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SOLUTION PROPERTIES AND ERROR BOUNDS FOR SEMI-INFINITE COMPLEMENTARITY PROBLEMS

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Abstract. In this paper, we deal with the semi-infinite complementarity problems (SICP), in which several important issues are covered, such as solvability, semismoothness of residual functions, and error bounds. In particular, we characterize the solution set by investigating the relationship between SICP and the classical complementarity problem. Furthermore, we show that the SICP can be equivalently reformulated as a typical semi-infinite min-max programming problem by employing NCP functions. Finally, we study the concept of error bounds and introduce its two variants, ε-error bounds and weak error bounds, where the concept of weak error bounds is highly desirable in that the solution set is not restricted to be nonempty.

1. **Introduction.** The classical nonlinear complementarity problem (NCP) is to find an $x \in \mathbb{R}^n$ such that

$$x \geq 0,\ F(x) \geq 0,\ x^T F(x) = 0,$$

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where F is a mapping from $\mathbb{R}^n$ into itself. Extensive references to the developments in this subject and their historical backgrounds can be found in Facchinei and Pang [12] and Harker and Pang [15].

Roughly speaking, the classical complementarity problem has been extended in two ways. The first one is generalizing the nonnegative orthant $\mathbb{R}^n_+$ to various other convex cone, such as second-order cone [1], semi-definiteness cone [16], or more generally, symmetric cone [27, 28]. An implicit assumption shared by these problems is that the data of the problem, such as the mapping F and the cone involved, are all fixed and completely independent of other parameters. Unfortunately, this is not always the case in reality. For example, in optimal control or engineering design fields [5], the data of the problem involves a time parameter; in non-cooperative games (e.g., generalized Nash equilibrium [11]), the strategy of each player is dependent on those of others. To address the problem of this type, we need to consider another generalized form as follows: find a vector $x \in \mathbb{R}^n$ such that

$$x \geq 0, \;\; F(x,w) \geq 0, \;\; x^T F(x,w) = 0, \quad w \in \Omega, \tag{1.1}$$

where Ω is a set in $\mathbb{R}^m$ and F is a mapping from $\mathbb{R}^n \times \Omega$ into $\mathbb{R}^n$. As the parameter w is a random variable with certain probability distribution, the above problem is said to be stochastic complementarity problem. Results of this type were first treated by Chen and Fukushima [6], and subsequent investigations were carried out by other authors; see [8, 13, 18, 17, 19]. By using stochastic approach, they could obtain a solution in the probability sense, which, however, is not a real solution to the original problem (1.1). Two most natural questions to ask are: (a) How to deal with the case in which the parameter w is not a random variable; (b) Under which conditions can we find an exact solution of (1.1).

Depending on the role played by the parameter w, the problem (1.1) is divided into two classes and the corresponding techniques are completely different. One, as mentioned above, is the stochastic complementarity problem, provided that w is regarded as a random variable. Otherwise, to avoid the confusion, it is preferable to refer to the problem (1.1) as a semi-infinite complementarity problem (SICP for short), because it shares the characterizations of semi-infinite programming and complementarity problem, i.e., the design vector x is finite-dimensional, but the number of the complementarity problems involved in (1.1) is infinite. In addition, if $F(\cdot,\cdot)$ is an affine function with respect to x, i.e., $F(x,w) = M(w)x + q(w)$, where $M(w) \in \mathbb{R}^{n\times n}$ and $q(w) \in \mathbb{R}^n$, then problem (1.1) is called a semi-infinite linear complementarity problem, abbreviated as $\mathrm{SILCP}(q(w), M(w), \Omega)$. Similarly, denote by $\mathrm{SINCP}(F(\cdot,w),\Omega)$ for the case where F is nonlinear with respect to x. Here it is worth mentioning that the relation between SICP and NCP is a different with that between SIP (semi-infinite programming) and NLP (nonlinear programming). In fact, SIP can reduce to NLP when the number of constraint functions is finite, but this is not shared by SICP and NCP except for Ω consisting of a single point.

In this paper, we begin with developing the solvability and feasibility of the semi-infinite complementarity problem. In particular, for the nonlinear case, we show that the solution set S^* coincides precisely with the intersection of the solution sets of two classical nonlinear complementarity problems $\mathrm{NCP}(F_{\max})$ and $\mathrm{NCP}(F_{\min})$, i.e., $S^* = \mathrm{SOL}(F_{\max}) \cap \mathrm{SOL}(F_{\min})$. However, for the linear case, the equation will fail if we only replace $F_{\max}(x)$ by $M_{\max}x + q_{\max}$ and $F_{\min}(x)$ by $M_{\min}x + q_{\min}$. In other words, for the semi-infinite linear complementarity problem, we have $S^* \subseteq \mathrm{SOL}(q_{\max}, M_{\max}) \cap \mathrm{SOL}(q_{\min}, M_{\min})$, and the inclusion can be strict unless some assumptions are made on the structure of the expansive matrix $(M(w), q(w))$, as will be illustrated by Theorem 2.3 and Example 2.4. Furthermore, we transform the semi-infinite complementarity problem into an equivalent semi-infinite min-max programming problem by utilizing NCP functions. This offers another explanation of why we call the problem (1.1) as semi-infinite complementarity problem. The semismoothness of residual functions are discussed as well.

The theory of error bounds provides a useful aid for understanding the connection between a residual function and the actual distance to the solution set, and hence plays an important role in convergence analysis and stopping criteria for many iterative algorithms; for comprehensive surveys of this topic, please refer to [23] and references therein. In the latter part of this paper, we discuss error bounds and introduce its two variants, ε-error bounds and weak error bounds. Specifically, we show that the ε-error bounds can be obtained by using the well-known error bounds for the classical complementarity problem $\mathrm{LCP}(q(w), M(w))$, where the parameter ε represents the degree of the approximation between S^* and $\mathrm{SOL}(q(w), M(w))$. Nevertheless, it should be realized that the existence of a vector x satisfying the complementarity conditions for all $w \in \Omega$ may be more restrictive, that is, the solution set S^* may be empty. This makes the utility of error bounds be somewhat limited, because in many situations it is possible to find a vector x such that the complementarity condition holds true for some w but not for others. As a remedy of this difficulty, we introduce the concept of weak error bounds, which makes sense even if the solution set is empty. Example 5.4 illustrates that the weak error bounds can be readily derived from error bounds, but the converse is not necessarily true.

The paper is organized as follows. In Section 2, we characterize the solution set and provide criteria for the feasibility of the SICP. In Section 3, we reformulate the SICP as typical semi-infinite min-max programming problems and address the semismoothness of residual functions. The concept of error bounds and its one variant, ε-error bounds, are treated in Section 4, whereas another variant, weak error bound, is discussed in Section 5. Some conclusions are drawn in Section 6.

