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THE SOLVABILITIES OF THREE OPTIMIZATION PROBLEMS ASSOCIATED WITH SECOND-ORDER CONE

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Abstract. In this paper, we study the solvabilities of three optimization problems associated with second-order cone, including the absolute value equations associated with second-order cone (SOCAVE), eigenvalue complementarity problem associated with second-order cone (SOCEiCP), and quadratic eigenvalue complementarity problem associated with second-order cone (SOCQEiCP). More specifically, we characterize under what conditions these optimizations have solution and unique solution, respectively.

1. Introduction

In this paper, we study the solvabilities of three optimization problems associated with second-order cone. The first optimization problem that our target is the so-called absolute value equations associated with second-order cone, abbreviated as SOCAVEs. For SOCAVEs, there have two types of them. The first type is in the form of

$$Ax - |x| = b. \tag{1.1}$$

Another one is a more general SOCAVE, which is in the form of

$$Ax + B|x| = b, \tag{1.2}$$

where $A, B \in \mathbb{R}^{n\times n}$, $B \neq 0$, and $b \in \mathbb{R}^n$. Note that, unlike the standard absolute value equation that is presented below, here $|x|$ means the absolute value of x coming from the square root of the Jordan product "$\circ$", associated with second-order cone (SOC), of x and x, that is, $|x| := (x \circ x)^{1/2}$. The second-order cone in $\mathbb{R}^n$ $(n \geq 1)$, also called the Lorentz cone or ice-cream cone, is defined as

$$\mathcal{K}^n := \left\{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\| \leq x_1\right\},$$

where $\|\cdot\|$ denotes the Euclidean norm. If $n = 1$, then $\mathcal{K}^1$ is the set of nonnegative reals $\mathbb{R}_+$. In general, a general second-order cone $\mathcal{K}$ could be the Cartesian product of SOCs, i.e.,

$$\mathcal{K} := \mathcal{K}^{n_1} \times \cdots \times \mathcal{K}^{n_r}.$$

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For simplicity, we focus on the single second-order cone $\mathcal{K}^n$ because all the analysis can be carried over to the setting of Cartesian product. More details about second-order cone, Jordan product, and $(\cdot)^{1/2}$ will be introduced in Section 2.

Indeed, the SOCAVE (1.1) (respectively, SOCAVE (1.2)) is a natural extension of the standard absolute value equation (AVE for short) as bellow:

$$Ax - |x| = b, \quad (\text{respectively, } Ax + B|x| = b) \tag{1.3}$$

where $|x|$ denotes the componentwise absolute value of vector $x \in \mathbb{R}^n$. It is known that the standard absolute value equation (1.3) was first introduced by Rohn in [44] and recently has been investigated by many researchers. For standard absolute value equation, there are two main research directions. One is on the theoretical side in which the corresponding properties of the solution for the AVE (1.3) are studied, see [21, 25, 28, 29, 32, 35, 42, 44, 52]. The other one focuses on the algorithm for solving the absolute value equation, see [5, 23, 30, 31, 33, 34, 45, 53, 54].

On the theoretical aspect, Mangasarian and Meyer [35] show that the AVE (1.3) is equivalent to the bilinear program, the generalized LCP (linear complementarity problem), and the standard LCP provided 1 is not an eigenvalue of A. Prokopyev [42] further improves the above equivalence which indicates that the AVE (1.3) can be equivalently recast as an LCP without any assumption on A and B, and also provides a relationship with mixed integer programming. In general, if solvable, the AVE (1.3) can have either unique solution or multiple (e.g., exponentially many) solutions. Indeed, various sufficient conditions on solvability and non-solvability of the AVE (1.3) with unique and multiple solutions are discussed in [35,42]. Moreover, Wu and Guo [52] further study the unique solvability of the AVE (1.3), and give some new and useful results for the unique solvability of the AVE (1.3).

Recently, the absolute value equation associated with second-order cone or circular cone are investigated in [22] and [27], respectively. In particular, Hu, Huang and Zhang [22] show that the SOCAVE (1.2) is equivalent to a class of second-order cone linear complementarity problems, and establish a result regarding the unique solvability of the SOCAVE (1.2). Along this direction, we further look into the SOCAVEs (1.1) and (1.2) in this paper, and achieve some new results about the existence of (unique) solution.

The second optimization problem that we focus on is the so-called second-order cone eigenvalue complementarity problem, SOCEiCP for short. More specifically, given two matrices $B, C \in \mathbb{R}^{n\times n}$, the SOCEiCP is to find $(x, y, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ such that

$$\text{SOCEiCP}(B, C): \begin{cases} y = \lambda Bx - Cx, \\ y \succeq_{\mathcal{K}^n} 0, \ x \succeq_{\mathcal{K}^n} 0, \\ x^T y = 0, \\ a^T x = 1, \end{cases} \tag{1.4}$$

where a is an arbitrary fixed point with $a \in \operatorname{int}(\mathcal{K}^n)$, and $x \succeq_{\mathcal{K}^n} 0$ means that $x \in \mathcal{K}^n$, a partial order. The SOCEiCP(B, C) given as in (1.4) comes naturally from the traditional eigenvalue complementarity problem [43, 47], which seeks to find $(x, y, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ such that

$$\mathrm{EiCP}(B, C): \begin{cases} y = \lambda Bx - Cx, \\ y \geq 0, \ x \geq 0, \\ x^T y = 0, \\ e^T x = 1, \end{cases}$$

where $B, C \in \mathbb{R}^{n \times n}$ and $e = (1, 1, \cdots, 1)^T \in \mathbb{R}^n$. Usually, the matrix B is assumed to be positive definite. The scalar λ is called a complementary eigenvalue and x is a complementary eigenvector associated to λ for the pair (B, C). The condition $x^T y = 0$ and the nonnegative requirements on x and y imply that either $x_i = 0$ or $y_i = 0$ for $1 \leq i \leq n$. These two variables are called complementary.

A natural extension of the EiCP goes to the quadratic eigenvalue complementarity problem (QEiCP), whose mathematical format is as below. Given $A, B, C \in \mathbb{R}^{n \times n}$, the QEiCP consists of finding $(x, y, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ such that

$$\mathrm{QEiCP}(A, B, C): \begin{cases} y = \lambda^2 Ax + \lambda Bx + Cx, \\ y \geq 0, \ x \geq 0, \\ x^T y = 0, \\ e^T x = 1, \end{cases}$$

where $e = (1, 1, \cdots, 1)^T \in \mathbb{R}^n$. It is clear that when $A = 0$, the QEiCP(A, B, C) reduces to the EiCP$(B, -C)$. The λ component of a solution to the QEiCP(A, B, C) is called a quadratic complementary eigenvalue for the pair (A, B, C), whereas the x component is called a quadratic complementary eigenvector for the pair (A, B, C).

Following the same idea for creating the SOCEiCP(B, C), the third optimization problem that we study in this paper is the so-called second-order cone quadratic eigenvalue complementarity problem (SOCQEiCP). In other words, given matrices $A, B, C \in \mathbb{R}^{n \times n}$, the SOCQEiCP seeks to find $(x, y, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ such that

$$\mathrm{SOCQEiCP}(A, B, C): \begin{cases} y = \lambda^2 Ax + \lambda Bx + Cx, \\ y \succeq_{\mathcal{K}^n} 0, \ x \succeq_{\mathcal{K}^n} 0, \\ x^T y = 0, \\ a^T x = 1, \end{cases} \tag{1.5}$$

with arbitrary fixed point $a \in \mathrm{int}(\mathcal{K}^n)$. The SOCEiCP (1.4) and the SOCQEiCP (1.5) have been investigated in [2, 3, 19]. The purpose of this paper aims to establish the solvabilities of the SOCEiCP (1.4) and the SOCQEiCP (1.5) by reformulating them as second-order cone complementarity problem (SOCCP) and a nonsmooth system of equations (see more details in Section 5).

We point out that the last normalization constraint appeared in the above EiCP, QEiCP, SOCEiCP, and SOCQEiCP has been introduced in order to prevent the x component of a solution to vanish. In other words, "for an arbitrary fixed point $a \in \mathrm{int}(\mathcal{K}^n)$, $x \in \mathcal{K}^n$ satisfying $a^T x > 0$ is equivalent to $x \neq 0$". To see this, we provide some arguments as below. First, it is trivial that $a^T x > 0$ implies that

$x \neq 0$. Now, suppose that $x = (x_1, x_2) \in \mathcal{K}^n$ which is nonzero. Then, there must have $x_1 > 0$. Using the definition of

$$\mathrm{int}(\mathcal{K}^n) = \left\{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\| < x_1\right\},$$

we have

$$a^T x = a_1 x_1 + \langle a_2, x_2 \rangle > |\langle a_2, x_2 \rangle| + \langle a_2, x_2 \rangle \geq 0.$$

This proves that $a^T x > 0$.

Another thing needs to be pointed out is that the normalization constraint $e^T x = 1$ is good enough for EiCP and QEiCP; moreover, this condition was also used in [2] for SOCEiCP. However, we show that it does not make sense in the settings of SOCEiCP and SOCQEiCP because $e \notin \mathrm{int}(\mathcal{K}^n)$. Indeed, for a counterexample, we consider $\lambda = 1$, $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in \mathcal{K}^2$, two matrices $C = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \in \mathbb{R}^{2\times 2}$ and $B := I \in \mathbb{R}^{2\times 2}$. Then, we have $\lambda Bx - Cx = \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \in \mathcal{K}^2$. Hence, $x^T(\lambda Bx - Cx) = 0$, but $e^T x = 0$. This is why, in this paper, we require a point $a \in \mathrm{int}(\mathcal{K}^n)$ such that $a^T x = 1$ to serve as the normalization constraint in SOCEiCP and SOCQEiCP.

To close this section, we say a few words about notations. As usual, $\mathbb{R}^n$ denotes the space of n-dimensional real column vectors. $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote the nonnegative and positive reals. For any $x, y \in \mathbb{R}^n$, the Euclidean inner product are denoted $\langle x, y \rangle = x^T y$, and the Euclidean norm $\|x\|$ are denoted as $\|x\| = \sqrt{\langle x, x \rangle}$. Given a matrix $A \in \mathbb{R}^{n\times n}$, $\|A\|_a$ denotes the arbitrary matrix norm, for example, $\|A\|_1$, $\|A\|_2$ and $\|A\|_\infty$. In addition, $\rho(A)$ means the spectral radius of A, that is, $\rho(A) := \max\{|\lambda| \mid \lambda \text{ is eigenvalue of } A\}$, and $M(\mathcal{K}^n) \subset \mathcal{K}^n$ denotes that for any $z \in \mathcal{K}^n$, we have $Mz \in \mathcal{K}^n$. For convenience, we say that a pair $(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}$ solves the SOCEiCP(B, C) when the triplet (x, y, λ) with $y = \lambda Bx - Cx$, is a solution to the SOCEiCP(B, C) in the sense defined in (1.4). Similarly, we say that $(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}$ solves the SOCQEiCP(A, B, C) when the same occurs with the triplet (x, y, λ), where $y = \lambda^2 Ax + \lambda Bx + Cx$.

2. Preliminaries

In this section, we recall some basic concepts and background materials regarding second-order cone and the absolute value of $x \in \mathbb{R}^n$, which will be extensively used in the subsequent analysis. More details can be found in [9, 14, 16, 17, 20, 22].

The official definition of second-order cone (SOC) is already defined in Section 1. We begin with introducing the concept of Jordan product. For any two vectors $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, the *Jordan product* of x and y associated with $\mathcal{K}^n$ is given by

$$x \circ y := \begin{bmatrix} x^T y \\ y_1 x_2 + x_1 y_2 \end{bmatrix}.$$

The Jordan product, unlike scalar or matrix multiplication, is not associative, which is a main source of complication in the analysis of optimization problems involved SOC, see [14, 16, 20] and references therein for more details. The identity element under this Jordan product is $e = (1, 0, \cdots, 0)^T \in \mathbb{R}^n$. With these definitions, x^2 means the Jordan product of x with itself, i.e., $x^2 := x \circ x$; while $x^{1/2}$ with $x \in \mathcal{K}^n$ denotes the unique vector in $\mathcal{K}^n$ such that $x^{1/2} \circ x^{1/2} = x$. In light of this, the vector $|x|$ in the SOCAVEs (1.1) and (1.2) is computed by

$$|x| := (x \circ x)^{1/2}.$$

However, by the definition of $|x|$, it is not easy to write out the expression of $|x|$ explicitly. Fortunately, there is another way to reach $|x|$ via spectral decomposition and projection onto second-order cone. We elaborate it as below. For $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, the *spectral decomposition* of x with respect to SOC is given by

$$x = \lambda_1(x) u_x^{(1)} + \lambda_2(x) u_x^{(2)}, \tag{2.1}$$

where $\lambda_i(x) = x_1 + (-1)^i \|x_2\|$ for $i = 1, 2$ and

$$u_x^{(i)} = \begin{cases} \frac{1}{2}\left(1, (-1)^i \frac{x_2^T}{\|x_2\|}\right)^T & \text{if} \quad \|x_2\| \neq 0, \\ \frac{1}{2}\left(1, (-1)^i \omega^T\right)^T & \text{if} \quad \|x_2\| = 0, \end{cases}$$

with $\omega \in \mathbb{R}^{n-1}$ being any vector satisfying $\|\omega\| = 1$. The two scalars $\lambda_1(x)$ and $\lambda_2(x)$ are called spectral values (or eigenvalues) of x; while the two vectors $u_x^{(1)}$ and $u_x^{(2)}$ are called the spectral vectors (or eigenvectors) of x. Moreover, it is obvious that the spectral decomposition of $x \in \mathbb{R}^n$ is unique if $x_2 \neq 0$.

