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Stationary point conditions for the FB merit function associated with symmetric cones

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ABSTRACT

For the symmetric cone complementarity problem, we show that each stationary point of the unconstrained minimization reformulation based on the Fischer–Burmeister merit function is a solution to the problem, provided that the gradient operators of the mappings involved in the problem satisfy column monotonicity or have the Cartesian P_0-property. These results answer the open question proposed in the article that appeared in Journal of Mathematical Analysis and Applications 355 (2009) 195–215.

1. Introduction

Let $\mathbb{A} = (\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$ be an n-dimensional Euclidean Jordan algebra (see Section 2 for the definition) and $\mathcal{K}$ be the symmetric cone in $\mathbb{V}$. We consider the following symmetric cone complementarity problem (SCCP) which is *to find a vector* $\zeta \in \mathbb{V}$ *such that*

$$F(\zeta) \in \mathcal{K}, \qquad G(\zeta) \in \mathcal{K}, \qquad \langle F(\zeta), G(\zeta)\rangle = 0, \tag{1}$$

where F and G are the differentiable mappings from $\mathbb{V}$ to $\mathbb{V}$. This class of problems provides a unified framework for the nonlinear complementarity problem (NCP) over the nonnegative orthant cone in $\mathbb{R}^n$, the second-order cone complementarity problem (SOCCP) and the semidefinite complementarity problem (SDCP), and becomes one of the main research interests in the current optimization field; see, e.g., [5,10,14,16,17,19].

Recently, there are active studies for merit functions (or complementarity functions) for the SCCP. For example, Liu, Zhang and Wang [14] extended a class of merit functions proposed in [8] for the NCP to the SCCP; Kong, Tuncel and Xiu [11] studied the implicit Lagrangian merit function for the SCCP; Kong, Sun and Xiu [10] proposed a regularized smoothing method by use of the natural residual complementarity function associated with symmetric cones; and Huang and Ni [6] developed a smoothing algorithm with the regularized CHKS smoothing function over symmetric cones. Along this line, we also extended the one-parametric class of merit functions in [7] to the SCCP [15]. Specifically, a function $\psi : \mathbb{V}\times\mathbb{V} \to \mathbb{R}_+$ is called a merit function associated with the cone $\mathcal{K}$ if

$$\psi(x, y) = 0 \Longleftrightarrow x \in \mathcal{K}, \quad y \in \mathcal{K}, \qquad \langle x, y\rangle = 0. \tag{2}$$

With such a function, the SCCP can be reformulated as an unconstrained minimization

$$\min_{\zeta\in\mathbb{V}} \Psi(\zeta) := \psi(F(\zeta), G(\zeta)), \tag{3}$$

in the sense that ζ^* solves (1) if and only if it is a solution of (3) with zero optimal value. Then, the effective unconstrained minimization methods can be applied for solving it.

A popular choice for ψ is the Fischer–Burmeister (FB) merit function ψ_{FB} defined as

$$\psi_{FB}(x, y) := \frac{1}{2}\|\phi_{FB}(x, y)\|^2 \quad \forall x, y \in \mathbb{V} \tag{4}$$

where $\phi_{FB} : \mathbb{V}\times\mathbb{V} \to \mathbb{V}$ is the FB complementarity function associated with $\mathcal{K}$, given by

$$\phi_{FB}(x, y) = (x^2 + y^2)^{1/2} - (x + y) \tag{5}$$

with $x^2 = x \circ x$ denoting the Jordan product of x and itself, and $x^{1/2}$ the unique square root of $x \in \mathcal{K}$, i.e., $x^{1/2} \circ x^{1/2} = x$. The function ψ_{FB} was first proved to be differentiable in [14], and later the authors of [12,15] independently showed that it is continuously differentiable everywhere with Lipschitz continuous gradients. However, it has been an open question: under what conditions every stationary point of the minimization problem

$$\min_{\zeta\in\mathbb{V}} \Psi_{FB}(\zeta) := \psi_{FB}(F(\zeta), G(\zeta)) \tag{6}$$

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is guaranteed to be a solution of (1). The main difficulty to establish such results is described in [15]. The study for stationary point conditions is extremely important in the merit function approach since, when applying effective gradient-type methods for solving the minimization reformulation problems, one at most expects to get a stationary point due to the nonconvexity of the merit functions.