A few words about our notations. All vectors are column vectors and superscript T denotes transpose. We denote by $\mathbb{R}^n$ the n-dimensional real vector space, by $\mathbb{R}^{n\times n}$ the space of $n \times n$ real matrices, and by $\mathbb{B}$ the unit ball. For a vector $x \in \mathbb{R}^n$, x_+ will denote the orthogonal projection on the nonnegative orthant $\mathbb{R}^n_+$, that is, $(x_+)_i = \max\{x_i, 0\}$ for all $i = 1, 2, \cdots, n$. The diameter of a set A, denoted by $\mathrm{diam}(A)$, is defined as the maximum distance between any pair of points in this set, that is, $\mathrm{diam}(A) = \max\limits_{x,y\in A} \|x - y\|$. Let S^* be the solution of the problem (1.1). For any fixed $w \in \Omega$, we denote by $\mathrm{SOL}(F(\cdot, w))$ the solution set of the classic nonlinear complementarity problem $\mathrm{NCP}(F(\cdot, w))$, and by $\mathrm{SOL}(q(w), M(w))$ the solution set of the classic linear complementarity problem $\mathrm{LCP}(q(w), M(w))$. It is well known that various matrix classes have played a key role in all aspects of the classical linear complementarity problem; see [10] for the details. For example, a matrix M is called (i) an S-matrix if there exists a vector $z > 0$ such that $Mz > 0$; (ii) a copositive matrix if $x \geq 0$ implies that $x^T Mx \geq 0$; (iii) an R_0-matrix if

$$x \geq 0,\ Mx \geq 0,\ x^T Mx = 0 \quad \Longrightarrow \quad x = 0.$$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be semismooth at x if f is locally Lipschitzian at x and the limit

$$\lim_{\substack{V\in\partial f(x+th')\\ h'\to h,t\downarrow 0}} \{Vh'\}$$

exists for any $h \in \mathbb{R}^n$, where ∂f denotes the generalized Jacobian defined by Clarke [9]. Recall also that f is said to be semidifferentiable at x if the limit

$$\lim_{\substack{h'\to h\\ t\downarrow 0}} \frac{f(x+th')-f(x)}{t}$$

exists for any $h \in \mathbb{R}^n$; see [26, Chapter 7] for more details.

2. **Solution properties.** The main aim of this section is to present the characterization of the solution set and provide criteria for the feasibility of SICP. Our analysis is based on relating SICP to two classical complementarity problems as defined, respectively, in (2.2) for the nonlinear case and in (2.4) for the linear case.

Theorem 2.1. *Consider the* SILCP$(q(w), M(w), \Omega)$. *Then*

$$S^* = \bigcap_{w\in\Omega} \mathrm{SOL}(q(w), M(w)). \tag{2.1}$$

Moreover, if $M(w_0)$ is an R_0-Matrix for some $w_0 \in \Omega$, then $S^* \neq \emptyset$ if and only if $\bigcap_{i=1}^{p}$ $\mathrm{SOL}(q(w_i), M(w_i)) \neq \emptyset$ for any finite many points $w_1, \cdots, w_p \in \Omega$. Furthermore, if $M(w)$ is a column sufficient matrix for each $w \in \Omega$, then at most n+1 points are needed to consider.

Proof. First, from definition, (2.1) is trivial. If $M(w_0)$ is an R_0-matrix, then the set $\mathrm{SOL}(q(w_0), M(w_0))$ is bounded [10, Proposition 3.9.23], which in turn implies the boundedness of S^*. On the other hand, since $\mathrm{SOL}(q(w), M(w))$ is closed for each w, so is S^*. Thus S^* is compact. Applying the finite intersection theorem of compact sets, we obtain the first part of the theorem. The column sufficiency of $M(w)$ implies that $\mathrm{SOL}(q(w), M(w))$ is convex by [10, Theorem 3.5.8]. Hence, the second part follows from Helly's Theorem [25, Corollary 21.3.2]. □

It is easy to see that the identity $S^* = \bigcap_{w\in\Omega} \mathrm{SOL}(F(\cdot, w))$ remains true for the nonlinear case SINCP$(F(\cdot, w), \Omega)$. However, it is not suggested the ripe possibilities, because we have to solve all of the classical complementarity problems one by one. To overcome this drawback, we assume that the set Ω is compact and the mapping F is continuous on $\mathbb{R}^n \times \Omega$, which ensure the well-definedness of the following function

$$F_{\max}(x) = \begin{pmatrix} \max_{w\in\Omega} F_1(x,w) \\ \vdots \\ \max_{w\in\Omega} F_n(x,w) \end{pmatrix} \quad \text{and} \quad F_{\min}(x) = \begin{pmatrix} \min_{w\in\Omega} F_1(x,w) \\ \vdots \\ \min_{w\in\Omega} F_n(x,w) \end{pmatrix}. \tag{2.2}$$

The following result shows that the solution set S^* coincides with the intersection of the solution sets of two classical nonlinear complementarity problems NCP$(F_{\max})$ and NCP$(F_{\min})$.

Theorem 2.2. *Consider the* SINCP$(F(\cdot, w), \Omega)$. *If* Ω *is compact and* F *is continuous on* $\mathbb{R}^n \times \Omega$, *then* $S^* = \mathrm{SOL}(F_{\max}) \cap \mathrm{SOL}(F_{\min})$.

Proof. Suppose $x^* \in S^*$. Then $F_{\max}(x^*) \geq 0$ by definition. Note that $(x^*)^T F(x^*, w) = 0$ is equivalent to $x_i^* F_i(x^*, w) = 0$ for all $i = 1, 2, \cdots, n$. Taking the pointwise supremum yields that $x_i^*(F_{\max}(x^*))_i = 0$. Hence, $x^* \in \mathrm{SOL}(F_{\max})$. Similarly, we can argue that $x^* \in \mathrm{SOL}(F_{\min})$, which says $S^* \subseteq \mathrm{SOL}(F_{\max}) \cap \mathrm{SOL}(F_{\min})$. Now we show the reverse inclusion. Let $x^* \in \mathrm{SOL}(F_{\max}) \cap \mathrm{SOL}(F_{\min})$. Because $F_{\min}(x^*) \geq 0$ and $x^* \geq 0$, we know for any $w \in \Omega$ and $i = 1, 2, \cdots, n$ that

$$F_i(x^*, w) \geq 0 \text{ and } x_i^* F_i(x^*, w) \geq 0. \tag{2.3}$$

On the other hand, since $x^* \in \mathrm{SOL}(F_{\max})$, we have $x_i^*(F_{\max}(x^*))_i = 0$, which in turn implies that $x_i^* F_i(x^*, w) \leq 0$. Combing this and (2.3) yields $x_i^* F_i(x^*, w) = 0$ for any $w \in \Omega$ and $i = 1, 2, \cdots, n$. This completes the proof. □

The above result makes it possible to characterize the solution of semi-infinite complementarity problem by checking two classical complementarity problems. We now turn our attention to the linear case SILCP$(q(w), M(w), \Omega)$. Let $a_{ij}(w)$ denote the (i, j)-entry of a matrix $M(w)$. Define

$$M_{\max} = \begin{pmatrix} \max\limits_{w\in\Omega} a_{11}(w) & \cdots & \max\limits_{w\in\Omega} a_{1n}(w) \\ & \vdots & \\ \max\limits_{w\in\Omega} a_{n1}(w) & \cdots & \max\limits_{w\in\Omega} a_{nn}(w) \end{pmatrix}, \tag{2.4}$$

$$M_{\min} = \begin{pmatrix} \min\limits_{w\in\Omega} a_{11}(w) & \cdots & \min\limits_{w\in\Omega} a_{1n}(w) \\ & \vdots & \\ \min\limits_{w\in\Omega} a_{n1}(w) & \cdots & \min\limits_{w\in\Omega} a_{nn}(w) \end{pmatrix},$$

and

$$q_{\max} = \begin{pmatrix} \max\limits_{w\in\Omega} q_1(w) \\ \vdots \\ \max\limits_{w\in\Omega} q_n(w) \end{pmatrix}, \qquad q_{\min} = \begin{pmatrix} \min\limits_{w\in\Omega} q_1(w) \\ \vdots \\ \min\limits_{w\in\Omega} q_n(w) \end{pmatrix}.$$

Motivated by Theorem 2.2, it is natural to speculate whether $S^* = \mathrm{SOL}(F_{\max}) \cap \mathrm{SOL}(F_{\min})$ remains true if we replace $F_{\max}(x)$ by $M_{\max}x + q_{\max}$ and $F_{\min}(x)$ by $M_{\min}x + q_{\min}$, i.e., does S^* equal $\mathrm{SOL}(q_{\max}, M_{\max}) \cap \mathrm{SOL}(q_{\min}, M_{\min})$? Unfortunately, the equality may fail unless some additional assumptions are made. The following theorem and example will elaborate more about this point.