Next, we talk about the projection onto second-order cone. Let x_+ be the projection of x onto $\mathcal{K}^n$, while x_- be the projection of $-x$ onto its dual cone of $\mathcal{K}^n$. Since second-order cone $\mathcal{K}^n$ is self-dual, the dual cone of $\mathcal{K}^n$ is itself, i.e., $(\mathcal{K}^n)^* = \mathcal{K}^n$. In fact, the explicit formula of projection of $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ onto $\mathcal{K}^n$ is characterized in [14, 16–18, 20] as below:

$$x_+ = \begin{cases} x & \text{if } x \in \mathcal{K}^n, \\ 0 & \text{if } x \in -\mathcal{K}^n, \\ u & \text{otherwise,} \end{cases} \qquad \text{where} \quad u = \begin{bmatrix} \frac{x_1 + \|x_2\|}{2} \\ \left(\frac{x_1 + \|x_2\|}{2}\right) \frac{x_2}{\|x_2\|} \end{bmatrix}.$$

Similarly, the expression of x_- is in the form of

$$x_- = \begin{cases} 0 & \text{if } x \in \mathcal{K}^n, \\ -x & \text{if } x \in -\mathcal{K}^n, \\ w & \text{otherwise,} \end{cases} \qquad \text{where} \quad w = \begin{bmatrix} -\frac{x_1 - \|x_2\|}{2} \\ \left(\frac{x_1 - \|x_2\|}{2}\right) \frac{x_2}{\|x_2\|} \end{bmatrix}.$$

Together with the spectral decomposition (2.1) of x, it can be verified that $x = x_+ - x_-$ and the expression of x_+ and x_- have the form:

$$\begin{aligned} x_+ &= (\lambda_1(x))_+ u_x^{(1)} + (\lambda_2(x))_+ u_x^{(2)}, \\ x_- &= (-\lambda_1(x))_+ u_x^{(1)} + (-\lambda_2(x))_+ u_x^{(2)}, \end{aligned}$$

where $(\alpha)_+ = \max\{0, \alpha\}$ for $\alpha \in \mathbb{R}$.

Based on the definitions and expressions of x_+ and x_-, we introduce another expression of $|x|$ associated with SOC. In fact, the alternative expression is obtained by the so-called SOC-function, which can be found in [10]. For any $x \in \mathbb{R}^n$, we define the *absolute value* $|x|$ of x with respect to SOC as $|x| := x_+ + x_-$. In fact, in the setting of SOC, the form $|x| = x_+ + x_-$ is equivalent to the form $|x| = (x \circ x)^{1/2}$. Combining the above expression of x_+ and x_-, it is easy to see that the expression of the absolute value $|x|$ is in the form of

$$\begin{aligned} |x| &= \big[(\lambda_1(x))_+ + (-\lambda_1(x))_+\big] u_x^{(1)} + \big[(\lambda_2(x))_+ + (-\lambda_2(x))_+\big] u_x^{(2)} \\ &= \big|\lambda_1(x)\big| u_x^{(1)} + \big|\lambda_2(x)\big| u_x^{(2)}. \end{aligned}$$

For the absolute value $|x|$ associated with SOC, Hu, Huang and Zhang [22] have obtained some properties as the following lemmas.

Lemma 2.1. *[22, Theorem 2.1] The generalized Jacobian of the absolute value function $|\cdot|$ is given as follows:*

(a) *Suppose that $x_2 = 0$. Then, $\partial|x| = \{tI \mid t \in \overline{\mathrm{sgn}}(x_1)\}$.*

(b) *Suppose that $x_2 \neq 0$.*

(i) *If $x_1 + \|x_2\| < 0$ and $x_1 - \|x_2\| < 0$, then* $\partial|x| = \{\nabla|x|\} = \left\{ \begin{bmatrix} -1 & 0^T \\ 0 & -I \end{bmatrix} \right\}$.

(ii) *If $x_1 + \|x_2\| > 0$ and $x_1 - \|x_2\| > 0$, then* $\partial|x| = \{\nabla|x|\} = \left\{ \begin{bmatrix} 1 & 0^T \\ 0 & I \end{bmatrix} \right\}$.

(iii) *If $x_1 + \|x_2\| > 0$ and $x_1 - \|x_2\| < 0$, then*

$$\partial|x| = \{\nabla|x|\} = \left\{ \begin{bmatrix} 0 & \frac{x_2^T}{\|x_2\|} \\ \frac{x_2}{\|x_2\|} & \frac{x_1}{\|x_2\|}\left(I - \frac{x_2 x_2^T}{\|x_2\|^2}\right) \end{bmatrix} \right\}.$$

(iv) *If $x_1 + \|x_2\| = 0$ and $x_1 - \|x_2\| < 0$, then*

$$\partial|x| = \left\{ \frac{1}{2} \begin{bmatrix} t-1 & (t+1)\frac{x_2^T}{\|x_2\|} \\ (t+1)\frac{x_2}{\|x_2\|} & -2I + (t+1)\frac{x_2 x_2^T}{\|x_2\|^2} \end{bmatrix} \,\Bigg|\, t \in \overline{\mathrm{sgn}}(x_1 + \|x_2\|) \right\}.$$

(v) *If $x_1 + \|x_2\| > 0$ and $x_1 - \|x_2\| = 0$, then*

$$\partial|x| = \left\{ \frac{1}{2} \begin{bmatrix} t+1 & (1-t)\frac{x_2^T}{\|x_2\|} \\ (1-t)\frac{x_2}{\|x_2\|} & 2I - (1-t)\frac{x_2 x_2^T}{\|x_2\|^2} \end{bmatrix} \,\Bigg|\, t \in \overline{\mathrm{sgn}}(x_1 - \|x_2\|) \right\},$$

where the function $\overline{\mathrm{sgn}}(\cdot)$ denotes that $\overline{\mathrm{sgn}}(a) = \begin{cases} \{1\} & \text{if } a > 0, \\ \{t \mid t \in [-1,1]\} & \text{if } a = 0, \\ \{-1\} & \text{if } a < 0. \end{cases}$

Lemma 2.2. *[22, Theorem 2.2] For any $V \in \partial|x|$, the absolute value of every eigenvalue of V is not greater than 1.*

Lemma 2.3. *[22, Theorem 2.3] For any $V \in \partial|x|$, we have $Vx = |x|$.*

3. Existence of solution to the SOCAVEs

This section is devoted to the existence and nonexistence of solution to the SOCAVE (1.1) and SOCAVE (1.2).

Theorem 3.1. *Let $C \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$.*

(a) *If the following system*

$$(C - I)z = b, \;\; z \in \mathcal{K}^n \tag{3.1}$$

has a solution, then for any $A = \pm C$ the SOCAVE (1.1) has a solution.

(b) *If the following system*

$$(C + B)z = b, \;\; z \in \mathcal{K}^n$$

has a solution, then for any $A = \pm C$ the SOCAVE (1.2) has a solution.

Proof. (a) Suppose that $z := (z_1, z_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ is a solution to the system (3.1), i.e.,

$$(C - I)z = b, \;\; z \in \mathcal{K}^n.$$

Since $z \in \mathcal{K}^n$, it follows that $z_1 \geq \|z_2\|$. Taking $x = \pm z$, which means $x = (\pm z_1, \pm z_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Using the definition of $|x|$, we see that

$$\begin{aligned} |x| &= \big|\lambda_1(x)\big| u_x^{(1)} + \big|\lambda_2(x)\big| u_x^{(2)} \\ &= \Big| \pm z_1 - \| \pm z_2\| \Big| \begin{bmatrix} \frac{1}{2} \\ -\frac{\pm z_2}{2\|z_2\|} \end{bmatrix} + \Big| \pm z_1 + \| \pm z_2\| \Big| \begin{bmatrix} \frac{1}{2} \\ \frac{\pm z_2}{2\|z_2\|} \end{bmatrix} \\ &= z. \end{aligned}$$

Plugging in $A = \pm C$ yields that

$$Ax - |x| = \pm Cx - z = (C - I)z = b.$$

This says that x is a solution to the SOCAVE (1.1).

(b) The arguments are similar to part (a). □

Theorem 3.2. *Suppose that $-b \in \mathcal{K}^n$ and $A(\mathcal{K}^n) \subseteq \mathcal{K}^n$ with $\rho(A) < 1$. Then, the SOCAVE (1.1) has a solution $x \in \mathcal{K}^n$.*

Proof. We consider the iterative scheme $x^{k+1} = Ax^k - b$ with $x^0 := -b$. Since $-b \in \mathcal{K}^n$, it follows that $x^k \in \mathcal{K}^n$ for every $k \in \mathbb{N}$. Hence, from the condition $\rho(A) < 1$, we can conclude that the sequence $\{x^k\}$ converges to a point x^* such that $x^* = Ax^* - b$. Combining with the closeness of $\mathcal{K}^n$, this yields $x^* \in \mathcal{K}^n$, which implies

$$Ax^* - |x^*| = Ax^* - x^* = b.$$

Thus, $x^* \in \mathcal{K}^n$ is a solution to the SOCAVE (1.1). □

Remark 3.3. In fact, if the condition $\rho(A) < 1$ in Theorem 3.2 is replaced by $\|A\|_a < 1$, where $\|A\|_a$ denotes an arbitrary matrix norm, then the result of Theorem 3.2 still holds.

Theorem 3.4. *Suppose that $0 \neq b \in \mathcal{K}^n$. Then, the following hold.*

(a) *If the spectral norm $\|A\| < 1$ with $\|A\| := \sqrt{\rho(A^H A)}$, then the SOCAVE (1.1) has no solution.*

(b) *If $\|A\| < 1$, $B(\mathcal{K}^n) \subset -\mathcal{K}^n$ and $\|Bx\| \geq \|x\|$ for any $x \in \mathcal{K}^n$, then the SOCAVE (1.2) has no solution.*

Proof. From $Ax - |x| = b$ and $0 \neq b \in \mathcal{K}^n$, it follows that $Ax - |x| \in \mathcal{K}^n$. This together with the fact $|x| \in \mathcal{K}^n$ implies $Ax + |x| \in \mathcal{K}^n$. Moreover, by the self-duality of $\mathcal{K}^n$, we see that

$$\begin{aligned} \|Ax\|^2 - \|x\|^2 &= \|Ax\|^2 - \||x|\|^2 \\ &= \langle Ax + |x|, Ax - |x| \rangle \\ &\geq 0. \end{aligned}$$

Hence, we have

$$\|x\| \leq \|Ax\| \leq \|A\|\|x\| < \|x\|,$$

where the last inequality is due to $\|A\| < 1$. This is a contradiction. Therefore, the SOCAVE (1.1) has no solution.

(b) The idea for the proof is similar to part (a), we present it for completeness. From $Ax + B|x| = b$ and $0 \neq b \in \mathcal{K}^n$, we know $Ax + B|x| \in \mathcal{K}^n$. Then, it follows from $B(\mathcal{K}^n) \subset -\mathcal{K}^n$ and $b \in \mathcal{K}^n$ that $Ax = b - B|x| \in \mathcal{K}^n$, which says $Ax - B|x| \in \mathcal{K}^n$. Moreover, by the self-duality of $\mathcal{K}^n$, we have

$$\|Ax\|^2 - \|B|x|\|^2 = \langle Ax + B|x|, Ax - B|x| \rangle \geq 0,$$

which implies

$$\|x\| > \|Ax\| \geq \|B|x|\| \geq \||x|\| = \|x\|,$$

where the first inequality is due to $\|A\| < 1$ and the last inequality is due to $\|Bx\| \geq \|x\|$ for any $x \in \mathcal{K}^n$. This is a contradiction. Hence, the SOCAVE (1.2) has no solution. □

4. The unique solvability for the SOCAVEs

In this section, we further investigate the unique solvability of the SOCAVE (1.1) and SOCAVE (1.2).

Theorem 4.1. (a) *If all singular values of A exceed 1, then the SOCAVE (1.1) has a unique solution.*

(b) *If all singular values of $A \in \mathbb{R}^{n\times n}$ exceed the maximal singular value of $B \in \mathbb{R}^{n\times n}$, then the SOCAVE (1.2) has a unique solution.*

Proof. (a) For any $V \in \partial|x|$, by Lemma 2.3, we have $|x| = Vx$, which implies that

$$Ax - |x| = Ax - Vx = (A - V)x,$$

i.e., the SOCAVE (1.1) becomes the equation $(A - V)x = b$. Moreover, by Lemma 2.1, we know that the real matrix V is symmetric. This leads to that the singular values of V are the absolute values of eigenvalue of V. On the other hand, from Lemma 2.2, it follows that all singular values of V are not greater than 1. Combining with the condition that all singular values of A exceed 1, we can assert that the matrix $A-V$ is nonsingular. If not, there exists $0 \neq x \in \mathbb{R}^n$ such that $(A-V)x = 0$, i.e., $Ax = Vx$. Hence, we have

$$\|x\|^2 < \langle Ax, Ax \rangle = \langle Vx, Vx \rangle \leq \|x\|^2,$$

which is a contradiction. Thus, the matrix $A - V$ is nonsingular, which says the equation $(A - V)x = b$ has a unique solution. Then, the proof is complete.

(b) The proof is similar to that for part (a), we present it for completeness. For any $V \in \partial|x|$, by Lemma 2.3 again, we have $|x| = Vx$; and hence

$$Ax + B|x| = (A + BV)x.$$

Moreover, we also know that all singular values of V are not greater than 1 due to Lemma 2.2. Applying the condition that all singular values of A exceed the maximal singular value of $B \in \mathbb{R}^{n\times n}$ and [22, Theorem 3.1], we obtain that the matrix $A + BV$ is nonsingular. Thus, the equation $(A + BV)x = b$ has a unique solution, which says the SOCAVE (1.2) has a unique solution. □

Remark 4.2. We point out that in [22], Hu, Huang and Zhang have shown that if all singular values of $A \in \mathbb{R}^{n\times n}$ exceed the maximal singular value of $B \in \mathbb{R}^{n\times n}$, the SOCAVE (1.2) has at least one solution for any $b \in \mathbb{R}^n$. In Theorem 4.1(b), we study when the SOCAVE (1.2) has a unique solution, which is a stronger result than the aforementioned one in [22], although the same condition is used. In other words, under the condition that all singular values of $A \in \mathbb{R}^{n\times n}$ exceed the maximal singular value of $B \in \mathbb{R}^{n\times n}$, it guarantees that the SOCAVE (1.2) not only has at least one solution, but also has a unique solution.