The main purpose of this paper is to settle down this open problem. By exploiting the classification of a simple Euclidean Jordan algebra and extending a weaker result than the first implication of [4, Prop. 3.4] to the setting of symmetric cones, we show that each stationary point of the minimization problem (6) is a solution to (1) if the gradient operators ∇F and $-\nabla G$ are column monotone. If the operator ∇G is invertible, this condition can be relaxed to the one that $\nabla G^{-1}\nabla F$ has the Cartesian P_0-property.

2. Preliminaries

This section recalls some results on Euclidean Jordan algebras that will be used in the subsequent section. More detailed expositions of Euclidean Jordan algebras can be found in Koecher's lecture notes [9] and the monograph by Faraut and Korányi [3].

A Euclidean Jordan algebra is a triple $(\mathbb{V}, \circ, \langle\cdot,\cdot\rangle_{\mathbb{V}})$ where $(\mathbb{V}, \langle\cdot,\cdot\rangle_{\mathbb{V}})$ is a finite-dimensional inner product space over the real number field $\mathbb{R}$ and $(x, y) \mapsto x \circ y : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ is a bilinear mapping satisfying the following conditions:

(i) $x \circ y = y \circ x$ for all $x, y \in \mathbb{V}$;
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathbb{V}$, where $x^2 = x \circ x$;
(iii) $\langle x \circ y, z\rangle_{\mathbb{V}} = \langle y, x \circ z\rangle_{\mathbb{V}}$ for all $x, y, z \in \mathbb{V}$.

Let $\mathbb{A} = (\mathbb{V}, \circ, \langle\cdot,\cdot\rangle_{\mathbb{V}})$ denote a Euclidean Jordan algebra. We assume that there is an element $e \in \mathbb{V}$ (called the *unit* element) such that $x \circ e = x$ for all $x \in \mathbb{V}$. By [3, Theorem III. 2.1], the set of squares $\mathcal{K} := \{x^2 \mid x \in \mathbb{V}\}$ is a symmetric cone. We write $x \succeq_{\mathcal{K}} y$ (respectively, $x \succ_{\mathcal{K}} y$) to mean $x - y \in \mathcal{K}$ (respectively, $x - y \in \text{int}\mathcal{K}$).

For $x \in \mathbb{V}$, let $m(x) := \min\{k : \{e, x, x^2, \ldots, x^k\}$ are linearly dependent$\}$ and define the *rank* of $\mathbb{A}$ by $r := \max\{m(x) : x \in \mathbb{V}\}$. Recall that an element $c \in \mathbb{V}$ is *idempotent* if $c^2 = c$, and it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. One says that a finite set $\{c_1, c_2, \ldots, c_k\}$ of primitive idempotents in $\mathbb{V}$ is a *Jordan frame* if

$$c_j \circ c_i = 0 \quad \text{if } j \neq i \text{ for all } j, i = 1, 2, \ldots, k, \quad \text{and} \quad \sum_{j=1}^{k} c_j = e.$$

Now we may state the second version of the spectral decomposition theorem.