Theorem 2.3. *Consider the* SILCP$(q(w), M(w), \Omega)$. *If* Ω *is compact and* $M(w)$ *and* $q(w)$ *are continuous on* Ω, *then*

$$S^* \supseteq \mathrm{SOL}(q_{\max}, M_{\max}) \cap \mathrm{SOL}(q_{\min}, M_{\min}).$$

Furthermore, suppose in each row of the expansive matrix $(M(w), q(w))$, *the minimum (and maximum) is attained by a common* $\hat{w}$ *(and* $\bar{w}$*), i.e., for each* $i = 1, 2, \ldots, n$, *there exist* $\hat{w}^i, \bar{w}^i \in \Omega$ *such that* $(M_{\min}, q_{\min})_i = \big(M(\hat{w}^i), q(\hat{w}^i)\big)_i$ *and*

$(M_{\max}, q_{\max})_i = \big(M(\bar{w}^i), q(\bar{w}^i)\big)_i$ *where the subscript* i *denotes the* i*-th row vector. Then the equality holds, i.e.,*

$$S^* = \mathrm{SOL}(q_{\max}, M_{\max}) \cap \mathrm{SOL}(q_{\min}, M_{\min}).$$

Proof. According to the rules of calculation dealing with maximization and minimization [26, Exercise 1.36], we have

$$\begin{array}{rcl} \max\limits_{w\in\Omega}(M(w)x + q(w)) & \leq & M_{\max}x + q_{\max} \\ \min\limits_{w\in\Omega}(M(w)x + q(w)) & \geq & M_{\min}x + q_{\min} \end{array} \tag{2.5}$$

for all $x \geq 0$. By applying (2.5) and following an argument similar to that for Theorem 2.2, we obtain the first part of the theorem. Noting that (2.5) holds as equality by invoking the hypothesis, the second part follows readily. □

Example 2.4. Consider the $\mathrm{SILCP}(q(w), M(w), \Omega)$ with

$$M(w) = \begin{pmatrix} 1-2w & -1 \\ 0 & -w \end{pmatrix}, \ q(w) = \begin{pmatrix} 1 \\ w \end{pmatrix}, \text{ and } \Omega = [0,1].$$

From a simple calculation, we have $S^* = \{(0,0)^T, (0,1)^T\}$. On the other hand,

$$M_{\max} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \ q_{\max} = \begin{pmatrix} 1 \\ 1 \end{pmatrix};$$

and

$$M_{\min} = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \ q_{\min} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then, $\mathrm{SOL}(q_{\max}, M_{\max}) = \{(0,0)^T\}$ and $\mathrm{SOL}(q_{\min}, M_{\min}) = \{(0,0)^T, (1,0)^T\}$. Thus, $S^* \supsetneq \mathrm{SOL}(q_{\max}, M_{\max}) \cap \mathrm{SOL}(q_{\min}, M_{\min})$, i.e., the inclusion is strict. Now we show that the equality holds true if $q(w)$ is replaced by a constant vector $\tilde{q} = (1,1)^T$, i.e., $q(w) = \tilde{q}$ for all $w \in \Omega$. Actually, in this case, only one entry in every row of the expansive matrix $(M(w), q)$ is dependent on w, and hence the hypothesis in Theorem 2.3 holds. By direct calculation again, we have $S^* = \{(0,0)\}$, $\mathrm{SOL}(q, M_{\max}) = \{(0,0)^T\}$, and $\mathrm{SOL}(q, M_{\min}) = \{(0,0)^T, (1,0)^T, (0,1)^T\}$ under this case. Therefore, $S^* = \mathrm{SOL}(q, M_{\max}) \cap \mathrm{SOL}(q, M_{\min})$.

Along the same lines as that in Theorems 2.2 and 2.3, we present a set of descriptions of the solution set S^*.

Corollary 2.5. *The following statements hold.*

(a): *Consider the* $\mathrm{SINCP}(F(\cdot, w), \Omega)$. *If* Ω *is compact and* $F(x, \cdot)$ *is continuous on* Ω *for each* x*, then*

$$\begin{array}{rcl} S^* & = & \mathrm{SOL}(\alpha F_{\min} + \beta F_{\max}) \cap \{x | F_{\min}(x) \geq 0\} \\ & = & \mathrm{SOL}(F_{\min}) \cap \{x | x^T F_{\max}(x) \leq 0\} \end{array}$$

where $\alpha \geq 0$ *and* $\beta > 0$.

(b): *Consider the* $\mathrm{SILCP}(q(w), M(w), \Omega)$. *If* Ω *is compact and* $M(w)$ *and* $q(w)$ *are continuous on* Ω*, then*

$$\begin{array}{rcl} S^* & \supseteq & \mathrm{SOL}(\alpha q_{\min} + \beta q_{\max}, \alpha M_{\min} + \beta M_{\max}) \cap \{x | M_{\min}x + q_{\min} \geq 0\} \\ & = & \mathrm{SOL}(q_{\min}, M_{\min}) \cap \{x | x^T (M_{\max}x + q_{\max}) \leq 0\} \end{array}$$

where $\alpha \geq 0$ and $\beta > 0$. Furthermore, if in each row of the expansive matrix $(M(w), q(w))$, the minimum (and maximum) is attained by a common $\hat{w}$ (and $\bar{w}$), then the equality holds.

Proof. As mentioned, the arguments are similar to those for Theorems 2.2 and 2.3. In view of this, we only provide sketch proof. To see part(a), we only need to check whether

$$S^* \supseteq \mathrm{SOL}(\alpha F_{\min} + \beta F_{\max}) \cap \{x | F_{\min}(x) \geq 0\} =: A$$

because the reverse inclusion is clear. Suppose $x^* \in A$ which says $F_{\min}(x^*) \geq 0$ and

$$\alpha F_{\min}(x^*) + \beta F_{\max}(x^*) \geq 0,\ x^* \geq 0,\ \big(\alpha F_{\min}(x^*) + \beta F_{\max}(x^*)\big)^T x^* = 0.$$

Then, we have

$$0 = \big(\alpha F_{\min}(x^*) + \beta F_{\max}(x^*)\big)^T x^* \geq (\alpha + \beta) F_{\min}(x^*)^T x^* \geq 0$$

which implies $F_{\min}(x^*)^T x^* = 0$. Plugging it into the first equality above yields $F_{\max}(x^*)^T x^* = 0$. This means $x^* \in \mathrm{SOL}(F_{\min}) \cap \mathrm{SOL}(F_{\max}) = S^*$ by Theorem 2.2. The proof for part (b) is not repeated here. □

We next turn our attention to the feasibility of SICP. The feasibility issue for the complementarity problem is always important because in some real life applications, such as in engineering design and economy, the data are restricted in domain region not whole space. Before proceeding, let us introduce the following concept, which reduces to that of S-matrix in classical complementarity problem when Ω only contains a single element.