Corollary 4.3. *If the matrix A is nonsingular and $\|A^{-1}\| < 1$, then the SOCAVE (1.1) has a unique solution.*

Proof. This is an immediate consequence of Theorem 4.1(a), whose proof is similar to that for [35, Proposition 4.1]. Hence, we omit it. □

Theorem 4.4. (a) *If the matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ satisfies*

$$|a_{ii}| > \sqrt{n} + \sum_{j\neq i} |a_{ij}| \quad \forall i \in \mathscr{N} := \{1, 2, \cdots, n\},$$

then for any $b \in \mathbb{R}^n$ the SOCAVE (1.1) has a unique solution.

(b) *If the matrices $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ satisfy*

$$|a_{ii}| > \|B\|_\infty \sqrt{n} + \sum_{j\neq i} |a_{ij}| \quad \forall i \in \mathscr{N} := \{1, 2, \cdots, n\},$$

then for any $b \in \mathbb{R}^n$ the SOCAVE (1.2) has a unique solution.

Proof. (a) Again, for any $V \in \partial|x|$, we know that $|x| = Vx$ and $\|V\| \leq 1$, which implies that the SOCAVE (1.1) is equal to the equation $(A - V)x = b$. Moreover, by the relationship between the spectral norm and the infinite norm, i.e.,

$$\|V\|_\infty \leq \sqrt{n}\|V\|,$$

it follows that $\|V\|_\infty \le \sqrt{n}$. Let $[w_{ij}] = W := A - V = [a_{ij} - v_{ij}]$. Then, we note that for any $i \in \mathscr{N} = \{1, 2, \cdots, n\}$,

$$\begin{aligned} |w_{ii}| &= |a_{ii} - v_{ii}| \ge |a_{ii}| - |v_{ii}| \\ &> \sqrt{n} + \sum_{j \ne i} |a_{ij}| - |v_{ii}| \\ &\ge \sqrt{n} + \sum_{j \ne i} |w_{ij}| - \sum_{j=1}^{n} |v_{ij}| \\ &\ge \sum_{j \ne i} |w_{ij}|, \end{aligned}$$

where the last inequality is due to $\|V\|_\infty \le \sqrt{n}$. This indicates that the matrix $A - V = W$ is a strictly diagonally dominant by row. Hence, the matrix $A - V$ is nonsingular, which leads to that the equation $(A - V)x = b$ has a unique solution. Thus, the SOCAVE (1.1) has a unique solution.

(b) The proof is similar to part (a) and we omit it here. ☐

Theorem 4.5. *If the matrix $A \in \mathbb{R}^{n \times n}$ can be expressed as*

$$A = \alpha I + M, \quad \textit{where} \quad M(\mathcal{K}^n) \subseteq \mathcal{K}^n \quad \textit{and} \quad \alpha - 1 > \rho(M),$$

then for any $b \in \mathbb{R}^n$, the SOCAVE (1.1) has a unique solution.

Proof. For any $x \in \mathcal{K}^n$ and $V \in \partial |x|$, we know that $x = |x| = Vx$ and $\|V\| \le 1$. Note that

$$Ax - |x| = (\alpha I - V)x + Mx = (\alpha - 1)|x| + Mx.$$

This implies that the matrix $\alpha I + M - V$ is a generalized M-matrix with respect to $\mathcal{K}^n$. Hence, we have the matrix $\alpha I + M - V$ is nonsingular. In addition, applying the fact that $Ax - |x| = (\alpha I + M - V)x$, it yields that the SOCAVE (1.1) has a unique solution. ☐

Lemma 4.6. *For any $x, y \in \mathbb{R}^n$, let $|x|$, $|y|$ be the absolute value coming from the square root of x^2 and y^2 under the Jordan product, respectively. Then, we have*

$$\big\| |x| - |y| \big\| \le \|x - y\|.$$

Proof. First, we note that

$$\begin{aligned} \|x - y\|^2 - \big\| |x| - |y| \big\|^2 &= \langle x - y, x - y \rangle - \langle |x| - |y|, |x| - |y| \rangle \\ &= 2\big(\langle |x|, |y| \rangle - \langle x, y \rangle\big) \\ &= 2\big(\langle x_+ + x_-, y_+ + y_- \rangle - \langle x_+ - x_-, y_+ - y_- \rangle\big) \\ &= 4\big(\langle x_+, y_- \rangle + \langle x_-, y_+ \rangle\big) \\ &\ge 0. \end{aligned}$$

With this, it is clear to see that $\big\| |x| - |y| \big\| \le \|x - y\|$. Then, the proof is complete. ☐

Theorem 4.7. *For any $\beta \in \mathbb{R}$, assume that the matrix $\beta I + A$ is nonsingular.*

(a) *If the matrix A satisfies*

$$\left\|(\beta I + A)^{-1}\right\| < \frac{1}{|\beta| + 1},$$

then the SOCAVE (1.1) has a unique solution.

(b) *If the matrices A and B satisfy*

$$\left\|(\beta I + A)^{-1}\right\| < \frac{1}{|\beta| + \|B\|},$$

then the SOCAVE (1.2) has a unique solution.

Proof. (a) For the SOCAVE (1.1), we know that

$$Ax - |x| = b \iff (\beta I + A)x = \beta x + |x| + b.$$

If the matrix $\beta I + A$ is nonsingular, then we further have

$$Ax - |x| = b \iff (\beta I + A)x = \beta x + |x| + b \iff x = (\beta I + A)^{-1}(\beta x + |x| + b).$$

In view of this, we consider the following iterative scheme

$$x^{k+1} = (\beta I + A)^{-1}(\beta x^k + |x^k| + b).$$

With this, it follows that

$$x^{k+1} - x^k = (\beta I + A)^{-1}\left[\beta(x^k - x^{k-1}) + (|x^k| - |x^{k-1}|)\right].$$

Hence, we have

$$\begin{aligned} \left\|x^{k+1} - x^k\right\| &= \left\|(\beta I + A)^{-1}\left[\beta(x^k - x^{k-1}) + (|x^k| - |x^{k-1}|)\right]\right\| \\ &\le \left\|(\beta I + A)^{-1}\right\| \left[|\beta|\|x^k - x^{k-1}\| + \||x^k| - |x^{k-1}|\|\right] \\ &\le \left\|(\beta I + A)^{-1}\right\| (|\beta| + 1)\|x^k - x^{k-1}\|, \end{aligned} \tag{4.1}$$

where the last inequality holds due to Lemma 4.6. This together with the assumption that $\left\|(\beta I + A)^{-1}\right\| < \frac{1}{|\beta|+1}$ yields the sequence $\{x^k\}$ converges to a solution of the SOCAVE (1.1).

Next, we verify the SOCAVE (1.1) has a unique solution. If there exist x^* and $\bar{x}$ that both satisfy the SOCAVE (1.1), then as done in (4.1) we have

$$\|x^* - \bar{x}\| \le \left\|(\beta I + A)^{-1}\right\| (|\beta| + 1)\|\|x^* - \bar{x}\|.$$

Since $\left\|(\beta I + A)^{-1}\right\| < \frac{1}{|\beta|+1}$, we obtain that $x^* = \bar{x}$. This says that the SOCAVE (1.1) has a unique solution. Thus, the proof is complete.

(b) The proof is similar to part (a) and we omit it here. ☐

5. The solvabilities of SOCEiCP and SOCQEiCP

In this section, we focus on the solvabilities of the other two optimization problems, SOCEiCP(B,C) and SOCQEiCP(A,B,C), which are given as in (1.4) and (1.5) respectively. In order to clearly describe our results, we need a few concepts which were introduced in [3, 4].

Definition 5.1. Let $\mathcal{K}^n$ be a single second-order cone.

(a) A matrix $A \in \mathbb{R}^{n\times n}$ is called $\mathcal{K}^n$-regular if $x^TAx \neq 0$ for all nonzero $x \succeq_{\mathcal{K}^n} 0$.

(b) A matrix $A \in \mathbb{R}^{n\times n}$ is called strictly $\mathcal{K}^n$-copositive if $x^TAx > 0$ for all nonzero $x \succeq_{\mathcal{K}^n} 0$.

(c) A triple (A, B, C) with $A, B, C \in \mathbb{R}^{n\times n}$ is called $\mathcal{K}^n$-hyperbolic if
$$(x^TBx)^2 \geq 4(x^TAx)(x^TCx)$$
for all nonzero $x \succeq_{\mathcal{K}^n} 0$.

(d) The class $R_0(\mathcal{K}^n) \subseteq \mathbb{R}^{n\times n}$ consists of those matrices $A \in \mathbb{R}^{n\times n}$ such that there exists no nonzero $x \in \mathcal{K}^n$ satisfying $Ax \in \mathcal{K}^n$ and $x^TAx = 0$.

(e) The class $S_0(\mathcal{K}^n) \subseteq \mathbb{R}^{n\times n}$ consists of those matrices $A \in \mathbb{R}^{n\times n}$ such that $Ax \in \mathcal{K}^n$ for at least a nonzero $x \in \mathcal{K}^n$.

(f) The class $R_0^{'}(\mathcal{K}^n) \subseteq \mathbb{R}^{n\times n}$ consists of those matrices $A \in \mathbb{R}^{n\times n}$ such that $x^TAx = 0$ for at least a nonzero $x \in \mathcal{K}^n$ satisfying $Ax \in \mathcal{K}^n$.

(g) The class $S_0^{'}(\mathcal{K}^n) \subseteq \mathbb{R}^{n\times n}$ consists of those matrices $A \in \mathbb{R}^{n\times n}$ such that there exists no nonzero $x \in \mathcal{K}^n$ satisfying $Ax \in \mathcal{K}^n$.

In fact, there exist some study in [3, 46, 48], which investigated the eigenvalues problems involved with general cones. The solvability results therein automatically include solvabilities of SOCEiCP(B,C) and SOCQEiCP(A,B,C) as special cases. For example, we extract some of them from [3, 46, 48], when the cone reduces to a SOC or is a general cone, and list them as below.

Proposition 5.2. *Let $\mathcal{K}^n$ be a single second-order cone and consider the SOCEiCP(B,C) given as in (1.4) and the SOCQEiCP(A,B,C) given as in (1.5).*

(a) *If $B \in \mathbb{R}^{n\times n}$ is strictly $\mathcal{K}^n$-copositive, then SOCEiCP(B, C) has solutions for any $C \in \mathbb{R}^{n\times n}$.*

(b) *If A is $\mathcal{K}^n$-regular and (A, B, C) is $\mathcal{K}^n$-hyperbolic, then SOCQEiCP(A, B, C) has solutions.*

(c) *The matrix $C \in R_0^{'}(\mathcal{K}^n)$ if and only if 0 is a quadratic complementary eigenvalue for SOCQEiCP(A, B, C).*

(d) *If $C \in S_0^{'}(\mathcal{K}^n)$ and A is strictly $\mathcal{K}^n$-copositive, there exist at least one positive and one negative quadratic complementary eigenvalue for SOCQEiCP(A, B, C).*

(e) *If $A \in S_0^{'}(\mathcal{K}^n)$ and C is strictly $\mathcal{K}^n$-copositive, there exist at least one positive and one negative quadratic complementary eigenvalue for SOCQEiCP(A, B, C).*

In view of the above existing solvability results in the literature, we aim to seek the solvabilities of SOCEiCP(B,C) and SOCQEiCP(A,B,C) via different approach. In this section, we will recast these problems as three reformulations, called **Reformulation I**, **Reformulation II** and **Reformulation III**.

The idea of Reformulation I is to recast these problems as a form of second-order cone complementarity problem (SOCCP), which is a natural extension of nonlinear complementarity problem (NCP). To proceed, we first recall the mathematical format of the SOCCP as follows. More details can be found in [6–9, 12–14, 16, 36–41, 50, 51]. Given a continuously differentiable mapping $F : \mathbb{R}^n \to \mathbb{R}^n$, the SOCCP($F$) is to find $x \in \mathbb{R}^n$ satisfying

$$\text{SOCCP}(F): \begin{cases} x \succeq_{\mathcal{K}^n} 0, \\ F(x) \succeq_{\mathcal{K}^n} 0, \\ x^T F(x) = 0. \end{cases} \tag{5.1}$$

It is well know that the KKT conditions of a second-order cone programming problem can be rewritten as a SOCCP(F). We now elaborate how to recast the SOCEiCP(B, C) as the SOCCP(F). Suppose that we are given the SOCEiCP(B, C) as in (1.4), where $B, C \in \mathbb{R}^{n\times n}$ and the matrix B is assumed to be positive definite. For any $x \in \mathbb{R}^n$ such that $x \neq 0$, plugging $w = \lambda Bx - Cx$ into the complementarity condition $x^T w = 0$ yields $\lambda = \frac{x^T Cx}{x^T Bx}$. Hence, we obtain

$$w = \frac{x^T Cx}{x^T Bx} Bx - Cx.$$

With this, for any $x \in \mathbb{R}^n$ such that $x \neq 0$, we define a mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ which is given by

$$F(x) := \frac{x^T Cx}{x^T Bx} Bx - Cx. \tag{5.2}$$

This mapping F is not good enough to be put into the SOCCP (5.1) because $F(0)$ is not defined yet. To this end, we show the following lemma to make up the value $F(0)$.