Theorem 2.1 (*[3, Theorem III. 1.2]*). *Let $\mathbb{A}$ be a Euclidean Jordan algebra with rank r. Then for every $x \in \mathbb{V}$, there exist a Jordan frame $\{c_1, c_2, \ldots, c_r\}$ and real numbers $\lambda_1(x), \ldots, \lambda_r(x)$, arranged in the decreasing order $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_r(x)$, such that*

$$x = \lambda_1(x)c_1 + \lambda_2(x)c_2 + \cdots + \lambda_r(x)c_r.$$

The numbers $\lambda_j(x)$ (counting multiplicities), which are uniquely determined by x, are called the eigenvalues *of x, and $\text{tr}(x) = \sum_{j=1}^{r} \lambda_j(x)$ is called the trace of x.*

Since, by [3, Prop. III.1.5], a Jordan algebra $(\mathbb{V}, \circ)$ with a unit element $e \in \mathbb{V}$ is Euclidean if and only if the symmetric bilinear form $\text{tr}(x \circ y)$ is positive definite, we may define another inner product on $\mathbb{V}$ by

$$\langle x, y\rangle := \text{tr}(x \circ y) \quad \forall x, y \in \mathbb{V}. \tag{7}$$

The inner product $\langle\cdot,\cdot\rangle$ is associative by [3, Prop. II.4.3], i.e., $\langle x, y \circ z\rangle = \langle y, x \circ z\rangle$ for any $x, y, z \in \mathbb{V}$. For any given $x \in \mathbb{V}$, let $\mathcal{L}(x)$ be the Lyapunov operator defined by

$$\mathcal{L}(x)y := x \circ y \quad \forall y \in \mathbb{V}.$$

Then, $\mathcal{L}(x)$ is symmetric with respect to the inner product $\langle\cdot,\cdot\rangle$ in the sense that

$$\langle \mathcal{L}(x)y, z\rangle = \langle y, \mathcal{L}(x)z\rangle \quad \forall y, z \in \mathbb{V}.$$

In what follows, we let $\|\cdot\|$ be the norm on $\mathbb{V}$ induced by this inner product, i.e.,

$$\|x\| := \sqrt{\langle x, x\rangle} = \sqrt{\text{tr}(x^2)} = \left(\sum_{j=1}^{r} \lambda_j^2(x)\right)^{1/2} \quad \forall x \in \mathbb{V}. \tag{8}$$

This definition implies that the unit element e in this paper has a length equal to $\sqrt{r}$.

Unless otherwise stated, in the rest of this paper, we assume that $\mathbb{A} = (\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$ is a simple Euclidean Jordan algebra of rank r and dimension n. By [3, Theorem V.3.7], $r \geq 2$.

Let $x \in \mathbb{V}$ have the spectral decomposition $x = \sum_{j=1}^{r} \lambda_j(x)c_j$, where $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_r(x)$ are the eigenvalues of x and $\{c_1, c_2, \ldots, c_r\}$ is the corresponding Jordan frame. By [3, Lemma IV. 1.3], the operators $\mathcal{L}(c_j), j = 1, 2, \ldots, r$ commute and admit a simultaneous diagonalization. For all $i, j \in \{1, 2, \ldots, r\}$, define the subspaces

$$\mathbb{V}_{ii} := \mathbb{R}c_i = \{x \in \mathbb{V} \mid x \circ c_i = x\},$$
$$\mathbb{V}_{ij} := \left\{x \in \mathbb{V} \mid x \circ c_i = \frac{1}{2}x = x \circ c_j\right\} \quad \text{when } i \neq j,$$

and let $\mathcal{C}_{ij}(x)$ be the orthogonal projection operator onto $\mathbb{V}_{ij}$. The following lemma gives the spectral decomposition of the operator $\mathcal{L}(x)$, whose proof can be found in [9].

Lemma 2.1. *Let $x \in \mathbb{V}$ have the spectral decomposition $x = \sum_{j=1}^{r} \lambda_j(x)c_j$. Then the linear symmetric operator $\mathcal{L}(x)$ has the spectral decomposition*

$$\mathcal{L}(x) = \sum_{j=1}^{r} \lambda_j(x)\mathcal{C}_{jj}(x) + \sum_{1\leq j<l\leq r} \frac{1}{2}\left(\lambda_j(x) + \lambda_l(x)\right)\mathcal{C}_{jl}(x) \tag{9}$$

with the spectrum $\sigma(\mathcal{L}(x))$ consisting of all distinct $\frac{1}{2}(\lambda_j(x) + \lambda_l(x))$ for $j, l = 1, \ldots, r$.