Definition 2.6. The matrix $M(w)$ is said to be a semi-infinite S-matrix relative to a set Ω if there exists a vector $z > 0$ such that $M(w)z > 0$ for all $w \in \Omega$.

With this preparation, the condition that guarantees the feasibility of SICP can be stated.

Theorem 2.7. *Consider the* SILCP$(q(w), M(w), \Omega)$*. Let Ω be compact and $M(\cdot)$ be continuous on Ω. Then $M(w)$ is a semi-infinite S-matrix relative to Ω if and only if the* SILCP$(q(w), M(w), \Omega)$ *is feasible for all $q \in C(\Omega)$, where $C(\Omega)$ denotes all continuous mapping on Ω.*

Proof. We first show "only if" part. From the facts that Ω is compact and that $M(w)z > 0$ for all $w \in \Omega$, there exists a sufficiently small scalar $\alpha > 0$ such that $M(w)z \geq \alpha e$ for all $w \in \Omega$ where $e = (1, 1, \cdots, 1)^T$. Now choose $\mu > 0$ with $\mu e > -q_{\min}$. Letting $\bar{\mu} = \frac{\mu}{\alpha}$, it follows that $\bar{\mu} z \geq 0$ and $M(w)(\bar{\mu} z) + q_{\min} \geq 0$. Therefore, $\bar{\mu} z$ is a feasible point. We next show "if" part. Let $q(w) := \tilde{q} < 0$ for all $w \in \Omega$. The feasibility of SILCP$(q(w), M(w), \Omega)$ (i.e. SILCP$(\tilde{q}, M(w), \Omega)$) means the existence of a vector $z \geq 0$ such that $M(w)z \geq -\tilde{q}$ for all $w \in \Omega$. Note that when $\lambda > 0$ small enough we have $z + \lambda e > 0$ and $\lambda M_{\min} e > \tilde{q}$. Therefore,

$$\begin{aligned} M(w)(z + \lambda e) &= M(w)z + \lambda M(w)e \\ &\geq M(w)z + \lambda M_{\min} e \\ &> M(w)z + \tilde{q} \\ &\geq 0. \end{aligned}$$

This completes the proof. □

As a direct consequence of Theorem 2.7, the following result furnishes a simple criterion for $\mathrm{SILCP}(q(w), M(w), \Omega)$ to be feasible.

Corollary 2.8. *Consider the* $\mathrm{SILCP}(q(w), M(w), \Omega)$. *Suppose* Ω *is compact and* $M(w)$ *is continuous on* Ω. *If* $M_{\min}$ *is an* S*-matrix, then* $\mathrm{SILCP}(q(w), M(w), \Omega)$ *is feasible for all* $q \in C(\Omega)$.

Proof. From (2.4) and the definition of S-matrix, it can be easily verified that if $M_{\min}$ is an S-matrix (i.e. $M_{\min}z > 0$ for some $z > 0$), then $M(w)z > 0$ for all $w \in \Omega$, implying that $M(w)$ is a semi-infinite S-matrix. Therefore, Theorem 2.7 is applicable. □

For simplicity, we write $\mathrm{SILCP}(0, M(w), \Omega)$ and $\mathrm{LCP}(0, M(w))$ as $\mathrm{SILCP}(M(w), \Omega)$ and $\mathrm{LCP}(M(w))$, respectively. In a similar manner, their corresponding solution sets are written as $\mathrm{SOL}(M(w), \Omega)$ and $\mathrm{SOL}(M(w))$, respectively.

Theorem 2.9. *Consider the* $\mathrm{SILCP}(M(w), \Omega)$. *Suppose* Ω *is compact and* $M(w)$ *is continuous on* Ω. *If* $M(w)$ *is a copositive matrix for each* $w \in \Omega$, *then*

$$\{x \in \mathbb{R}^n_+ | M^T_{\max} x \le 0\} \subseteq S^*. \tag{2.6}$$

Proof. Since $M(w)$ is copositive matrix, we have $\{x \in \mathbb{R}^n_+ | M(w)^T x \le 0\} \subseteq \mathrm{SOL}(M(w))$ by Theorem 3.8.13 in [10]. Hence $\bigcap_{w\in\Omega} \{x \in \mathbb{R}^n_+ | M(w)^T x \le 0\} \subseteq \bigcap_{w\in\Omega} \mathrm{SOL}(M(w))$. Using the facts that $S^* = \bigcap_{w\in\Omega} \mathrm{SOL}(M(w))$ by Theorem 2.1 and that $M^T_{\max} x \ge M(w)x$ for all $x \in \mathbb{R}^n_+$ and $w \in \Omega$, we get the result immediately. □

The aforementioned result indicates that we can find a solution of $\mathrm{SILCP}(M(w), \Omega)$ by checking the left set in (2.6).

3. **Equivalent reformulation.** In this section, we show that the SICP can be equivalently reformulated as typical semi-infinite min-max programming problems. To begin, we recall that a function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is an NCP function, if it has the property

$$\phi(a, b) = 0 \quad \Longleftrightarrow \quad a \ge 0,\ b \ge 0,\ ab = 0.$$

In the last decade, NCP functions have been used as a powerful tool for dealing with the classical complementarity problem, because it allows us to reformulate the complementarity problems as equations or minimization problems. Such formulations are very beneficial for both analytical and computational purpose. Indeed, powerful theories from classical analysis of systems of equations can be applied to treat the classical complementarity problem for developing the existence of solutions and for analyzing these solutions; efficient algorithms for solving equations and optimization problems can be employed and extended to solve the classical complementarity problem. For an excellent study of this topic, please refer to [12].

Analogous to the classical complementarity problem, we obtain the equivalent formulation of the SICP as a system of equations:

$$x \in S^* \quad \Longleftrightarrow \quad \Phi(x, w) = 0 \ \forall w \in \Omega,$$

where $\Phi : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ is defined by

$$\Phi(x, w) = \begin{pmatrix} \phi(F_1(x, w), x_1) \\ \vdots \\ \phi(F_n(x, w), x_n) \end{pmatrix}.$$

A straightforward choice of a residual function is

$$r(x) = \max_{w \in \Omega} \|\Phi(x, w)\|^2.$$

Clearly, to solve the semi-infinite complementarity problem is the same as to find a root of $r(x) = 0$, or equivalently, to find an optimal solution of the following minimization problem with zero objective value:

$$\min_{x \in \mathbb{R}^n} \max_{w \in \Omega} \|\Phi(x, w)\|^2.$$

Noting that this minimization problem is a typical semi-infinite min-max programming problem [24] (also called min-max programming in some literature), it offers another explanation of why we call the problem (1.1) as semi-infinite complementarity problem. Note also that the residual functions involved for semi-infinite complementarity problem are expressed by pointwise supremum of a family of functions. Although such functions fail to preserve smoothness, they enjoy some other nice properties, such as semidifferentiable and semismoothness. Toward this end, let us introduce the following concept.