Lemma 5.3. *Consider the SOCEiCP(B, C) given as in (1.4) where B is positive definite. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be defined as in (5.2) where $x \neq 0$. Then, $\lim_{x\to 0} F(x) = 0$.*

Proof. Since B is positive definite, from Cholesky factorization, there exists an invertible lower triangle matrix L with positive diagonal entries such that $B = LL^T$. Hence, for $x \neq 0$, we have

$$x^T Bx = x^T LL^T x = \left(L^T x\right)^T \left(L^T x\right)$$

and

$$x^T Cx = x^T LL^{-1} C(L^T)^{-1} L^T x = \left(L^T x\right)^T \left(L^{-1} C(L^{-1})^T\right) \left(L^T x\right).$$

For convenience, we denote $D := L^{-1}C(L^{-1})^T$ and let $M := \|D\|_{sup} = \max_{1\le i,j\le n} |d_{ij}|$ be the supremum norm of D, where d_{ij} means the (i, j)-entry of D. In addition, for $x \neq 0$, we denote $y = (y_1, \cdots, y_n)^T := L^T x$. Then, we obtain

$$\left|\frac{x^T Cx}{x^T Bx}\right| = \left|\frac{y^T Dy}{y^T y}\right| \le \frac{\sum_{i,j=1}^n |d_{ij}||y_i||y_j|}{\sum_{i=1}^n |y_i|^2}.$$

By Cauchy's inequality $|y_i||y_j| \le \frac{y_i^2+y_j^2}{2}$, we see that

$$\frac{\sum_{i,j=1}^n |d_{ij}||y_i||y_j|}{\sum_{i=1}^n |y_i|^2} \le \frac{M \cdot \sum_{i,j=1}^n \frac{y_i^2+y_j^2}{2}}{\sum_{i=1}^n y_i^2} = \frac{M}{2} \cdot \frac{n\sum_{i=1}^n y_i^2 + n\sum_{j=1}^n y_j^2}{\sum_{i=1}^n y_i^2} = nM$$

which says

$$\left| \frac{x^TCx}{x^TBx} \right| \le nM.$$

This further implies that

$$\|F(x)\| \le \left| \frac{x^TCx}{x^TBx} \right| \cdot \|Bx\| + \|Cx\| \le (nM)\|Bx\| + \|Cx\|.$$

Applying the continuity of linear transformation B and C proves $\lim_{x\to 0} F(x) = 0$. $\square$

Very often, the mapping F in the SOCCP(F) is required to be differentiable. Therefore, in view of Lemma 5.3, it is natural to redefine $F(x)$ as

$$F(x) = \begin{cases} \frac{x^TCx}{x^TBx} Bx - Cx & \text{if } x \ne 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{5.3}$$

This enables that the mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuous. Indeed, it is clear that the mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is even smooth except for 0. In other words, F may not be differentiable at 0. To see this, we give an example as below. For $n = 2$, we take $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ which is positive definite with $b_{12} > 0$ and $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ with $c_{22} \ne 0$. Because B is positive definite, the entries b_{11}, b_{22} are positive. If we consider the first term of $F(x)$ as in (5.2), i.e., $\frac{x^TCx}{x^TBx} Bx$, it can be written out as

$$\left(\frac{c_{11}x_1^2 + (c_{12}+c_{21})x_1x_2 + c_{22}x_2^2}{b_{11}x_1^2 + (b_{12}+b_{21})x_1x_2 + b_{22}x_2^2} \right) \cdot \begin{bmatrix} b_{11}x_1 + b_{12}x_2 \\ b_{21}x_1 + b_{22}x_2 \end{bmatrix}.$$

If we denote

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} := \frac{x^TCx}{x^TBx} Bx,$$

using the fact

$$\lim_{x_1 \to 0} \left[\left(\frac{c_{11}x_1^2 + (c_{12}+c_{21})x_1x_2 + c_{22}x_2^2}{b_{11}x_1^2 + (b_{12}+b_{21})x_1x_2 + b_{22}x_2^2} \right) \cdot \left(b_{11} + \frac{b_{12}x_2}{x_1} \right) \right] = \infty,$$

we see that $\frac{\partial f_1}{\partial x_1}(0)$ does not exist. This means f is not differentiable at 0, and hence $F(x) = f(x) - Cx$ is not differentiable at 0.

Next, we provide two technical lemmas in order to express the Jacobian matrix of $F(x)$ for $x \ne 0$.

Lemma 5.4. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$ $(1 \le i \le n)$ are real-valued differentiable functions. Denote $G(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_n(x) \end{bmatrix}$. Then, the scalar product function $f(x)G(x) = \begin{bmatrix} f(x)g_1(x) \\ f(x)g_2(x) \\ \vdots \\ f(x)g_n(x) \end{bmatrix}$ is a differentiable function on $\mathbb{R}^n$ and its Jacobian matrix $\nabla(f(x)G(x))$ is expressed as*

$$\nabla\big(f(x)G(x)\big) \;=\; \nabla f(x)(G(x))^T + f(x)\nabla G(x).$$

Proof. The proof comes from direct computation as below.

$$\begin{aligned}
&\nabla\big(f(x)G(x)\big) \\
= &\begin{pmatrix} (\frac{\partial f}{\partial x_1}\cdot g_1 + f\cdot\frac{\partial g_1}{\partial x_1})(x) & (\frac{\partial f}{\partial x_1}\cdot g_2 + f\cdot\frac{\partial g_2}{\partial x_1})(x) & \cdots & (\frac{\partial f}{\partial x_1}\cdot g_n + f\cdot\frac{\partial g_n}{\partial x_1})(x) \\ (\frac{\partial f}{\partial x_2}\cdot g_1 + f\cdot\frac{\partial g_1}{\partial x_2})(x) & (\frac{\partial f}{\partial x_2}\cdot g_2 + f\cdot\frac{\partial g_2}{\partial x_2})(x) & \cdots & (\frac{\partial f}{\partial x_2}\cdot g_n + f\cdot\frac{\partial g_n}{\partial x_2})(x) \\ \vdots & \vdots & \ddots & \vdots \\ (\frac{\partial f}{\partial x_n}\cdot g_1 + f\cdot\frac{\partial g_1}{\partial x_n})(x) & (\frac{\partial f}{\partial x_n}\cdot g_2 + f\cdot\frac{\partial g_2}{\partial x_n})(x) & \cdots & (\frac{\partial f}{\partial x_n}\cdot g_n + f\cdot\frac{\partial g_n}{\partial x_n})(x) \end{pmatrix} \\
= &\begin{pmatrix} \frac{\partial f}{\partial x_1}(x) \\ \frac{\partial f}{\partial x_2}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{pmatrix} \big[g_1(x)\ g_2(x)\ \cdots\ g_n(x)\big] + f(x) \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(x) & \frac{\partial g_2}{\partial x_1}(x) & \cdots & \frac{\partial g_n}{\partial x_1}(x) \\ \frac{\partial g_1}{\partial x_2}(x) & \frac{\partial g_2}{\partial x_2}(x) & \cdots & \frac{\partial g_n}{\partial x_2}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_1}{\partial x_n}(x) & \frac{\partial g_2}{\partial x_n}(x) & \cdots & \frac{\partial g_n}{\partial x_n}(x) \end{pmatrix} \\
= &\nabla f(x)(G(x))^T + f(x)\nabla G(x).
\end{aligned}$$

□

Lemma 5.5. *Consider the SOCEiCP(B, C) given as in (1.4) where B is positive definite. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be defined as in (5.3). Then, F is smooth except for* 0 *and its Jacobian matrix is expressed as*

$$\nabla F(x) = \big[(C + C^T)xx^TB - (B + B^T)xx^TC\big]\frac{xx^TB^T}{(x^TBx)^2} + \frac{x^TCx}{x^TBx}B^T - C^T.$$

Proof. Denote $f(x) = \frac{x^TCx}{x^TBx}$ and $g(x) = Bx$. Then, $F(x) = f(x)g(x) - Cx$. For $x \neq 0$, we know

$$\begin{aligned}
\nabla f(x) &= \frac{\nabla(x^TCx)\cdot(x^TBx) - (x^TCx)\cdot\nabla(x^TBx)}{(x^TBx)^2} \\
&= \frac{(C + C^T)x\cdot(x^TBx) - (x^TCx)\cdot(B + B^T)x}{(x^TBx)^2} \\
&= \frac{\big[(C + C^T)xx^TB - (B + B^T)xx^TC\big]x}{(x^TBx)^2}.
\end{aligned}$$

Then, this together with Lemma 5.4 lead to the desired result. □

Now, we sum up the relation between SOCEiCP(B,C) and SOCCP(F) in the below theorem and we call it **Reformulation I for SOCEiCP**.

Theorem 5.6 (**Reformulation I for SOCEiCP**). *Consider the SOCEiCP(B,C) given as in (1.4) where B is positive definite. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be defined as in (5.3). Then, the following hold.*

(a) *If (x^*, λ^*) solves the SOCEiCP(B,C), then x^* solves the SOCCP(F).*

(b) *Conversely, if $\bar{x}$ is a nonzero solution of the SOCCP(F), then (x^*, λ^*) solves the SOCEiCP(B,C) with $\lambda^* = \frac{\bar{x}^TC\bar{x}}{\bar{x}^TB\bar{x}}$ and $x^* = \frac{\bar{x}}{a^T\bar{x}}$.*

Proof. Part (a) is trivial and we only need to prove part (b). Suppose that $\bar{x}$ is a nonzero solution to the SOCCP(F) with F given as in (5.3). Then, we have $\frac{\bar{x}^TC\bar{x}}{\bar{x}^TB\bar{x}} \cdot B\bar{x} - C\bar{x} \in \mathcal{K}^n$, $\bar{x} \in \mathcal{K}^n$, and $\bar{x}^T\left(\frac{\bar{x}^TC\bar{x}}{\bar{x}^TB\bar{x}}B\bar{x} - C\bar{x}\right) = 0$. Since $a \in \mathrm{int}(\mathcal{K}^n)$ and $\bar{x} \in \mathcal{K}^n$, it yields $\frac{1}{a^T\bar{x}} > 0$ by the same arguments as on page 4. From all the above, we conclude that

$$
\begin{aligned}
y^* &:= \lambda^*Bx^* - Cx^* = \tfrac{1}{a^T\bar{x}}\left[\left(\tfrac{\bar{x}^TC\bar{x}}{\bar{x}^TB\bar{x}}\right)B\bar{x} - C\bar{x}\right] \in \mathcal{K}^n,\\
x^* &:= \tfrac{1}{a^T\bar{x}}\bar{x} \in \mathcal{K}^n,\\
a^Tx^* &= \tfrac{a^T\bar{x}}{a^T\bar{x}} = 1,\\
(x^*)^Ty^* &= \left(\tfrac{1}{a^T\bar{x}}\right)^2\left[\bar{x}^T\left(\tfrac{\bar{x}^TC\bar{x}}{\bar{x}^TB\bar{x}}B\bar{x} - C\bar{x}\right)\right] = 0.
\end{aligned}
$$

Thus, (x^*, λ^*) solves the SOCEiCP(B,C). □

Next, we consider the SOCQEiCP(A,B,C) given as in (1.5), where $A, B, C \in \mathbb{R}^{n\times n}$ such that the matrix A is positive definite (hence A is $\mathcal{K}^n$-regular) and the triplet (A,B,C) is $\mathcal{K}^n$-hyperbolic. For any $x \in \mathbb{R}^n$ with $x \neq 0$, plugging $w = \lambda^2Ax + \lambda Bx + Cx$ into the complementarity condition $x^Tw = 0$ yields $(x^TAx)\lambda^2 + (x^TBx)\lambda + (x^TCx) = 0$. Thus, λ can be obtained by solving this quadratic equation, i.e.,

$$\lambda_1(x) = \frac{-(x^TBx) + \sqrt{(x^TBx)^2 - 4(x^TAx)(x^TCx)}}{2(x^TAx)}, \tag{5.4}$$

$$\lambda_2(x) = \frac{-(x^TBx) - \sqrt{(x^TBx)^2 - 4(x^TAx)(x^TCx)}}{2(x^TAx)}. \tag{5.5}$$

Then, for $x \neq 0$, we define $F_i : \mathbb{R}^n \to \mathbb{R}^n$ as

$$F_i(x) = \lambda_i^2(x)Ax + \lambda_i(x)Bx + Cx, \tag{5.6}$$

where $i = 1, 2$. In order to guarantee the well-definedness of $F_i(0)$ for $i = 1, 2$, we need to look into $\lim\limits_{x\to 0} F_i(x)$.