To close this section, we recall the smoothness of FB merit function ψ_{FB} defined by (4) and (5), whose proof can be found in [14, Lemma 12] and [15, Prop. 4.3].

Lemma 2.2. *Let ψ_{FB} be defined by (4) and (5). Then, ψ_{FB} is continuously differentiable everywhere. Furthermore, $\nabla_x\psi_{\text{FB}}(0, 0) = \nabla_y\psi_{\text{FB}}(0, 0) = 0$; and if $(x, y) \neq (0, 0)$,*

$$\nabla_x\psi_{\text{FB}}(x, y) = \left[\mathcal{L}(x)\mathcal{L}^{-1}(z) - \mathcal{I}\right]\phi_{\text{FB}}(x, y),$$
$$\nabla_y\psi_{\text{FB}}(x, y) = \left[\mathcal{L}(y)\mathcal{L}^{-1}(z) - \mathcal{I}\right]\phi_{\text{FB}}(x, y),$$

where $z = (x^2 + y^2)^{1/2}$, and $\mathcal{I}$ denotes the identity operator from $\mathbb{V}$ to $\mathbb{V}$.

3. Main result

First of all, we present a new representation for the elements in $\mathbb{V}$. Let $\mathbb{V}_e$ denote the subspace generated by the unit element e, and $\mathbb{V}_e^{\perp}$ the orthogonal complementarity of $\mathbb{V}_e$. Note that the unit element e of $\mathbb{A}$ is unique. Hence, any $x \in \mathbb{V}$ can be uniquely written as $\lambda_x e + x_e$ with $\lambda_x \in \mathbb{R}$ and $x_e \in \mathbb{V}_e^{\perp}$. Moreover, we have the following result.

Lemma 3.1. *For $z = \lambda_z e + z_e \in \mathbb{V}$ with $\lambda_z \in \mathbb{R}$ and $z_e \in \mathbb{V}_e^{\perp}$, the following results hold.*

(a) $\text{tr}(z) = r\lambda_z$ *and* $\|z\|^2 = r\lambda_z^2 + \|z_e\|^2$.

(b) *If* $z \in \mathcal{K}$, *then* $\sqrt{r^2 - r}\lambda_z \geq \|z_e\|$. *If in addition* $z \neq 0$, *then* $\lambda_z > 0$ *and* $\|z_e\| > 0$.
(c) *When* $r = 2$, $\mathrm{tr}(\mathcal{L}^2(z_e)) = \|z_e\|^2$.

Proof. (a) The result is direct by the definition of $\langle \cdot, \cdot \rangle$ and the fact that $\|e\|^2 = r$.
(b) Since $z \in \mathcal{K}$, we have $\mathrm{tr}(z) \geq \|z\|$. This by part (a) implies $(r^2-r)\lambda_z^2 \geq \|z_e\|^2$, and the first part then follows. Since $r \geq 2$, from the inequality $\sqrt{r^2 - r}\lambda_z \geq \|z_e\|$ we obtain $\lambda_z \geq 0$, and $\|z_e\| = 0$ whenever $\lambda_z = 0$. This shows $\lambda_z > 0$ and $\|z_e\| > 0$ if $0 \neq z \in \mathcal{K}$.
(c) Since $\mathcal{L}(e) = \mathcal{I}$, we have $\mathcal{L}(z) = \lambda_z \mathcal{I} + \mathcal{L}(z_e)$, which together with Lemma 2.1 implies

$$\mathcal{L}(z_e) = \sum_{j=1}^{r}(\lambda_j(z) - \lambda_z)\mathcal{C}_{jj}(z) + \sum_{1 \leq j < l \leq z} \frac{1}{2}\left(\lambda_j(z) + \lambda_l(z) - 2\lambda_z\right)\mathcal{C}_{jl}(z).$$