Definition 3.1. We say that the semi-infinite strict complementarity condition holds at x if $\min\{F_i(x, w), x_i\} = 0$ and $\max\{F_i(x, w), x_i\} > 0$ for all $w \in \Omega$ and $i = 1, 2, \cdots, n$.

In the case of Ω consisting of a single element, the definition reduces to the strict complementarity condition for the classical complementarity problem. Here we list several NCP-functions which we will focus on:

$$\begin{aligned}
\phi_1(a, b) &= \min(a, b), \\
\phi_2(a, b) &= \sqrt{a^2 + b^2} - (a + b), \\
\phi_3(a, b) &= \sqrt{\{[\phi_2(a, b)]_+\}^2 + \alpha[(ab)_+]^2}, \quad \alpha > 0, \\
\phi_4(a, b) &= \phi_2(a, b) - \alpha a_+ b_+, \quad \alpha > 0, \\
\phi_5(a, b) &= \sqrt{[\phi_2(a, b)]^2 + \alpha(a_+ b_+)^2}, \quad \alpha > 0, \\
\phi_6(a, b) &= \sqrt{[\phi_2(a, b)]^2 + \alpha[(ab_+)]^4}, \quad \alpha > 0 \\
\phi_7(a, b) &= \sqrt{[\phi_2(a, b)]^2 + \alpha[(ab)_+]^2}, \quad \alpha > 0,
\end{aligned}$$

and the corresponding residual functions constructed via ϕ_i is denoted by r_i for $i = 1, 2, \cdots, 7$. The semismoothness of the residual functions r_i for $i = 1, 2, \cdots, 7$ are given in Theorem 3.3 for which the following lemma is needed.

Lemma 3.2. [26, Theorem 10.31] and [21, Theorem 3.2] *Let Y be a compact subset in $\mathbb{R}^m$. Consider the max-function $\theta(x) = \max_{y \in Y} g(x, y)$. If the gradient $\nabla_x g(\cdot, \cdot)$ is continuous on $\mathbb{R}^n \times Y$, then θ is semidifferentiable and semismooth.*

Theorem 3.3. *Consider the* SINCP$(F(\cdot, w), \Omega)$*. Suppose Ω is compact and F is continuously differentiable on $\mathbb{R}^n \times \Omega$. Then, the following conclusions hold.*

(a): *If the strictly semi-infinite complementarity condition holds at every point in a certain neighborhood of* x*, then* r_1 *is semidifferentiable and semismooth at* x*.*

(b): *The function* r_i *for* $i = 2, 3, \cdots, 7$ *is semidifferentiable and semismooth.*

Proof. The desired results follow from Lemma 3.2 and the facts of $(\phi_i)^2$ for $i = 2, 3, \cdots, 7$ being continuously differentiable [29] as well as $(\phi_1)^2$ being continuously differentiable in the presence of the strict complementarity condition [12]. □

4. **Error bounds.** We say that a residual function $r(x)$ is a global (local) error bound for SICP if there exists some constant $c > 0$ (and $\varepsilon > 0$) such that for each $x \in \mathbb{R}^n$ (when $r(x) \le \varepsilon$)

$$\operatorname{dist}(x, S^*) \le cr(x), \tag{4.1}$$

where $\operatorname{dist}(x, S^*) = \inf\{\|x - x^*\| \mid x^* \in S^*\}$. The theory of error bounds has a wide range of applications in different areas, for example, sensitivity analysis and the convergence analysis of the numerical methods; see [23], which presents an excellent survey of the theory of error bound and its relations to other issues. In particular, the question about the error bounds for the classical complementarity problem has been answered elegantly; see [12, Chapter 6]. Taking this fact into account, it is not difficult to treat the case where S^* happens to be $\operatorname{SOL}(F(\cdot, w))$ for some $w \in \Omega$. For any fixed $w \in \Omega$, denote by $r(x, w)$ the residual function of $\operatorname{NCP}(F(\cdot, w))$.

Theorem 4.1. *Consider the* $\operatorname{SINCP}(F(\cdot, w), \Omega)$*. Suppose the solution set* S^* *is nonempty and* $S^* = \operatorname{SOL}(F(\cdot, w_0))$ *for some* $w_0 \in \Omega$*. If* $r(x, w_0)$ *is a global (or local) error bound for the* $\operatorname{NCP}(F(\cdot, w_0))$*, then* $r(x) = \max_{w \in \Omega} r(x, w)$ *is a global (or local) error bound for the* $\operatorname{SINCP}(F(\cdot, w), \Omega)$*.*

Proof. When $r(x, w_0)$ is a global error bound, there exists $c > 0$ such that

$$\operatorname{dist}(x, \operatorname{SOL}(F(\cdot, w_0))) \le cr(x, w_0) \quad \forall x \in \mathbb{R}^n,$$

which, together with the identity $S^* = \operatorname{SOL}(F(\cdot, w_0))$ by hypothesis, implies that

$$\operatorname{dist}(x, S^*) = \operatorname{dist}(x, \operatorname{SOL}(F(\cdot, w_0))) \le cr(x, w_0) \le cr(x), \quad \forall x \in \mathbb{R}^n.$$

This completes the proof. □

To deal with the general case, we introduce the concept of *ε-error bounds*: Given $\varepsilon \ge 0$, we say that a residual function $r(x)$ is an *ε-error bound* for SICP if there exists $c > 0$ such that

$$\operatorname{dist}(x, S^*) \le cr(x) + \varepsilon \quad \forall x \in \mathbb{R}^n.$$

Obviously, if $\varepsilon = 0$, this definition reduces to the error bound defined by (4.1).

Theorem 4.2. *Consider the* $\operatorname{SILCP}(q(w), M(w), \Omega)$*. Suppose the solution set* S^* *is nonempty. If* $M(w_0)$ *is an* R_0*-matrix for some* $w_0 \in \Omega$*, then there exist* $c > 0$ *and* $\varepsilon > 0$ *with* $\varepsilon \le \operatorname{diam}\big(\operatorname{SOL}(q(w_0), M(w_0))\big)$ *such that*

$$\operatorname{dist}(x, S^*) \le cr(x) + \varepsilon$$

where $r(x) = \max_{w \in \Omega} \|\min\big(x, M(w)x + q(w)\big)\|$.

Proof. Since $M(w_0)$ is an R_0-matrix, $\mathrm{SOL}(q(w_0), M(w_0))$ is bounded. This means the existence of $\varepsilon > 0$ such that $\mathrm{SOL}(q(w_0), M(w_0)) \subseteq S^* + \varepsilon\mathbb{B}$. Consequently,

$$\mathrm{dist}(x, S^*) \leq \mathrm{dist}(x, \mathrm{SOL}(q(w_0), M(w_0))) + \varepsilon, \quad \forall x \in \mathbb{R}^n. \tag{4.2}$$

Noting that $S^* \subseteq \mathrm{SOL}(q(w_0), M(w_0))$, a simple upper bound of ε is the diameter of the set $\mathrm{SOL}(q(w_0), M(w_0))$. For the classical linear complementarity problem $\mathrm{LCP}(q(w_0), M(w_0))$, by [20, Theorem 2.1], there exists $c > 0$ such that

$$\mathrm{dist}(x, \mathrm{SOL}(q(w_0), M(w_0))) \leq c \,\|\min\big(x, M(w_0)x + q(w_0)\big)\| \quad \forall x \in \mathbb{R}^n,$$

from which and (4.2) the desired result follows. □

As evident from the above proof, the parameter ε represents the degree of approximation of the sets S^* and $\mathrm{SOL}(q(w_0), M(w_0))$. This is illustrated by Example 4.3 below.