Lemma 5.7. *Consider the SOCQEiCP(A,B,C) given as in (1.5) where A is positive definite. Let $F_i : \mathbb{R}^n \to \mathbb{R}^n$ be defined as in (5.6) where $x \neq 0$. Then, we have $\lim\limits_{x\to 0} F_i(x) = 0$ for $i = 1, 2$.*

Proof. Since A is positive definite, by Cholesky factorization, there exists an invertible lower triangle matrix L with positive diagonal entries such that $A = LL^T$. Using the same techniques in the proof of Lemma 5.3, for $x \neq 0$, we obtain

$$\begin{array}{rcl} x^TAx &=& (L^Tx)^T(L^Tx), \\ x^TBx &=& (L^Tx)^T(L^{-1}B(L^{-1})^T)(L^Tx), \\ x^TCx &=& (L^Tx)^T(L^{-1}C(L^{-1})^T)(L^Tx). \end{array}$$

For convenience, we denote $D := L^{-1}B(L^{-1})^T$, $E := L^{-1}C(L^{-1})^T$, $M_1 := \|D\|_{sup} = \max_{1\leq i,j\leq n} |d_{ij}|$ be the supremum norm of D, and $M_2 := \|E\|_{sup} = \max_{1\leq i,j\leq n} |e_{ij}|$ be the supremum norm of E, where d_{ij} is the (i,j)-entry of D and e_{ij} is the (i,j)-entry of E. In addition, we also denote $y = (y_1, \cdots, y_n)^T := L^Tx$. Using the same techniques in the proof of Lemma 5.3, we obtain

$$\begin{array}{rcl} |x^TBx| &=& |y^TDy| \leq nM_1 \displaystyle\sum_{i=1}^n y_i^2, \\ |x^TCx| &=& |y^TEy| \leq nM_2 \displaystyle\sum_{i=1}^n y_i^2. \end{array}$$

Hence, for each i and for $x \neq 0$, we see that

$$\begin{array}{rl} & |\lambda_i(x)| \\ \leq & \dfrac{|x^TBx| + \sqrt{|x^TBx|^2 + 4|x^TBx||x^TCx|}}{2|x^TAx|} \\ \leq & \dfrac{nM_1\sum_{i=1}^n y_i^2 + \sqrt{(nM_1\sum_{i=1}^n y_i^2)^2 + 4(nM_1\sum_{i=1}^n y_i^2)(nM_2\sum_{i=1}^n y_i^2)}}{2\sum_{i=1}^n y_i^2} \\ \leq & M_3 := \left(\dfrac{1+\sqrt{5}}{2}\right) n \max\{M_1, M_2\}. \end{array}$$

This yields

$$\|F_i(x)\| \leq M_3^2\|Ax\| + M_3\|Bx\| + \|Cx\|,$$

for each i and $x \neq 0$. Then, by the continuity of linear transformation A, B, and C, the desired result follows. □

Again, in view of Lemma 5.7, we need to do something to construct a differentiable mapping F_i. In other words, we redefine $F_i(x)$ by

$$F_i(x) = \begin{cases} \lambda_i^2(x)Ax + \lambda_i(x)Bx + Cx & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{5.7}$$

where $\lambda_i(x)$, $i = 1, 2$ are given as in (5.4)-(5.5). From Lemma 5.7, it is clear that the mapping $F_i : \mathbb{R}^n \to \mathbb{R}^n$ is continuous for $i = 1, 2$. In fact, the mapping $F_i : \mathbb{R}^n \to \mathbb{R}^n$ is smooth except for 0. To see this fact, we give an example as follows. For $n = 2$, we take $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ which is positive definite and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ such that $b_{22} \neq 0$. Because A is positive definite, the entries a_{22} are positive. Now for each $i = 1, 2$, we consider the first two terms of $F_i(x)$ described as in (5.6), i.e.,

$$\lambda_i^2(x) \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} + \lambda_i(x) \begin{bmatrix} b_{11}x_1 + b_{12}x_2 \\ b_{21}x_1 + b_{22}x_2 \end{bmatrix} := \begin{bmatrix} g_{i1}(x) \\ g_{i2}(x) \end{bmatrix}.$$

It can be verified that

$$\lim_{x_1 \to 0} \frac{g_{i2}(x)}{x_1} = \infty,$$

which implies that $\frac{\partial g_{i2}}{\partial x_1}(0)$ does not exist. Therefore, $F_i(x)$ is not differentiable at 0.

For $x \neq 0$, the Jacobian matrix of $F_i(x)$ in (5.7) is computed as below.

Lemma 5.8. *Consider the SOCQEiCP(A, B, C) given as in (1.5) where A is positive definite. Let $F_i : \mathbb{R}^n \to \mathbb{R}^n$ be defined as in (5.7) for $i = 1, 2$. Then, F_i is smooth except for 0 and its Jacobian matrix is expressed as*

$$\nabla F_i(x) = \nabla \lambda_i(x) \left(2\lambda_i(x) x^T A^T + x^T B^T\right) + \lambda_i^2(x) A^T + \lambda_i(x) B^T + C^T,$$

where

$$\begin{aligned} \nabla \lambda_1(x) \quad = \quad & \frac{1}{2x^T A x} (Bx + B^T x) \Big((D(x))^{-\frac{1}{2}} (x^T B x) - 1 \Big) \\ & - \frac{1}{x^T A x} \cdot (D(x))^{-\frac{1}{2}} \left[(Ax + A^T x)(x^T C x) + (Cx + C^T x)(x^T A x) \right] \\ & + \frac{1}{2(x^T A x)^2} \left[x^T B x - \sqrt{D(x)} \right] (Ax + A^T x), \end{aligned}$$

$$\begin{aligned} \nabla \lambda_2(x) \quad = \quad & -\frac{1}{2x^T A x} (Bx + B^T x)((D(x))^{\frac{1}{2}} (x^T B x) + 1) \\ & + \frac{1}{x^T A x} (D(x))^{-\frac{1}{2}} \left[(Ax + A^T x)(x^T C x) + (Cx + C^T x)(x^T A x) \right] \\ & + \frac{1}{2(x^T A x)^2} \left[x^T B x + \sqrt{D(x)} \right] (Ax + A^T x), \end{aligned}$$

and $D(x) := (x^T B x)^2 - 4(x^T A x)(x^T C x)$.

Proof. The proof is routine check by applying chain rule. First, we denote $D(x) := (x^T Bx)^2 - 4(x^T Ax)(x^T Cx)$. Then, it can be verified that

$$\begin{aligned}
\nabla\lambda_1(x) &= \frac{1}{2(x^TAx)^2}\left[-\nabla(x^TBx)+\nabla(\sqrt{D(x)})\right](x^TAx) \\
&\quad -\frac{1}{2(x^TAx)^2}\left[-x^TBx+\sqrt{D(x)}\right]\nabla(x^TAx) \\
&= \frac{1}{2x^TAx}(-Bx-B^Tx)+\frac{1}{2x^TAx}(D(x))^{-\frac{1}{2}}(x^TBx)(Bx+B^Tx) \\
&\quad -\frac{1}{x^TAx}(D(x))^{-\frac{1}{2}}\left[(Ax+A^Tx)(x^TCx)+(Cx+C^Tx)(x^TAx)\right] \\
&\quad -\frac{1}{2(x^TAx)^2}\left[-x^TBx+\sqrt{D(x)}\right](Ax+A^Tx) \\
&= \frac{1}{2x^TAx}(Bx+B^Tx)\left[(D(x))^{-\frac{1}{2}}(x^TBx)-1\right] \\
&\quad -\frac{1}{x^TAx}(D(x))^{-\frac{1}{2}}\left[(Ax+A^Tx)(x^TCx)+(Cx+C^Tx)(x^TAx)\right] \\
&\quad +\frac{1}{2(x^TAx)^2}\left[x^TBx-\sqrt{D(x)}\right](Ax+A^Tx)
\end{aligned}$$

and

$$\begin{aligned}
\nabla\lambda_2(x) &= -\frac{1}{2x^TAx}(Bx+B^Tx)\left](D(x))^{\frac{1}{2}}(x^TBx)+1\right] \\
&\quad +\frac{1}{x^TAx}(D(x))^{-\frac{1}{2}}\left[(Ax+A^Tx)(x^TCx)+(Cx+C^Tx)(x^TAx)\right] \\
&\quad +\frac{1}{2(x^TAx)^2}\left[x^TBx+\sqrt{D(x)}\right](Ax+A^Tx).
\end{aligned}$$

Applying Lemma 5.4 for each i, we have

$$\begin{aligned}
& \nabla F_i(x) \\
= \;& 2\lambda_i(x)\nabla\lambda_i(x)\cdot(x^TA^T)+\lambda_i^2(x)A^T+\nabla\lambda_i(x)\cdot(x^TB^T)+\lambda_i(x)B^T+C^T \\
= \;& \nabla\lambda_i(x)(2\lambda_i(x)x^TA^T+x^TB^T)+\lambda_i^2(x)A^T+\lambda_i(x)B^T+C^T.
\end{aligned}$$

Then, the proof is complete. □

Again, we sum up the relation between SOCQEiCP(A, B, C) and SOCCP(F_i) for $i = 1, 2$ in the below theorem, and we call it **Reformulation I for SOCQEiCP**.

Theorem 5.9 (**Reformulation I for SOCQEiCP**).
Consider the SOCQEiCP(A, B, C) given as in (1.5) where A is positive definite. Let $F_i : \mathbb{R}^n \to \mathbb{R}^n$ be defined as in (5.7) for $i = 1, 2$. Then, the following hold.

(a) *If (x^*, λ^*) solves the SOCQEiCP(A, B, C), then x^* solves either SOCCP(F_1) or SOCCP(F_2).*
(b) *Conversely, if $\bar{x}$ is a nonzero solution to the SOCCP(F_i) for $i = 1, 2$, then (x^*, λ^*) solves the SOCQEiCP(A, B, C) with $x^* = \frac{\bar{x}}{a^T\bar{x}}$ and $\lambda^* = \lambda_i(\bar{x})$ $(i = 1, 2)$ defined as in (5.4)-(5.5).*

Proof. Again, part (a) is trivial. We only need to prove part (b). Suppose that $\bar{x}$ is a nonzero solution of SOCCP(F_i) with F_i as in (5.7), $i = 1, 2$. Then, for each i, we know that $\lambda_i^2(\bar{x})A\bar{x} + \lambda_i(\bar{x})B\bar{x} + C\bar{x} \in \mathcal{K}^n$, $\bar{x} \in \mathcal{K}^n$, and $\bar{x}^T(\lambda_i^2(\bar{x})A\bar{x} + \lambda_i(\bar{x})B\bar{x} + C\bar{x}) = 0$. Since $a \in \text{int}(\mathcal{K}^n)$ and $\bar{x} \in \mathcal{K}^n$, we have $\frac{1}{a^T\bar{x}} > 0$ by similar arguments as on page 4. From all the above, we conclude that

$$\begin{array}{rcl} y^* & := & (\lambda^*)^2 Ax^* + \lambda^* Bx^* + Cx^* = \frac{1}{a^T\bar{x}} \left(\lambda_i^2(\bar{x})A\bar{x} + \lambda_i(\bar{x})B\bar{x} + C\bar{x}\right) \in \mathcal{K}^n, \\ y^* & := & \frac{1}{a^T\bar{x}}\bar{x} \in \mathcal{K}^n, \\ a^T x^* & = & \frac{a^T\bar{x}}{a^T\bar{x}} = 1, \\ (x^*)^T y^* & = & \left(\frac{1}{a^T\bar{x}}\right)^2 \left[\bar{x}^T(\lambda_i^2(\bar{x})A\bar{x} + \lambda_i(\bar{x})B\bar{x} + C\bar{x})\right] = 0. \end{array}$$

Thus, (x^*, λ^*) solves the SOCQEiCP(A, B, C). □

Remark 5.10. Note that the mapping F as in (5.3) and (5.7) are continuous and non-monotone [19], moreover, the term $x^T F(x) = 0$ is indeed redundant in the SOCCP (5.1). Hence, it is difficult to apply some usual techniques in the literature (shown as in [18]) to solve (5.1). Thus, we consider other approach for solving (5.1), which was proposed [1]. In particular, the SOCCP (5.1) can be replaced by solving the following problem

$$\begin{array}{c} \min \sum_{j=1}^{n} \theta_r(x_j) + \theta_r(F_j(x)) - 1 \\ \text{s.t } x \succeq_{\mathcal{K}^n} 0, \;\; F(x) \succeq_{\mathcal{K}^n} 0, \end{array}$$

where $\theta : \mathbb{R} \to (-\infty, 1)$ satisfies

$$\theta(t) < 0 \text{ if } t < 0, \quad \theta(0) = 0, \quad \lim_{t\to\infty} \theta(t) = 1,$$

and $\theta_r(t) = \theta(\frac{t}{r})$ for $r > 0$.

We now introduce the second approach to SOCEiCP and SOCQEiCP, which recasts them as a SOCLCP and another SOCCP. The SOC linear complementarity problem (SOCLCP) is to find a vector $x \in \mathbb{R}^n$ such that

$$\text{SOCLCP}(M, q) : \begin{cases} x \succeq_{\mathcal{K}^n} 0, \\ y \succeq_{\mathcal{K}^n} 0, \\ y = Mx + q, \\ x^T y = 0, \end{cases} \tag{5.8}$$

where the vector $q \in \mathbb{R}^n$ and the matrix $M \in \mathbb{R}^{n\times n}$. We shall denote it by SOCLCP(M, q). For the matrix C defined in the SOCEiCP (1.4), we will consider SOCLCP($-C$, 0) as follows:

$$\text{SOCLCP}(-C, 0) : \begin{cases} x \in \mathcal{K}^n, \\ -Cx \in \mathcal{K}^n, \\ x^T(-Cx) = 0. \end{cases}$$

For $(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}$, we define functions F_3 and F_4 as follows:

$$F_3(x, \lambda) := (\lambda Bx - Cx, a^T x - 1) \tag{5.9}$$

and

$$F_4(x,\lambda) := (-\lambda Bx - Cx, a^T x - 1) \tag{5.10}$$

where B, C are $n\times n$ matrices given as in the SOCEiCP(1.4) and $a \in \mathrm{int}(\mathcal{K}^n)$. With these two functions, F_3, F_4 defined as in (5.9) and (5.10) respectively, we consider their corresponding SOCCPs as follows:

$$\mathrm{SOCCP}(F_3): \begin{cases} x \in \mathcal{K}^n, \lambda \geq 0, \\ \lambda Bx - Cx \in \mathcal{K}^n, a^T x - 1 \geq 0, \\ x^T(\lambda Bx - Cx) + \lambda(a^T x - 1) = 0, \end{cases} \tag{5.11}$$

and

$$\mathrm{SOCCP}(F_4): \begin{cases} x \in \mathcal{K}^n, \lambda \geq 0, \\ -\lambda Bx - Cx \in \mathcal{K}^n, a^T x - 1 \geq 0, \\ x^T(-\lambda Bx - Cx) + \lambda(a^T x - 1) = 0, \end{cases} \tag{5.12}$$

We also need a technical lemma for this approach.