Since $\mathcal{C}_{jl}(z)$ for all $j, l = 1, 2, \ldots, r$ are orthogonal projection operators, we have

$$\mathcal{L}^2(z_e) = \sum_{j=1}^{r}(\lambda_j(z) - \lambda_z)^2\mathcal{C}_{jj}(z) + \sum_{1 \leq j < l \leq r} \frac{1}{4}\left(\lambda_j(z) + \lambda_l(z) - 2\lambda_z\right)^2\mathcal{C}_{jl}(z).$$

Note that when $r = 2$, part (a) implies $\lambda_1(z) + \lambda_2(z) - 2\lambda_z = 0$, and therefore we have

$$\begin{aligned}\mathrm{tr}(\mathcal{L}^2(z_e)) &= (\lambda_1(z) - \lambda_z)^2 + (\lambda_2(z) - \lambda_z)^2 \\ &= \|z\|^2 - 2\mathrm{tr}(z)\lambda_z + 2\lambda_z^2 \\ &= \|z\|^2 - 2\lambda_z^2 = \|z_e\|^2\end{aligned}$$

where the last equality is due to part (a). Thus, the proof is complete. □

To achieve the main result of this paper, the key is to establish the implication that

$$z^2 \succ_{\mathcal{K}} x^2 + y^2 \implies c\,[\mathcal{L}(z) - \mathcal{L}(x)]\,[\mathcal{L}(z) - \mathcal{L}(y)] \succ [\mathcal{L}(z) - \mathcal{L}(x) - \mathcal{L}(y)]^2 \tag{10}$$

for all $x, y \in \mathbb{V}$ and $z \succ_{\mathcal{K}} 0$, where $c > 0$ is a constant, and for the operators $\mathcal{G}, \mathcal{H} : \mathbb{V} \to \mathbb{V}$, $\mathcal{G} \succ \mathcal{H}$ means $\langle x, (\mathcal{G} - \mathcal{H})x\rangle_{\mathbb{V}} > 0$ for any $0 \neq x \in \mathbb{V}$ and $\mathcal{G} \succeq \mathcal{H}$ means $\langle x, (\mathcal{G}-\mathcal{H})x\rangle_{\mathbb{V}} \geq 0$ for any $x \in \mathbb{V}$. The following proposition tries to establish such an implication.

Proposition 3.1. *For any* $x = \lambda_x e + x_e, y = \lambda_y e + y_e \in \mathbb{V}$ *and* $z = \lambda_z e + z_e \in \mathrm{int}\mathcal{K}$, *if* $r\lambda_z^2 \geq \|z_e\|^2$ *and* $\mathrm{tr}[\mathcal{L}^2(z_e) - \mathcal{L}^2(x_e) - \mathcal{L}^2(y_e)] \geq r^{-1}(\|z_e\|^2 - \|x_e\|^2 - \|y_e\|^2)$, *then*

$$z^2 \succ_{\mathcal{K}} x^2 + y^2 \implies \mathcal{L}^2(z) - \mathcal{L}^2(x) - \mathcal{L}^2(y) \succ 0 \tag{11}$$

which is equivalent to saying that

$$z^2 \succ_{\mathcal{K}} x^2 + y^2 \implies 2\,[\mathcal{L}(z) - \mathcal{L}(x)]\,[\mathcal{L}(z) - \mathcal{L}(y)] \succ [\mathcal{L}(z) - \mathcal{L}(x) - \mathcal{L}(y)]^2\,. \tag{12}$$

Moreover, the two implications remain true when "$\succ$" is replaced by "$\succeq$".