Example 4.3. Consider the $\mathrm{SILCP}(q(w), M(w), \Omega)$ with

$$M(w) = \begin{pmatrix} 1 & w-1 \\ w-1 & w \end{pmatrix}, \; q(w) = \begin{pmatrix} 1-w \\ 0 \end{pmatrix}, \text{ and } \Omega = [0, 1].$$

It is easy to see that $\mathrm{SOL}(q_{\max}, M_{\max}) = \mathrm{SOL}(q_{\min}, M_{\min}) = \mathrm{SOL}(q(1), M(1)) = \{(0,0)\}$ and $\mathrm{SOL}(q(0), M(0)) = \{(x_1, x_2) | x_1 = 0, \; 0 \leq x_2 \leq 1\}$. From Theorem 2.3, we have $\{(0,0)\} = \mathrm{SOL}(q_{\max}, M_{\max}) \cap \mathrm{SOL}(q_{\min}, M_{\min}) \subseteq S^* \subseteq \mathrm{SOL}(q(1), M(1)) = \{(0,0)\}$, which in turn means that $S^* = \{(0,0)\}$. Clearly, we have $\mathrm{diam}(\mathrm{SOL}(q(0), M(0))) = 1$ and hence $\mathrm{SOL}(q(0), M(0)) \subseteq S^* + \mathbb{B}$. On the other hand, it can be easily verified that $M(0)$ is an R_0-matrix, and hence $\mathrm{LCP}(q(0), M(0))$ has a global error bound. By simple calculation, we get $\mathrm{dist}(x, \mathrm{SOL}(q(0), M(0))) \leq \sqrt{3}\|\min\big(x, M(0)x + q(0)\big)\|$ for all $x \in \mathbb{R}^2$. In summary, the inequality in Theorem 4.2 holds true by taking $c = \sqrt{3}$ and $\varepsilon = 1$.

Theorem 4.4. *Consider the* $\mathrm{SILCP}(q(w), M(w), \Omega)$. *Suppose* Ω *is compact and* $M(w)$ *is continuous on* Ω. *If the solution set* S^* *is nonempty, then*

$$r(x) \leq c\,\mathrm{dist}(x, S^*) \quad \forall x \in \mathbb{R}^n,$$

where $r(x) = \max_{w \in \Omega} \|\min\big(x, M(w)x + q(w)\big)\|$ *and* $c = 2 + \max_{w \in \Omega} \|M(w)\|$.

Proof. Letting $x \in \mathbb{R}^n$ be arbitrary and $\bar{x}$ be a projection of x onto S^*, we get

$$\begin{aligned}
& r(x) \\
= \;& \max_{w \in \Omega} \|\min\big(x, M(w)x + q(w)\big)\| \\
= \;& \max_{w \in \Omega} \|\min\big(x, M(w)x + q(w)\big) - \min\big(\bar{x}, M(w)\bar{x} + q(w)\big)\| \\
= \;& \max_{w \in \Omega} \|x - \big(x - M(w)x - q(w)\big)_+ - \bar{x} + \big(\bar{x} - M(w)\bar{x} - q(w)\big)_+\| \\
\leq \;& \|x - \bar{x}\| + \max_{w \in \Omega} \|\big(x - M(w)x - q(w)\big)_+ - \big(\bar{x} - M(w)\bar{x} - q(w)\big)_+\| \\
\leq \;& 2\|x - \bar{x}\| + \max_{w \in \Omega} \|M(w)(x - \bar{x})\| \\
\leq \;& (2 + \max_{w \in \Omega} \|M(w)\|)\|x - \bar{x}\|,
\end{aligned}$$

where the second inequality follows from the nonexpansivity of the projection mapping [26, Corollary 12.20]. □

The foregoing theorem shows that the order of the distance from any point x to the solution set S^* is at least as big as $r(x)$. Therefore, in order to be an error bound, a residual function must bound $r(x)$. More precisely, we know from Theorem 4.4 that

$$r(x) \leq c_1 \text{dist}(x, S^*), \text{ for some } c_1 > 0. \tag{4.3}$$

Hence if other residual function, say $\eta(x)$, is an error bound (i.e., $\text{dist}(x, S^*) \leq c_2\eta(x)$), we must have

$$r(x) \leq c_1 \text{dist}(x, S^*) \leq c_1 c_2 \eta(x).$$

This means that in order to be an error bound, other residual function must bound $r(x)$. In addition, if $r(x)$ is an error bound (i.e., $\text{dist}(x, S^*) \leq c_3 r(x)$), then it follows from (4.3) that

$$\frac{1}{c_1} r(x) \leq \text{dist}(x, S^*) \leq c_3 r(x).$$

Hence $r(x)$ can be used as an estimate to the distance $\text{dist}(x, S^*)$ since the latter is non-computable (or difficult) in some cases.

Theorem 4.5. *Consider the* $\text{SILCP}(q(w), M(w), \Omega)$. *Suppose the matrices* $M_{\max}$ *and* $M_{\min}$ *are both positive semidefinite and one of them is an* R_0*-matrix. If* $\text{SOL}(q_{\min}, M_{\min}) \cap \text{SOL}(q_{\max}, M_{\max}) \neq \emptyset$*, then there exist* $\varepsilon > 0$ *and* $c > 0$ *such that*

$$\text{dist}(x, S^*) \leq c(r_{\min}(x) + r_{\max}(x)) \quad \forall x \text{ satisfying } r_{\min}(x) + r_{\max}(x) \leq \varepsilon,$$

where $r_{\min}(x) = \|\min\left(x, M_{\min}x + q_{\min}\right)\|$ *and* $r_{\max}(x) = \|\min\left(x, M_{\max}x + q_{\max}\right)\|$.