Lemma 5.11. *Suppose that $x := (x_1, x_2) \in \mathcal{K}^n$, $y := (y_1, y_2) \in \mathcal{K}^n$. Then, the following hold.*

(a): $x^T y \geq 0$.

(b): *If $x \in \mathrm{int}(\mathcal{K}^n)$, then $x^T y > 0 \Longleftrightarrow y \neq 0$.*

(c): *If $x \neq 0$ and $y \neq 0$, then $x^T y = 0 \Longleftrightarrow x_1 = \|x_2\|$ and $y = \alpha(x_1, -x_2)$, where α is a positive constant. Similarly, if $x \neq 0$ and $y \neq 0$, then $x^T y = 0 \Longleftrightarrow y_1 = \|y_2\|$ and $x = \beta(x_1, -x_2)$, where β is a positive constant.*

Proof. (a) By the definition of $\mathcal{K}^n$ and Schwarz's inequality, the result follows by

$$x^T y = x_1 y_1 + x_2^T y_2 \geq x_1 y_1 - \|x_2\|\|y_2\| \geq x_1 y_1 - x_1 y_1 = 0.$$

(b) It is clear by straightforward checking, or see [9].

(c) By the assumption, we know that $x \neq 0$ implies that $x_1 > 0$. Otherwise, $\|x_2\| \leq x_1 = 0$ implies that $x = 0$. Similarly, we know that $y_1 > 0$. Using the assumption that $x^T y = 0$, the definition of $\mathcal{K}^n$ and Schwarz's inequality, we obtain

$$x_1 y_1 = |-x_1 y_1| = |x_2^T y_2| \leq \|x_2\|\|y_2\| \leq x_1 y_1. \tag{5.13}$$

This implies that

$$x_1 y_1 = \|x_2\|\|y_2\| \tag{5.14}$$

By the equality (5.14) and $x_1, y_1 > 0$, we know that $\|x_2\|, \|y_2\| > 0$. This fact together with $\|x_2\| \leq x_1$, $\|y_2\| \leq y_1$ and the equality (5.14) yields

$$x_1 = \|x_2\| \tag{5.15}$$

and

$$y_1 = \|y_2\|. \tag{5.16}$$

Combining all the above equalities (5.13),(5.14),(5.15) and (5.16), we know that

$$x_2^T y_2 = -x_1 y_1 = -\|x_2\|\|y_2\|.$$

This implies the equality holds in the Schwarz's inequality. Thus, there exists a constant k such that

$$y_2 = kx_2. \tag{5.17}$$

Since $k\|x_2\|^2 = x_2^T y_2 = -x_1 y_1 < 0$, we obtain that $k < 0$. Choosing $\alpha = -k$ says that $\alpha > 0$ and $y_2 = -\alpha x_2$. Now, applying the equalities (5.15),(5.16) and (5.17) leads to

$$y_1 = \|y_2\| = \|-\alpha x_2\| = |-\alpha|\|x_2\| = \alpha\|x_2\| = \alpha x_1,$$

which gives $y = \alpha(x_1, -x_2)$.

The other direction is trivial, so the proof is complete. $\square$

We conclude the relation between SOCEiCP(B, C) and the SOCCP, SOCLCP in the following theorem, and we call it **Reformulation II for SOCEiCP**.

Theorem 5.12 (**Reformulation II for SOCEiCP**).
Let F_3 and F_4 be defined as in (5.9) and (5.10), respectively. Suppose that (x^*, λ^*) solves the SOCEiCP(B,C) defined as in (1.4). Then, the following hold.

(a) *If $\lambda^* > 0$, then (x^*, λ^*) solves the SOCCP(F_3).*
(b) *If $\lambda^* < 0$, then $(x^*, -\lambda^*)$ solves the SOCCP(F_4).*
(c) *If $\lambda^* = 0$, then x^* solves the SOCLCP($-C$,0).*

Conversely, consider the SOCLCP given as in (5.8), the SOCCP(F_3) given as in (5.11), and the SOCCP(F_4) given as in (5.12).

(d) *If $\lambda^* \neq 0$ and (x^*, λ^*) solves the SOCCP(F_3), then (x^*, λ^*) solves the SOCEiCP(B,C).*
(e) *If $\lambda^* \neq 0$ and (x^*, λ^*) solves the SOCCP(F_4), then $(x^*, -\lambda^*)$ solves the SOCEiCP(B,C).*
(f) *If x^* solves the SOCLCP($-C, 0$) and $x^* \neq 0$, then $(\frac{x^*}{a^T x^*}, 0)$ solves the SOCEiCP(B, C).*

Proof. From the assumption that (x^*, λ^*) solves the SOCEiCP(B,C), we observe that

$$x^* \in \mathcal{K}^n \quad \text{and} \quad F_3(x^*, \lambda^*) = F_4(x^*, -\lambda^*) = (\lambda^* Bx^* - Cx^*, 0) \in \mathcal{K}^n \times \mathcal{K}.$$

(a) If $\lambda^* > 0$, then $(x^*, \lambda^*) \in \mathcal{K}^n \times \mathcal{K}$, $F_3(x^*, \lambda^*) \in \mathcal{K}^n \times \mathcal{K}$ and

$$(x^*, \lambda^*) \cdot F_3(x^*, \lambda^*) = (x^*)^T(\lambda^* Bx^* - Cx^*) + \lambda^*(a^T x^* - 1) = 0.$$

This says that (x^*, λ^*) solves the SOCCP(F_3).

(b) If $\lambda^* < 0$, then $(x^*, -\lambda^*) \in \mathcal{K}^n \times \mathcal{K}$, $F_4(x^*, -\lambda^*) \in \mathcal{K}^n \times \mathcal{K}$ and

$$(x^*, -\lambda^*) \cdot F_4(x^*, -\lambda^*) = (x^*)^T(\lambda^* Bx^* - Cx^*) - \lambda^*(a^T x^* - 1) = 0.$$

This says that $(x^*, -\lambda^*)$ solves the SOCCP(F_4).

(c) If $\lambda^* = 0$, then $(x^*, \lambda^*) = (x^*, 0) \in \mathcal{K}^n \times \mathcal{K}$, $\lambda^* Bx^* - Cx^* = -Cx^* \in \mathcal{K}^n$ and $(x^*)^T(-Cx^*) = 0$. This says that $(x^*, 0)$ solves the SOCLCP($-C$,0).

(d) If $\lambda^* \neq 0$ and (x^*, λ^*) solves the SOCCP(F_3), then we see that

$$\begin{cases} x^* \in \mathcal{K}^n, \lambda^* > 0, \\ \lambda^* Bx^* - Cx^* \in \mathcal{K}^n, a^T x^* - 1 \geq 0, \\ (x^*)^T(\lambda^* Bx^* - Cx^*) + \lambda^*(a^T x^* - 1) = 0. \end{cases}$$

By Lemma 5.11, it implies that $(x^*)^T(\lambda^* Bx^* - Cx^*) \geq 0$ and $a^T x^* - 1 = 0$. Hence, (x^*, λ^*) solves the SOCEiCP(B, C).

(e) If $\lambda^* \neq 0$ and (x^*, λ^*) solves the SOCCP(F_4), then we see that

$$\begin{cases} x^* \in \mathcal{K}^n, \lambda^* > 0, \\ -\lambda Bx^* - Cx^* \in \mathcal{K}^n, a^T x^* - 1 \geq 0, \\ (x^*)^T(\lambda^* Bx^* - Cx^*) + \lambda^*(a^T x^* - 1) = 0. \end{cases}$$

By Lemma 5.11 again, it implies that $(x^*)^T(-\lambda^* Bx^* - Cx^*) \geq 0$ and $a^T x^* - 1 = 0$. Hence, $(x^*, -\lambda^*)$ solves the SOCEiCP(B, C).

(f) If x^* solves SOCLCP$(-C, 0)$ and $x^* \neq 0$, then $(\frac{x^*}{a^T x^*}, 0)$ solves the SOCEiCP(B, C) trivially. □

To deal with SOCQEiCP, we have to define other functions. For $(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}$, we define functions F_5 and F_6 as follows:

$$F_5(x, \lambda) = (\lambda^2 Ax + \lambda Bx + Cx, a^T x - 1) \tag{5.18}$$

and

$$F_6(x, \lambda) = (\lambda^2 Ax - \lambda Bx + Cx, a^T x - 1), \tag{5.19}$$

where A, B, C are $n \times n$ matrices given as in the SOCQEiCP (1.5) and $a \in \mathrm{int}(\mathcal{K}^n)$. With these two new functions F_5, F_6 defined as in (5.18) and (5.19), respectively, we consider their corresponding SOCCPs as below:

$$\text{SOCCP}(F_5): \begin{cases} x \in \mathcal{K}^n, \lambda \geq 0, \\ \lambda^2 Ax - \lambda Bx + Cx \in \mathcal{K}^n, a^T x - 1 \geq 0, \\ x^T(\lambda^2 Ax - \lambda Bx + Cx) + \lambda(a^T x - 1) = 0, \end{cases} \tag{5.20}$$

and

$$\text{SOCCP}(F_6): \begin{cases} x \in \mathcal{K}^n, \lambda \geq 0, \\ \lambda^2 Ax - \lambda Bx + Cx \in \mathcal{K}^n, a^T x - 1 \geq 0, \\ x^T(\lambda^2 Ax - \lambda Bx + Cx) + \lambda(a^T x - 1) = 0, \end{cases} \tag{5.21}$$

Likewise, we present the relation between SOCQEiCP(B, C) and the SOCCP, SOCLCP in the following theorem, and we call it **Reformulation II for SOCQEiCP**.

Theorem 5.13 (**Reformulation II for SOCQEiCP**).
Let F_5 and F_6 be defined as in (5.18) and (5.19), respectively. Suppose that (x^*, λ^*) solves the SOCQEiCP(A,B,C) defined as in (1.5). Then, the following hold.

(a) *If $\lambda^* > 0$, then (x^*, λ^*) solves the SOCCP(F_5).*
(b) *If $\lambda^* < 0$, then $(x^*, -\lambda^*)$ solves the SOCCP(F_6).*
(c) *If $\lambda^* = 0$, then x^* solves the SOCLCP(C,0).*

Conversely, consider the SOCLCP given as in (5.8), the SOCCP(F_5) given as in (5.20), and the SOCCP(F_6) given as in (5.21).

(d) *If $\lambda^* \neq 0$ and (x^*, λ^*) solves the SOCCP(F_5), then (x^*, λ^*) solves the SOCQEiCP(A,B,C).*
(e) *If $\lambda^* \neq 0$ and (x^*, λ^*) solves the SOCCP(F_6), then $(x^*, -\lambda^*)$ solves the SOCQEiCP(A,B,C).*
(f) *If x^* solves SOCLCP($C, 0$) and $x^* \neq 0$, then $(\frac{x^*}{a^T x^*}, 0)$ solves the SOCCP(F_6).*

Proof. The arguments are quite similar to those for Theorem 5.12. For completeness, we also present them. From the assumption that (x^*, λ^*) solves the SOCQEiCP(A,B,C), we have the following observations: $x^* \in \mathcal{K}^n$ and $F_5(x^*, \lambda^*) = F_6(x^*, -\lambda^*) = ((\lambda^*)^2 Ax^* + \lambda^* Bx^* + Cx^*, 0) \in \mathcal{K}^n \times \mathcal{K}$.
(a) If $\lambda^* > 0$, then $(x^*, \lambda^*) \in \mathcal{K}^n \times \mathcal{K}$, $F_5(x^*, \lambda^*) \in \mathcal{K}^n \times \mathcal{K}$ and

$$(x^*, \lambda^*) \cdot F_5(x^*, \lambda^*) = (x^*)^T[(\lambda^*)^2 Ax^* + \lambda^* Bx^* + Cx^*] + \lambda^*(a^T x^* - 1) = 0.$$

This says that (x^*, λ^*) solves the SOCCP(F_5).

(b) If $\lambda^* < 0$, then $(x^*, -\lambda^*) \in \mathcal{K}^n \times \mathcal{K}$, $F_5(x^*, -\lambda^*) \in \mathcal{K}^n \times \mathcal{K}$ and

$$(x^*, -\lambda^*) \cdot F_5(x^*, -\lambda^*) = [(\lambda^*)^2 Ax^* + \lambda^* Bx^* + Cx^*] - \lambda^*(a^T x^* - 1) = 0.$$

This says that $(x^*, -\lambda^*)$ solves the SOCCP(F_5).

(c) If $\lambda^* = 0$, then $(x^*, \lambda^*) = (x^*, 0) \in \mathcal{K}^n \times \mathcal{K}$, $(\lambda^*)^2 Ax^* + \lambda^* Bx^* + Cx^* = Cx^* \in \mathcal{K}^n$ and $(x^*)^T Cx^* = 0$. This says that $(x^*, 0)$ solves the SOCLCP(C,0).

(d) If $\lambda^* \neq 0$ and (x^*, λ^*) solves the SOCCP(F_5), then we know that

$$\begin{cases} x^* \in \mathcal{K}^n, \lambda^* > 0, \\ (\lambda^*)^2 Ax^* + \lambda^* Bx^* + Cx^* \in \mathcal{K}^n, a^T x^* - 1 \geq 0, \\ (x^*)^T[(\lambda^*)^2 Ax^* + \lambda^* Bx^* + Cx^*] + \lambda^*(a^T x^* - 1) = 0. \end{cases}$$

By Lemma 5.11, this implies that $(x^*)^T[(\lambda^*)^2 Ax^* + \lambda^* Bx^* + Cx^*] \geq 0$ and $a^T x^* - 1 = 0$. Hence, (x^*, λ^*) solves the SOCQEiCP(A, B, C).