Proof. We adopt the proof technique of [4, Prop. 3.4]. First, consider the case where $z = (x^2 + y^2 + \delta e)^{1/2}$ for some $\delta > 0$. Fix any $x, y \in \mathbb{V}$ with $x = \lambda_x e + x_e$ and $y = \lambda_y e + y_e$ where $\lambda_x, \lambda_y \in \mathbb{R}$ and $x_e, y_e \in \mathbb{V}_e^{\perp}$. From $z^2 = x^2 + y^2 + \delta e$ and $z = \lambda_z e + z_e$, we have

$$\lambda_z^2 e + 2\lambda_z z_e + z_e^2 = \lambda_x^2 e + 2\lambda_x x_e + x_e^2 + \lambda_y^2 e + 2\lambda_y y_e + y_e^2 + \delta e.$$

Noting that $z_e^2, x_e^2, y_e^2 \in \mathbb{V}_e$ and $x_e, y_e, z_e \in \mathbb{V}_e^{\perp}$, we obtain from the last equality that

$$\begin{aligned}&\lambda_z z_e = \lambda_x x_e + \lambda_y y_e \quad \text{and} \\ &\lambda_z^2 e + z_e^2 = \lambda_x^2 e + x_e^2 + \lambda_y^2 e + y_e^2 + \delta e.\end{aligned} \tag{13}$$

From the first equality of (13), $\lambda_z\mathcal{L}(z_e) - \lambda_x\mathcal{L}(x_e) - \lambda_y\mathcal{L}(y_e) = 0$, which implies that

$$\mathcal{L}^2(z) - \mathcal{L}^2(x) - \mathcal{L}^2(y) = \left(\lambda_z^2 - \lambda_x^2 - \lambda_y^2\right)\mathcal{L}(e) + \mathcal{L}^2(z_e) - \mathcal{L}^2(x_e) - \mathcal{L}^2(y_e).$$

Thus, to prove (11), it suffices to prove that for any $0 \neq h = \lambda_h e + h_e \in \mathbb{V}$,

$$\left(\lambda_z^2 - \lambda_x^2 - \lambda_y^2\right)\|h\|^2 + \|z_e \circ h\|^2 - \|x_e \circ h\|^2 - \|y_e \circ h\|^2 > 0,$$

which, by noting that $z_e^2, x_e^2, y_e^2 \in \mathbb{V}_e$, $h = \lambda_h e + h_e$ and $h_e \in \mathbb{V}_e^{\perp}$, is equivalent to

$$(\lambda_z^2 - \lambda_x^2 - \lambda_y^2)\|h\|^2 + \lambda_h^2(\|z_e\|^2 - \|x_e\|^2 - \|y_e\|^2) + (\|z_e \circ h_e\|^2 - \|x_e \circ h_e\|^2 - \|y_e \circ h_e\|^2) > 0. \tag{14}$$

Since $\lambda_z > 0$ by Lemma 3.1(b), from the two equalities in (13) we have

$$r\lambda_z^2 + \frac{\|\lambda_x x_e + \lambda_y y_e\|^2}{\lambda_z^2} = \|z\|^2 = r\lambda_x^2 + \|x_e\|^2 + r\lambda_y^2 + \|y_e\|^2 + r\delta.$$

Multiplying the two sides with λ_z^2 and adding $\lambda_y^2\|x_e\|^2 + \lambda_x^2\|y_e\|^2$ simultaneously yields

$$(\lambda_x^2 + \lambda_y^2)(\|x_e\|^2 + \|y_e\|^2) + r\lambda_z^4 - \lambda_z^2(r\lambda_x^2 + \|x_e\|^2 + r\lambda_y^2 + \|y_e\|^2) > \|\lambda_y x_e - \lambda_x y_e\|^2,$$