Proof. The positive semidefiniteness of the matrices $M_{\max}$ and $M_{\min}$ implies the polyhedron of the solution sets $\text{SOL}(q_{\min}, M_{\min})$ and $\text{SOL}(q_{\max}, M_{\max})$, see [10, Theorem 3.1.7]. Since the intersection of these two sets is nonempty, it follows from [3, Corollary 3, pp.147] that the collection set $\{\text{SOL}(q_{\min}, M_{\min}), \text{SOL}(q_{\max}, M_{\max})\}$ is bounded linear regularity, that is, for every bounded subset D, there exists $k > 0$ such that, for any $x \in D$,

$$\begin{aligned} & \text{dist}(x, \text{SOL}(q_{\min}, M_{\min}) \cap \text{SOL}(q_{\max}, M_{\max})) \\ \leq \; & k \max\{\text{dist}(x, \text{SOL}(q_{\min}, M_{\min})), \text{dist}(x, \text{SOL}(q_{\max}, M_{\max}))\}. \end{aligned} \tag{4.4}$$

We know from [20] that the positive semidefiniteness of $M_{\min}$ implies the existence of $\varepsilon_1 > 0$ and $c_1 > 0$ such that

$$\text{dist}(x, \text{SOL}(q_{\min}, M_{\min})) \leq c_1 r_{\min}(x) \quad \forall x \text{ satisfying } r_{\min}(x) \leq \varepsilon_1. \tag{4.5}$$

Similarly, for the matrix $M_{\max}$, there exist $\varepsilon_2 > 0$ and $c_2 > 0$ such that

$$\text{dist}(x, \text{SOL}(q_{\max}, M_{\max})) \leq c_2 r_{\max}(x) \quad \forall x \text{ satisfying } r_{\max}(x) \leq \varepsilon_2. \tag{4.6}$$

Suppose, without loss of generality, that the matrix $M_{\min}$ is an R_0-matrix. Thus, the level set $\{x | r_{\min}(x) \leq \varepsilon\}$ is bounded (see e.g., [12, Proposition 9.1.26]) which further implies the boundedness of the set $\{x | r_{\min}(x) + r_{\max}(x) \leq \varepsilon\}$. Letting $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, $c = k\max\{c_1, c_2\}$ and specializing D as $\{x | r_{\min}(x) + r_{\max}(x) \leq \varepsilon\}$, we obtain from (4.4)-(4.6) that

$$\text{dist}(x, \text{SOL}(q_{\min}, M_{\min}) \cap \text{SOL}(q_{\max}, M_{\max})) \leq c(r_{\min}(x) + r_{\max}(x))$$

for all x satisfying $r_{\min}(x)+r_{\max}(x) \le \varepsilon$. Since $\mathrm{SOL}(q_{\min}, M_{\min}) \cap \mathrm{SOL}(q_{\max}, M_{\max}) \subseteq S^*$ by Theorem 2.3, it follows that

$$\mathrm{dist}(x, S^*) \le \mathrm{dist}(x, \mathrm{SOL}(q_{\min}, M_{\min}) \cap \mathrm{SOL}(q_{\max}, M_{\max})) \quad \forall x \in \mathbb{R}^n.$$

Combing the last two inequalities yields the desired result. □

5. **Weak error bounds.**

So far, we have studied several fundamental issues in the study of SICP. Needless to say, most of the results involve the assumption that the solution set is nonempty. In this section, we will introduce the concept of weak error bounds, which makes sense even if the solution set is empty. First, however, we consider the level-boundedness of the residual function $r(x) = \max\limits_{w\in\Omega} \|\min\big(x, M(w)x + q(w)\big)\|^2$; that is, the level set $\{x | r(x) \le \varepsilon\}$ is bounded for every $\varepsilon \ge 0$. This property is very flexible in providing a criterion for the existence of the solutions, and is crucial to the applications of many algorithms because the convergence of the iterative algorithms usually occurs in a limiting sense.

Definition 5.1. The matrix $M(w)$ is said to be a semi-infinite R_0-matrix relative to a set Ω if the $\mathrm{SILCP}(M(w), \Omega)$ has zero as its unique solution, that is,

$$x \ge 0,\ M(w)x \ge 0,\ x^T M(w)x = 0,\ \forall w \in \Omega \quad \Longrightarrow \quad x = 0.$$

In particular, if Ω is a singleton, the definition reduces to the standard definition of R_0-matrix for the classical complementarity problem. At first glance, the above definition may seem a bit artificial and restrictive. However, the following result shows the failure of this recognition.

Theorem 5.2. *Consider the* $\mathrm{SILCP}(q(w), M(w), \Omega)$*. If $M(w_0)$ is an R_0-matrix for some $w_0 \in \Omega$, then $M(w)$ is a semi-infinite R_0-matrix relative to Ω.*

Proof. Since $M(w_0)$ is an R_0-matrix, we have $\mathrm{SOL}(M(w_0)) = \{0\}$, from which and the fact $S^* \subseteq \bigcap\limits_{w\in\Omega} \mathrm{SOL}(M(w), \Omega) \subseteq \mathrm{SOL}(M(w_0))$ the desired result follows. □

Theorem 5.3 below asserts that the matrix being a semi-infinite R_0-matrix is a necessary and sufficient condition for the residual function $r(x)$ to be level-bounded.

Theorem 5.3. *Consider the* $\mathrm{SILCP}(q(w), M(w), \Omega)$*. Suppose Ω is compact and $M(w)$ and $q(w)$ are continuous. Then, $r(x) = \max\limits_{w\in\Omega} \|\min\big(x, M(w)x + q(w)\big)\|^2$ is level-bounded if and only if the matrix $M(w)$ is a semi-infinite R_0-matrix relative to Ω.*

Proof. We first prove the sufficiency. Suppose on the contrary that there exists a sequence $\|x_n\| \to \infty$ as $n \to \infty$, but $\{r(x_n)\}$ is bounded. We can assume, by passing to a subsequence if necessary, that $\frac{x_n}{\|x_n\|}$ converge to the limit x_0 with $\|x_0\| = 1$. Taking into account the continuity of $q(\cdot)$ and $M(\cdot)$ and the compactness of Ω, we see that $r(x)$ is continuous (see e.g., [24, Corollay 5.4.2]) and $q(w)$ is bounded on Ω. Hence, $\lim\limits_{n\to\infty} \frac{r(x_n)}{\|x_n\|} = 0$ and $\lim\limits_{n\to\infty} \frac{q(w)}{\|x_n\|} = 0$ for all $w \in \Omega$. Since

$$\frac{r(x_n)}{\|x_n\|^2} = \max_{w\in\Omega} \|\min\big(\frac{x_n}{\|x_n\|}, \frac{M(w)x_n + q(w)}{\|x_n\|}\big)\|^2,$$

taking the limit gives

$$\max_{w\in\Omega} \| \min\big(x_0, M(w)x_0\big)\|^2 = 0.$$

This means that the SILCP$(M(w),\Omega)$ has x_0, a nonzero vector, as a solution, contracting the definition of the semi-infinite R_0-matrix.

Now let us show the necessity. Suppose on the contrary that the SILCP$(M(w),\Omega)$ has a nonzero vector x as a solution. Let $I(x) = \{i|x_i = 0\}$ and $J(x) = \{i|x_i > 0\}$. The compactness of Ω and the continuity of q ensure that $q(w)$ is bounded on Ω. Thus there exists a scalar $K > 0$ such that, for any $k \geq K$,

$$kx_i \geq q_i(w) \text{ for all } w \in \Omega \text{ and } i \in J(x). \tag{5.1}$$

Given any $k \geq K$, we have

$$\begin{aligned} r(kx) &= \max_{w\in\Omega} \| \min\big(kx, kM(w)x + q(w)\big)\|^2 \\ &\leq \sum_{i=1}^{n} \max_{w\in\Omega} \big[\min\big(kx_i, (kM(w)x)_i + q_i(w)\big)\big]^2. \end{aligned} \tag{5.2}$$

We now consider the following two cases.