(e) If $\lambda^* \neq 0$ and (x^*, λ^*) solves the SOCCP(F_6), then we know that

$$\begin{cases} x^* \in \mathcal{K}^n, \lambda^* > 0, \\ (-\lambda^*)^2 Ax^* + (-\lambda^*) Bx^* + Cx^* \in \mathcal{K}^n, a^T x^* - 1 \geq 0, \\ (x^*)^T[(-\lambda^*)^2 Ax^* + (-\lambda^*) Bx^* + Cx^*] + \lambda^*(a^T x^* - 1) = 0. \end{cases}$$

By Lemma 5.11, this implies that $(x^*)^T[(-\lambda^*)^2 Ax^* + (-\lambda^*) Bx^* + Cx^*] \geq 0$ and $a^T x^* - 1 = 0$. Hence, $(x^*, -\lambda^*)$ solves the SOCEiCP(A,B,C).

(f) If x^* solves the SOCLCP($C, 0$) and $x^* \neq 0$, then $(\frac{x^*}{a^T x^*}, 0)$ solves the SOCQEiCP(A,B,C) trivially. □

Remark 5.14. Let $y = (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}$ and F_i be defined as in (5.9), (5.10), (5.18), (5.19). The SOCCP(F_i), $i = 3, 4, 5, 6$ can be written as

$$\text{SOCCP}(F_i) : \begin{cases} y \in \mathcal{K}^n \times \mathcal{K}^1 \\ F(y) \in \mathcal{K}^n \times \mathcal{K}^1, \\ y^T F(y) = 0. \end{cases}$$

Since F_i, $i = 3, 4, 5, 6$ is continuously differentiable, it is easy to look for an algorithm to solve the SOCCP(F_i), which is better than the SOCCP as in **Reformulation I**.

At last, we present the third approach to SOCEiCP and SOCQEiCP, which recasts them as a nonsmooth system of equations. Note that this approach was also investigated in [2], but the SOCEiCP (expression (13) given in [2]) is not correct because the authors replace the condition $x \neq 0$ by $e^T x > 0$. This is not appropriate like what are commented in Section 1. More specifically, for such approach, it needs to find a function $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ such that

$$\phi(x, y) = 0 \quad \Longleftrightarrow \quad \langle x, y \rangle = 0, x \in \mathcal{K}^n, y \in \mathcal{K}^n. \tag{5.22}$$

Such function is usually called a C-function or SOCCP-function. There are already many C-functions in the literature [13–15, 38, 41]. In this approach, we employ two popular ones, the *natural residual function*, also called the *min-function*, denoted by $\phi_{\rm NR}$, and the *Fischer-Burmeister function*, denoted by $\phi_{\rm FB}$. They are defined as below, respectively,

$$\phi_{\rm NR}(x, y) := x - (x - y)_+ = x - P_{\mathcal{K}^n}(x - y), \tag{5.23}$$

$$\phi_{\rm FB}(x, y) := (x + y) - (x^2 + y^2)^{\frac{1}{2}}, \tag{5.24}$$

where $P_{\mathcal{K}^n}$ denotes the projection mapping onto $\mathcal{K}^n$.

Before moving on, we also recall the concepts of the B-subdifferential and (strong) semismoothness, which will be used later. Given a mapping $H : \mathbb{R}^n \to \mathbb{R}^m$, if H is locally Lipschitz continuous, then the set

$$\partial_B H(z) := \left\{ V \in \mathbb{R}^{m \times n} \,|\, \exists \{z^k\} \subseteq D_H \text{ s.t. } z^k \to z, H^{'}(z^k) \to V \right\}$$

is nonempty and is called the *B-subdifferential* of H at z, where $D_H \subseteq \mathbb{R}^n$ denotes the set of points at which H is differentiable. The convex hull $\partial H(z) := conv\partial_B H(z)$ is called the generalized Jacobian in Clarke sense [11]. A mapping $H : \mathbb{R}^n \to \mathbb{R}^m$ is said to be semismooth at x if H is directionally differentiable at x; and for all $V \in \partial H(x + h)$ and $h \to 0$, there holds

$$Vh - H^{'}(x; h) = o(\|h\|).$$

The mapping H is said to be *strongly semismooth* at x if H is semismooth at x; and for all $V \in \partial H(x + h)$ and $h \to 0$, there holds

$$Vh - H^{'}(x; h) = o(\|h\|^2).$$

The mapping H is called *(strongly) semismooth* if it is (strongly) semismooth everywhere.

In light of [15, Proposition 4.3], [24, Lemmas 2.4 and 2.5] and [10, Propositions 4 and 5], we list some results about the natural residual function and the Fischer-Burmeister function.

Lemma 5.15. *Let $\phi_{\rm NR}$ be defined as in (5.23). Then, $\phi_{\rm NR}$ is strongly semismooth with*

$$\partial_B \phi_{\rm NR}(x, y) = \left\{ \begin{bmatrix} I - V & V \end{bmatrix} \in \mathbb{R}^{n \times 2n} \,|\, V \in P_{\mathcal{K}^n}(x - y) \right\},$$

for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$.

Lemma 5.16. *Let $\phi_{\rm FB}$ be defined as in (5.24). Then, $\phi_{\rm FB}$ is strongly semismooth. For any $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$, we denote $w=(w_1,w_2):=x^2+y^2$ and $z=(z_1,z_2):=(x^2+y^2)^{\frac{1}{2}}$. Then, each element in $\partial_B\phi_{\rm FB}$ is described by*

$$\big[I-V_x\ \ I-V_y\big]$$

with V_x and V_y having the following representation:

(a) *If $x^2+y^2\in{\rm int}(\mathcal{K}^n)$, then $V_x=L_z^{-1}L_x$ and $V_y=L_z^{-1}L_y$.*

(b) *If $x^2+y^2\in{\rm bd}(\mathcal{K}^n)$, and $(x,y)\neq(0,0)$, then*

$$V_x\in\left\{\frac{1}{2\sqrt{2w_1}}\begin{pmatrix}1 & \bar{w}_2^T\\ \bar{w}_2 & 4I-3\bar{w}_2\bar{w}_2^T\end{pmatrix}L_x+\frac{1}{2}\begin{pmatrix}1\\ -\bar{w}_2\end{pmatrix}u^T\right\}$$

and

$$V_y\in\left\{\frac{1}{2\sqrt{2w_1}}\begin{pmatrix}1 & \bar{w}_2^T\\ \bar{w}_2 & 4I-3\bar{w}_2\bar{w}_2^T\end{pmatrix}L_y+\frac{1}{2}\begin{pmatrix}1\\ -\bar{w}_2\end{pmatrix}v^T\right\},$$

for some $u=(u_1,u_2), v=(v_1,v_2)\in\mathbb{R}\times\mathbb{R}^{n-1}$ satisfying $|u_1|\le\|u_2\|\le1$ and $|v_1|\le\|v_2\|\le1$, where $\bar{w}_2=\frac{w_2}{\|w_2\|}$.

(c) *If $(x,y)=(0,0)$, then $V_x\in\{L_{\hat{x}}\}$, $V_y\in\{L_{\hat{y}}\}$ for some $\hat{x}$, $\hat{y}$ with $\|\hat{x}\|^2+\|\hat{y}\|^2=1$, or*

$$V_x\in\left\{\frac{1}{2}\begin{pmatrix}1\\ \bar{w}_2\end{pmatrix}\xi^T+\frac{1}{2}\begin{pmatrix}1\\ -\bar{w}_2\end{pmatrix}u^T+2\begin{pmatrix}0 & 0\\ (I-\bar{w}_2\bar{w}_2^T)s_2 & (I-\bar{w}_2\bar{w}_2^T)s_1\end{pmatrix}\right\},$$

$$V_y\in\left\{\frac{1}{2}\begin{pmatrix}1\\ \bar{w}_2\end{pmatrix}\eta^T+\frac{1}{2}\begin{pmatrix}1\\ -\bar{w}_2\end{pmatrix}v^T+2\begin{pmatrix}0 & 0\\ (I-\bar{w}_2\bar{w}_2^T)\omega_2 & (I-\bar{w}_2\bar{w}_2^T)\omega_1\end{pmatrix}\right\},$$

for some $u=(u_1,u_2)$, $v=(v_1,v_2)$, $\xi=(\xi_1,\xi_2)$, $\eta=(\eta_1,\eta_2)\in\mathbb{R}\times\mathbb{R}^{n-1}$ such that $|u_1|\le\|u_2\|\le1$, $|v_1|\le\|v_2\|\le1$, $|\xi_1|\le\|\xi_2\|\le1$, $|\eta_1|\le\|\eta_2\|\le1$, $\bar{w}_2\in\mathbb{R}^{n-1}$ satisfying $\|\bar{w}_2\|=1$, and $s=(s_1,s_2),\omega=(\omega_1,\omega_2)\in\mathbb{R}\times\mathbb{R}^{n-1}$ satisfying $\|s\|^2+\|\omega\|^2\le\frac{1}{2}$.

From (5.22), it is clear to see that when $\phi:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is a C-function, the SOCEiCP(B,C) can be reformulated as a nonsmooth system of equations:

$$\Phi(z)=\Phi(x,y,\lambda):=\begin{bmatrix}\phi(x,y)\\ \lambda Bx-Cx-y\\ a^Tx-1\end{bmatrix}=0. \tag{5.25}$$

Here $\Phi:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^{2n+1}$. We shall denote the above Φ by $\Phi_{\rm NR}$ and $\Phi_{\rm FB}$ when ϕ represents the natural residual function $\phi_{\rm NR}$ and the Fischer-Burmeister function $\phi_{\rm FB}$, respectively. With these notations, the B-subdifferential of $\Phi_{\rm NR}$ and $\Phi_{\rm FB}$, are written out as below lemmas.

Lemma 5.17. *The function $\Phi_{\rm NR}$ is semismooth. Moreover, the B-subdifferential of $\Phi_{\rm NR}$ at $z=(x,y,\lambda)$ is described by*

$$\begin{aligned}\partial_B\Phi_{\rm NR}(z)&=\partial_B\Phi_{\rm NR}(x,y,\lambda)\\ &=\left\{\begin{pmatrix}I-V & V & 0\\ \lambda B-C & -I & Bx\\ a^T & 0 & 0\end{pmatrix}\;\middle|\; V\in\partial_B P_{\mathcal{K}^n}(x-y)\right\}.\end{aligned}$$

Proof. This is a direct consequence of Lemma 5.15. $\square$

Lemma 5.18. *The function $\Phi_{\rm FB}$ is semismooth. Moreover, the B-subdifferential of $\Phi_{\rm FB}$ at $z = (x, y, \lambda)$ is described by*

$$\begin{aligned}\partial_B \Phi_{\rm FB}(z) &= \partial_B \Phi_{\rm FB}(x, y, \lambda) \\ &= \left\{ \left(\begin{array}{ccc} I - V_x & I - V_y & 0 \\ \lambda B - C & -I & Bx \\ a^T & 0 & 0 \end{array} \right) \middle| \begin{bmatrix} I - V_x & I - V_y \end{bmatrix} \in \partial_B \phi_{\rm FB}(z) \right\},\end{aligned}$$

where V_x and V_y are given in Lemma 5.16.

Proof. This is a direct consequence of Lemma 5.16. $\square$

Now, we are ready to conclude the relation between SOCEiCP(B, C) and the nonsmooth system of equations in the following theorem. We will call it **Reformulation III for SOCEiCP**.

Theorem 5.19 (**Reformulation III for SOCEiCP**)**.** *Let $\Phi(z) = \Phi(x, y, \lambda)$ be defined as in (5.25) and consider the SOCEiCP(B,C) given as in (1.4). Then, the following hold.*

(a) *If (x^*, λ^*) solves the SOCEiCP(B,C), then $\Phi(x^*, y^*, \lambda^*) = 0$ with $y^* = \lambda^* B x^* - C x^*$.*

(b) *Conversely, if (x^*, y^*, λ^*) is a solution of the nonsmooth system of equations $\Phi(z) = 0$, i.e., $\Phi(x^*, y^*, \lambda^*) = 0$, then (x^*, λ^*) solves the SOCEiCP(B,C).*

Furthermore, if Φ represents $\Phi_{\rm NR}$ or $\Phi_{\rm FB}$, then the B-subdifferential of Φ exists with

$$\partial_B \Phi_{\rm NR}(x^*, y^*, \lambda^*) = \left\{ \left(\begin{array}{ccc} I - V & V & 0 \\ \lambda B - C & -I & Bx^* \\ a^T & 0 & 0 \end{array} \right) \middle| V \in \partial_B P_{\mathcal{K}^n}(x^* - y^*) \right\}$$

and

$$\begin{aligned}&\partial_B \Phi_{\rm FB}(x^*, y^*, \lambda^*) \\ &\quad = \left\{ \left(\begin{array}{ccc} I - V_{x^*} & I - V_{y^*} & 0 \\ \lambda B - C & -I & Bx^* \\ a^T & 0 & 0 \end{array} \right) \middle| \begin{bmatrix} I - V_{x^*} & I - V_{y^*} \end{bmatrix} \in \partial_B \phi_{\rm FB}(x^*, y^*) \right\},\end{aligned}$$

where V_{x^*} and V_{y^*} are given in Lemma 5.16, respectively.

Proof. The results follow by the definition of the SOCEiCP, Lemma 5.17 and Lemma 5.18. $\square$

As for SOCQEiCP(A, B, C), we observe that when $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a C-function, the SOCQEiCP(A, B, C) can be reformulated as another nonsmooth system of equations:

$$\Psi(z) = \Psi(x, y, \lambda) := \begin{bmatrix} \phi(x, y) \\ \lambda^2 Ax + \lambda Bx + Cx - y \\ a^T x - 1 \end{bmatrix} = 0. \tag{5.26}$$

Here $\Psi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{2n+1}$. Again, we shall denote Ψ by $\Psi_{\rm NR}$ and $\Psi_{\rm FB}$ when ϕ means $\phi_{\rm NR}$ and $\phi_{\rm FB}$, respectively. In these cases, the B-subdifferential of $\Psi_{\rm NR}$ and $\Psi_{\rm FB}$ are expressed as lemmas.