which is equivalent to $(\lambda_z^2-\lambda_x^2-\lambda_y^2)(r\lambda_z^2-\|x_e\|^2-\|y_e\|^2) > \|\lambda_y x_e-\lambda_x y_e\|^2$. This means that both $\lambda_z^2 - \lambda_x^2 - \lambda_y^2$ and $r\lambda_z^2 - \|x_e\|^2 - \|y_e\|^2$ are positive or both are negative. If both are negative, we must have $\|x\|^2 + \|y\|^2 > 2r\lambda_z^2$, which by the assumption $r\lambda_z^2 \geq \|z_e\|^2$ yields the contradiction $\|z\|^2 > \|x\|^2 + \|y\|^2 > 2r\lambda_z^2 \geq r\lambda_z^2 + \|z_e\|^2 = \|z\|^2$. Thus, we get

$$\lambda_z^2 > \lambda_x^2 + \lambda_y^2 \quad \text{and} \quad r\lambda_z^2 > \|x_e\|^2 + \|y_e\|^2. \tag{15}$$

Using the first equality of (13) and the second inequality of (15), for any $s_e \in \mathbb{V}_e^{\perp}$,

$$\begin{aligned}&\left\langle s_e, \left[\mathcal{L}^2(z_e) - \mathcal{L}^2(x_e) - \mathcal{L}^2(y_e)\right] s_e\right\rangle \\ &= \|z_e \circ s_e\|^2 - \|x_e \circ s_e\|^2 - \|y_e \circ s_e\|^2 \\ &= \frac{\|\lambda_x x_e \circ s_e + \lambda_y y_e \circ s_e\|^2}{\lambda_z^2} - \left(\|x_e \circ s_e\|^2 + \|y_e \circ s_e\|^2\right) \\ &= \frac{(\lambda_x^2 + \lambda_y^2 - \lambda_z^2)\left(\|x_e \circ s_e\|^2 + \|y_e \circ s_e\|^2\right)}{\lambda_z^2} - \frac{\|s_e \circ (\lambda_x y_e - \lambda_y x_e)\|^2}{\lambda_z^2} \leq 0.\end{aligned}$$

This shows that $\mathcal{L}^2(z_e) - \mathcal{L}^2(x_e) - \mathcal{L}^2(y_e)$ is negative semidefinite on $\mathbb{V}_e^{\perp}$. Therefore,

$$\begin{aligned}&\|z_e \circ h_e\|^2 - \|x_e \circ h_e\|^2 - \|y_e \circ h_e\|^2 \\ &= \langle h_e, [\mathcal{L}^2(z_e) - \mathcal{L}^2(x_e) - \mathcal{L}^2(y_e)]h_e\rangle \\ &\geq \langle h_e, \mathrm{tr}(\mathcal{L}^2(z_e) - \mathcal{L}^2(x_e) - \mathcal{L}^2(y_e))\mathcal{I}h_e\rangle \\ &\geq r^{-1}\|h_e\|^2(\|z_e\|^2 - \|x_e\|^2 - \|y_e\|^2),\end{aligned}$$

where the last inequality is due to the given assumption. Along with (15), we have that

$$\begin{aligned}(\lambda_z^2-\lambda_x^2-\lambda_y^2)\|h\|^2+\lambda_h^2(\|z_e\|^2-\|x_e\|^2-\|y_e\|^2)\\ \quad+(\|z_e\circ h_e\|^2-\|x_e\circ h_e\|^2-\|y_e\circ h_e\|^2)\\ \geq \lambda_h^2(\|z\|^2-\|x\|^2+\|y\|^2)+r^{-1}\|h_e\|^2(\|z\|^2-\|x\|^2-\|y\|^2)\\ = r^{-1}\|h\|^2(\|z\|^2-\|x\|^2-\|y\|^2)>0.\end{aligned}$$

This shows that (14) holds, and the implication in (11) is true for any $x, y\in\mathbb{V}$ and $z=(x^2+y^2+\delta e)^{1/2}$. Using the same arguments of [4, Prop. 3.4] yields that (11) holds.

