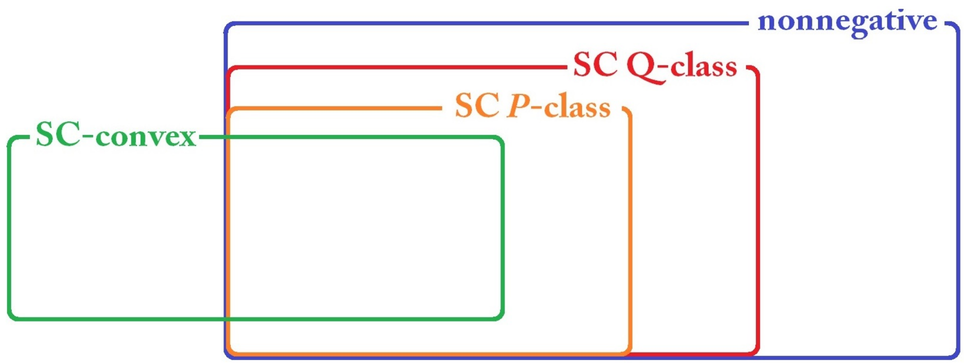

FIGURE 2. Relationship between SC P-class/Q-class and SC convex.

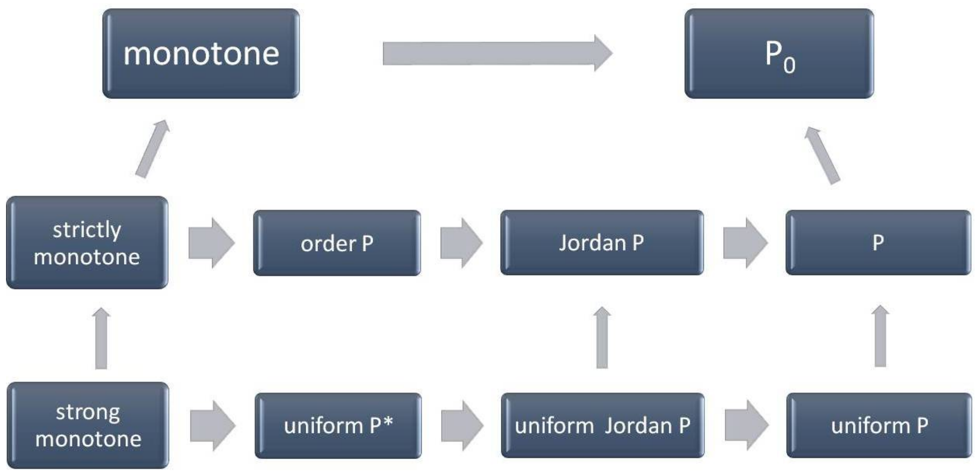

FIGURE 3. Relationship between monotone and P-property.

However, for vector-valued functions associated with symmetric cone, we pointed out some classification: the concept of SC P-class (or SC Q-class) is a natural extension from the SC-convexity. It is totally different from the "P-property", which is a concept generalized from the monotonicity, see Figure 3. For more details about P-property, we refer to [22].

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