Lemma 5.20. *The function $\Psi_{\rm NR}$ is semismooth. Moreover, the B-subdifferential of $\Psi_{\rm NR}$ at $z = (x, y, \lambda)$ is described by*

$$\begin{aligned} \partial_B \Psi_{\rm NR}(z) &= \partial_B \Psi_{NR}(x, y, \lambda) \\ &= \left\{ \left(\begin{array}{ccc} I - V & V & 0 \\ \lambda^2 A + \lambda B + C & -I & 2\lambda A x + Bx \\ a^T & 0 & 0 \end{array} \right) \middle| V \in \partial_B P_{\mathcal{K}^n}(x - y) \right\}. \end{aligned}$$

Proof. By direct computation and Lemma 5.15, the proof is trivial. □

Lemma 5.21. *The function $\Psi_{\rm FB}$ is semismooth. Moreover, the B-subdifferential of $\Psi_{\rm FB}$ at $z = (x, y, \lambda)$ is described by*

$$\begin{aligned} &\partial_B \Psi_{\rm FB}(z) = \partial_B \Psi_{\rm FB}(x, y, \lambda) \\ &= \left\{ \left(\begin{array}{ccc} I - V_x & I - V_y & 0 \\ \lambda^2 A + \lambda B + C & -I & 2\lambda A x + Bx \\ a^T & 0 & 0 \end{array} \right) \middle| \begin{bmatrix} I - V_x & I - V_y \end{bmatrix} \in \partial_B \phi_{\rm FB}(z) \right\}, \end{aligned}$$

where V_x and V_y are given in Lemma 5.16.

Proof. By direct computation and Lemma 5.16, the proof is trivial. □

Now, we are ready to conclude the relation between SOCQEiCP(A,B, C) and the nonsmooth system of equations in the following theorem. We call it **Reformulation III for SOCQEiCP**.

Theorem 5.22 (Reformulation III for SOCQEiCP). *Let $\Psi(z) = \Psi(x, y, \lambda)$ be defined as in (5.26) and consider the SOCQEiCP(A,B,C) given as in (1.5). Then, the following hold.*

(a) *If (x^*, λ^*) solves the SOCQEiCP(A,B,C), then $\Psi(x^*, y^*, \lambda^*) = 0$ with $y^* = (\lambda^*)^2 A x^* + \lambda^* B x^* + C x^*$.*

(b) *Conversely, if (x^*, y^*, λ^*) is a solution of the nonsmooth system of equations $\Psi(z) = 0$, i.e., $\Psi(x^*, y^*, \lambda^*) = 0$, then (x^*, λ^*) solves the SOCQEiCP(A,B,C).*

Furthermore, if Ψ represents $\Psi_{\rm NR}$ or $\Psi_{\rm FB}$, then the B-subdifferential of Ψ exists with

$$\begin{aligned} &\partial_B \Psi_{\rm NR}(x^*, y^*, \lambda^*) \\ &= \left\{ \left(\begin{array}{ccc} I - V & V & 0 \\ (\lambda^*)^2 A + \lambda^* B + C & -I & 2\lambda^* A x^* + B x^* \\ a^T & 0 & 0 \end{array} \right) \middle| V \in \partial_B P_{\mathcal{K}^n}(x^* - y^*) \right\}. \end{aligned}$$

and

$$\partial_B \Psi_{\mathrm{FB}}(x^*, y^*, \lambda^*) = \left\{ \begin{pmatrix} I - V_{x^*} & I - V_{y^*} & 0 \\ (\lambda^*)^2 A + \lambda^* B + C & -I & 2\lambda^* Ax + Bx \\ a^T & 0 & 0 \end{pmatrix} \;\middle|\; \begin{bmatrix} I - V_{x^*} & I - V_{y^*} \end{bmatrix} \in \partial_B \phi_{\mathrm{FB}}(x^*, y^*) \right\},$$

where V_{x^*} and V_{y^*} are given in Lemma 5.16, respectively.

Proof. Applying Lemma 5.20 and Lemma 5.21, the proof is trivial. □

6. Concluding remarks

In this paper, the existence of solutions for three types of optimization problems involving SOC is studied. First, we look into the absolute value equations associated with SOC, which are natural extensions of the standard absolute value equations. For the absolute value equation associated with SOC, we have characterized under what condition, the SOCAVEs have solution and unique solution. Such results are new to the literature and will be helpful for further study of the SOCAVEs. In addition, we study the solvabilities of two types of eigenvalue complementarity problems, i.e., the SOCEiCP(B, C) and the SOCQEiCP(A, B, C). Our approach is based on reformulating them as various second-order cone complementarity problems, which is a novel thinking different from the existing ways. Such an idea may pave a way to seeking new algorithms for solving the SOCEiCP(B, C) and the SOCQEiCP(A, B, C). We leave them as our future research topics.

References

[1] L. Abdallah, M. Haddou and T. Migot, *Solving absolute value equation using complementarity and smoothing functions*, J. Comput. Appl. Mth. **327** (2018), 196–207.

[2] S. Adly and H. Rammal, *A new method for solving second-order cone eigenvalue complementarity problems*, J. Optim. Theory Appl. **165** (2015), 563–585.

[3] C. Brás, M. Fukushima, A. Iusem, and J. Júdice, *On the quadratic eigenvalue complementarity problem over a general convex cone*, Appl. Math. Comput. **271** (2015), 391–403.

[4] C. Brás, A. Iusem, and J. Júdice, *On the quadratic eigenvalue complementarity problem*, J. Global Optim. **66** (2016), 153–171.

[5] L. Caccetta, B. Qu, and G.-L. Zhou, *A globally and quadratically convergent method for absolute value equations*, Comput. Optim. Appl. **48** (2011), 45–58.

[6] J.-S. Chen, *Two classes of merit functions for the second-order cone complementarity problem*, Math. Method Oper. Res. **64** (2006), 495–519.

[7] J.-S. Chen, *A new merit function and its related properties for the second-order cone complementarity problem*, Pac. J. Optim. **2** (2006), 167–179.

[8] J.-S. Chen, *Conditions for error bounds and bounded level sets of some merit functions for SOCCP*, J. Optim. Theory Appl. **135** (2007), 459–473.

[9] J.-S. Chen, *SOC Functions and Their Applications*, Springer Optimization and Its Applications 143, Springer, Singapore, 2019.

[10] J.-S. Chen, X. Chen, and P. Tseng, *Analysis of nonsmooth vector-valued functions associated with second-order cones*, Math. Program. **101** (2004), 95–117.

[11] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983. Reprinted by SIAM, Philadelphia, 1990.

[12] J.-S. Chen and S.-H. Pan, *A descent method for solving reformulation of the second-order cone complementarity problem*, J. Comput. Appl. Math. **213** (2008), 547–558.

[13] J.-S. Chen and S.-H. Pan, *A one-parametric class of merit functions for the second-order cone complementarity problem*, Comput. Optim. Appl. **45** (2010), 581–606.

[14] J.-S. Chen and S.-H. Pan, *A survey on SOC complementarity functions and solution methods for SOCPs and SOCCPs*, Pac. J. Optim. **8** (2012), 33–74.

[15] X. D. Chen, D. Sun, and J. Sun, *Complementarity functions and numerical experiments for second-order cone complementarity problems*, Comput. Optim. Appl **25** (2003), 39–56.

[16] J.-S. Chen and P. Tseng, *An unconstrained smooth minimization reformulation of second-order cone complementarity problem*, Math. Program. **104** (2005), 293–327.

[17] U. Faraut and A. Koranyi, *Anlysis on Symmetric Cones*, Oxford Mathematical Monographs, Oxford University Press, New York, 1994.

[18] F. Facchinei and J. Pang, *Finite-dimensional Variational Inequalities and Complementarity Problems*, Springer, New York, 2003.

[19] L. Fernandes, M. Fukushima, J. Júdice, and H. Sherali, *The second-order cone eigenvalue complementarity problem*, Optim. Method Softw. **31** (2016), 24–52.

[20] M. Fukushima, Z.-Q. Luo, and P. Tseng, *Smoothing functions for second-order-cone complimentarity problems*, SIAM J. Optim. **12** (2002), 436–460.

[21] S.-L. Hu and Z.-H. Huang, *A note on absolute value equations*, Optim. Lett. **4** (2010), 417–424.

[22] S.-L. Hu, Z.-H. Huang, and Q. Zhang, *A generalized Newton method for absolute value equations associated with second order cones*, J. Comput. Appl. Math. **235** (2011), 1490–1501.

[23] X.-Q. Jiang and Y. Zhang, *A smoothing-type algorithm for absolute value equations*, J. Ind. Manag. Optim. **9** (2013), 789–798.

[24] C. Kanzow, I. Ferenczi, and M. Fukushima, *On the local convergence of semismooth Newton methods for linear and nonlinear second-order cone programs without strict complementarity*, SIAM J. Optim. **20** (2009), 297–320.

[25] S. Ketabchi and H. Moosaei, *Minimum norm solution to the absolute value equation in the convex case*, J. Optim. Theory Appl. **154** (2012), 1080–1087.

[26] L. Kong, L. Tunel, and N. Xiu, *Fischer-Burmeister complementarity function on Euclidean Jordan algebras*, Pac. J. Optim. **6** (2007), 423–440.

[27] X.-H. Miao, J.-T. Yang, and S.-L. Hu, *A generalized Newton method for absolute value equations associated with circular cones*, Appl. Math. Comput. **269** (2015), 155–168.

[28] O. L. Mangasarian, *Absolute value programming*, Comput. Optim. Appl. **36** (2007), 43–53.

[29] O. L. Mangasarian, *Absolute value equation solution via concave minimization*, Optim. Lett. **1** (2007), 3–5.

[30] O. L. Mangasarian, *A generalized Newton method for absolute value equations*, Optim. Lett. **3** (2009), 101–108.

[31] O. L. Mangasarian, *Knapsack feasibility as an absolute value equation solvable by successive linear programming*, Optim. Lett. **3** (2009), 161–170.

[32] O. L. Mangasarian, *Primal-dual bilinear programming solution of the absolute value equation*, Optim. Lett. **6** (2012), 1527–1533.

[33] O. L. Mangasarian, *Absolute value equation solution via dual complementarity*, Optim. Lett. **7** (2013), 625–630.

[34] O. L. Mangasarian, *Absolute value equation solution via linear programming*, J. Optim. Theory Appl. **161** (2014), 870–876.

[35] O. L. Mangasarian and R. R. Meyer, *Absolute value equation*, Linear Algebra Appl. **419** (2006), 359–367.

[36] S.-H. Pan and J.-S. Chen, *A damped Gauss-Newton method for the second-order cone complementarity problem*, Appl. Math. Optim. **59** (2009), 293–318.

[37] S.-H. Pan and J.-S. Chen, *A regularization method for the second-order cone complementarity problems with the Cartesian P_0-property*, Nonlinear Anal. Theory Methods Appl. **70(4)** (2009), 1475–1491.

[38] S.-H. Pan and J.-S. Chen, *A semismooth Newton method for SOCCPs based on a one-parametric class of complementarity functions*, Comput. Optim. Appl. **45** (2010), 59–88.

[39] S.-H. Pan and J.-S. Chen, *A linearly convergent derivative-free descent method for the second-order cone complementarity problem*, Optimization **59** (2010), 1173–1197.

[40] S.-H. Pan and J.-S. Chen, *A least-square semismooth Newton method for the second-order cone complementarity problem*, Optim. Method Softw. **26** (2011), 1–22.

[41] S.-H. Pan, S. Kum, Y. Lim, and J.-S. Chen, *On the generalized Fischer-Burmeister merit function for the second-order cone complementarity problem*, Math. Comput. **83** (2014), 1143–1171.

[42] O. A. Prokopyev, *On equivalent reformulations for absolute value equations*, Comput. Optim. Appl. **44** (2009), 363–372.

[43] M. Queiroz, J. Júdice, and C. Humes, *The symmetric eigenvalue complementarity problem*, Math. Comput. **73** (2003), 1849–1863.

[44] J. Rohn, *A theorem of the alternative for the equation $Ax + B|x| = b$*, Linear Multilinear A. **52** (2004), 421–426.

[45] J. Rohn, *An algorithm for solving the absolute value equation*, Eletron. J. Linear Al. **18** (2009), 589–599.

[46] A. Seeger and Torki, *On eigenvalues induced by a cone constraint*, Linear Algebra Appl. **372** (2003), 181–206.

[47] A. Seeger, *Eigenvalue analysis of equilibrium processes defined by linear complementarity conditions*, Linear Algebra Appl. **292** (1999), 1–14.

[48] A. Seeger, *Quadratic eigenvalue problems under conic constraints*, SIAM J. Matrix Anal. Appl. **32** (2011), 700–721.

[49] D. Sun and J. Sun, *Strong semismoothness of the Fischer-Burmeister SDC and SOC complementarity functions*, Math. Program. **103** (2005), 575–581.

[50] J. Tao and M. Gowda, *Some P-properties for nonlinear transformations on Euclidean Jordan algebras*, Math. Method Oper. Res. **30** (2005), 985–1004.

[51] J. Wu and J.-S. Chen, *A proximal point algorithm for the monotone second-order cone complementarity problem*, Comput. Optim. Appl. **51** (2012), 1037–1063.

[52] S. L. Wu and P. Guo, *On the unique solvability of the absolute value equation*, J. Optim. Theory Appl. **169** (2016), 705–712.

[53] S. Yamanaka and M. Fukushima, *A brancd and bound method for the absolute value programs*, Optimization **63** (2014), 305–319.

[54] C. Zhang and Q.-J. Wei, *Global and finite convergence of a generalized Newton method for absolute value equations*, J. Optim. Theory Appl. **143** (2009), 391–403.

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