Case 1. If $i \in J(x)$, then $(M(w)x)_i = 0$. It follows from (5.1) that

$$\max_{w\in\Omega} \big[\min\big(kx_i, k(M(w)x)_i + q_i(w)\big)\big]^2 = \max_{w\in\Omega} q_i(w)^2. \tag{5.3}$$

Case 2. If $i \in I(x)$, then, by a simple calculation, we have

$$\big[\min\big(kx_i, k(M(w)x)_i + q_i(w)\big)\big]^2 = 0 \quad \text{if } k(M(w)x)_i + q_i(w) \geq 0,$$

and

$$\big[\min\big(kx_i, k(M(w)x)_i + q_i(w)\big)\big]^2 \leq q_i(w)^2 \quad \text{if } k(M(w)x)_i + q_i(w) < 0,$$

where the inequality in the latter case comes from the fact that

$$q_i(w) \leq k(M(w)x)_i + q_i(w) < 0.$$

Thus,

$$\max_{w\in\Omega} \big[\min\big(kx_i, k(M(w)x)_i + q_i(w)\big)\big]^2 \leq \max_{w\in\Omega} q_i(w)^2. \tag{5.4}$$

Putting the facts (5.2),(5.3), and (5.4) together, it follows that

$$r(kx) \leq \sum_{i=1}^{n} \max_{w\in\Omega} q_i(w)^2 < \infty$$

for all $k \geq K$. This contradicts the level-boundedness of $r(x)$. □

As shown in our previous discussion, solving the semi-infinite complementarity problem is equivalent to finding a vector x such that $x \in \text{SOL}(F(\cdot,w))$ for all $w \in \Omega$. However, in many situations, it is possible to find a vector x such that $x \in \text{SOL}(F(\cdot,w))$ for some w but not for others. In this case, it is necessary and interesting to give a quantitative measure of the closeness of each $x \in \mathbb{R}^n$ to each individual set $\text{SOL}(F(\cdot,w))$ in terms of residual functions. In other words, we wish to find $c > 0$ such that

$$\text{dist}(x, \text{SOL}(F(\cdot,w))) \leq cr(x) \ \forall w \in \Omega, \ \forall x \in \mathbb{R}^n,$$

or equivalently,

$$\max_{w\in\Omega} \operatorname{dist}(x, \mathrm{SOL}(F(\cdot, w))) \le cr(x) \quad \forall x \in \mathbb{R}^n, \tag{5.5}$$

which is referred as weak error bounds. The importance of introducing this concept is twofold. First, in (5.5), the solution set S^* is not assumed to be nonempty, as required in (4.1). Second, the weak error bound can be easily derived from the error bound because we always have

$$\max_{w\in\Omega} \operatorname{dist}(x, \mathrm{SOL}(F(\cdot, w))) \le \operatorname{dist}(x, S^*) \quad \forall x \in \mathbb{R}^n, \tag{5.6}$$

due to $S^* \subseteq \mathrm{SOL}(F(\cdot, w))$ for all $w \in \Omega$. However, the converse is not necessarily true unless some more restrictive conditions are imposed, for example, the linear regularity of the collection $\{\mathrm{SOL}(F(\cdot, w))|w \in \Omega\}$; for further details on this subject, see [2, 3, 4, 22, 30]. In addition, the inequality in (5.6) can also be strict. The following example illustrates this point.

Example 5.4. Consider the $\mathrm{SILCP}(q(w), M(w), \Omega)$ with

$$M(w) = \begin{pmatrix} w \ 1-2w \\ -w \ 1-w \end{pmatrix}, \ q(w) = \begin{pmatrix} w \\ 0 \end{pmatrix}, \text{ and } \Omega = \{0, 1\}.$$

Clearly, we have $\mathrm{SOL}(q(0), M(0)) = \{(x_1, x_2)|x_1 \ge 0, x_2 = 0\}$ and $\mathrm{SOL}(q(1), M(1)) = \{(x_1, x_2)|x_1 = 0, 0 \le x_2 \le 1\}$, and hence $S^* = \{(0, 0)\}$, according to the identity that $S^* = \mathrm{SOL}(q(0), M(0)) \cap \mathrm{SOL}(q(1), M(1))$. Letting $x = (1, 1)$ yields

$$\begin{aligned} &\max\{\operatorname{dist}(x, \mathrm{SOL}(q(0), M(0))), \operatorname{dist}(x, \mathrm{SOL}(q(1), M(1)))\} \\ &= 1 < \sqrt{2} = \operatorname{dist}(x, S^*). \end{aligned}$$

Given $d = (d_1, \cdots, d_n) \in \mathbb{R}^n$, we write $d \in [0, 1]^n$ to means $d_i \in [0, 1]$ for all $i = 1, 2, \cdots, n$. It is known that a matrix A is an P-Matrix if and only if $I - D + DA$ is nonsingular for any diagonal matrix $D = diag(d)$ with $0 \le d_i \le 1$, see [14]. This fact will be used in the proof for the following theorem which gives a significant refinement of [8, Theorem 3.2], because not only the finiteness of the index set Ω is dropped but also the error bounds constant is computable.

Theorem 5.5. *Consider the* $\mathrm{SILCP}(q(w), M(w), \Omega)$. *Suppose* $M(w)$ *is continuous and* Ω *is compact. For each* $w \in \Omega$, *let* $M(w)$ *be an* P*-matrix and denote by* $x^*(w)$ *the unique solution to* $\mathrm{LCP}(q(w), M(w))$. *Then,*

$$\max_{w\in\Omega} \|x - x^*(w)\| \le c \max_{w\in\Omega} \| \min\big(x, M(w)x + q(w)\big)\| \quad \forall x \in \mathbb{R}^n,$$

where $c = \max\limits_{\substack{d\in[0,1]^n \\ w\in\Omega}} \|(I - D + DM(w))^{-1}\|$ *and* $D = \operatorname{diag}(d_1, d_2, \cdots, d_n)$.

Proof. Given any $w \in \Omega$, it follows from [7] that

$$\begin{aligned} & \|x - x^*(w)\| \\ \le \quad & \max_{d\in[0,1]^n} \|(I - D + DM(w))^{-1}\| \| \min\big(x, M(w)x + q(w)\big)\|, \quad \forall x \in \mathbb{R}^n. \end{aligned}$$

Since $M(w)$ is continuous over the compact set Ω, and $M(w)$ is an P-Matrix by hypothesis, then $c = \max\limits_{\substack{d\in[0,1]^n \\ w\in\Omega}} \|(I - D + DM(w))^{-1}\|$ is well defined. The desired conclusion follows by taking the pointwise supremum over the index set Ω. $\square$

6. **Conclusions.** Several fundamental issues have been discussed in this paper. The emphasis is on the solvability, feasibility, semismoothness of residual functions, and error bounds. Overall speaking, the present work makes the following contributions. First, we characterize the solution set by investigating its relationship to the solution sets of two classical complementarity problems, rather than resorting to the fact $S^* = \bigcap_{w\in\Omega} \mathrm{SOL}(F(\cdot, w))$ (see Theorems 2.2 and 2.3). Second, we introduce the concept of weak error bounds, which has particularly attractive in the case where the solution set is empty. In addition, some important concepts in the study of classical complementarity problem have been extended to the context of semi-infinite complementarity problem (see Definitions 2.6, 3.1, and 5.1). Several questions merit further investigation: (a) Under which conditions the solution set S^* is nonempty? (b) How can we propose an efficient algorithm for solving SICP by using the differentiability properties of the residual functions? (c) How can we give an upper bound for the parameter ε as tight as possible in the concept of ε-error bounds?

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