We next prove that the implication in (11) is equivalent to that of (12). Suppose that the implication in (11) holds. Fix any $0\neq h\in\mathbb{V}$. By the symmetry of $\mathcal{L}(x)$, clearly,

$$\begin{aligned}&\langle h, [\mathcal{L}(z)-\mathcal{L}(x)]\,[\mathcal{L}(z)-\mathcal{L}(y)]\,h\rangle\\ &\quad=\langle h, [\mathcal{L}(z)-\mathcal{L}(y)]\,[\mathcal{L}(z)-\mathcal{L}(x)]\,h\rangle\,.\end{aligned}$$

Let $\mathcal{S}(x,y)$ denote the symmetric part of $[\mathcal{L}(z)-\mathcal{L}(x)][\mathcal{L}(z)-\mathcal{L}(y)]$. Then,

$$\langle h, [\mathcal{L}(z)-\mathcal{L}(x)]\,[\mathcal{L}(z)-\mathcal{L}(y)]\,h\rangle=\langle h, \mathcal{S}(x,y)h\rangle\,.$$

Using the definition of $\mathcal{S}(x,y)$, a simple computation yields that

$$\begin{aligned}\mathcal{S}(x,y)&=\frac{1}{2}\,[\mathcal{L}(y)-\mathcal{L}(z)]\,[\mathcal{L}(x)-\mathcal{L}(z)]\\ &\quad+\frac{1}{2}\,[\mathcal{L}(x)-\mathcal{L}(z)]\,[\mathcal{L}(y)-\mathcal{L}(z)]\\ &=\frac{1}{2}\,[\mathcal{L}(z)-\mathcal{L}(x)-\mathcal{L}(y)]^2\\ &\quad+\frac{1}{2}\left[\mathcal{L}^2(z)-\mathcal{L}^2(x)-\mathcal{L}^2(y)\right].\end{aligned}$$

The last two equations along with Eq. (11) imply the implication in (12). Conversely, if the implication in (12) holds, from the last two equations we obtain the implication in (11). The last part follows by the continuity of the operators. □

By Lemma 3.1(b)–(c), when $r=2$, the assumptions of Proposition 3.1 automatically hold, and we recover the first implication of [4, Prop. 3.4], or its equivalent result as below.

Corollary 3.1. *Suppose that $r=2$. Then, for any $x,y\in\mathbb{V}$ and $z\succ_{\mathcal{K}}0$, it holds that*

$$\begin{aligned}z^2\succ_{\mathcal{K}}x^2+y^2\Longrightarrow\ & 2\,[\mathcal{L}(z)-\mathcal{L}(x)]\,[\mathcal{L}(z)-\mathcal{L}(y)]\\ \succ\ & [\mathcal{L}(z)-\mathcal{L}(x)-\mathcal{L}(y)]^2\,.\end{aligned}\tag{16}$$

Moreover, the implication remains true when "$\succ$" is replaced by "$\succeq$".

When $r\geq 3$, the assumption $r\lambda_z^2\geq\|z_e\|^2$ in Proposition 3.1 may not hold. Also, it is hard to verify whether $\mathrm{tr}(\mathcal{L}^2(z_e)-\mathcal{L}^2(x_e)-\mathcal{L}^2(y_e))\geq r^{-1}(\|z_e\|^2-\|x_e\|^2-\|y_e\|^2)$ holds or not. In other words, by use of Proposition 3.1 it is difficult to achieve our goal for $r\geq 3$. However, as will be shown by Proposition 3.2, an implication as in (10) can be established for $r\geq 3$ by extending the proof of [18, Lemma 6.3(c)] to another three classes of matrix algebras.

Proposition 3.2. *Suppose that $r\geq 3$. Then, for any $x,y\in\mathbb{V}$ and $z\succ_{\mathcal{K}}0$,*

$$\begin{aligned}z^2\succ_{\mathcal{K}}x^2+y^2\Longrightarrow\ & 4\,[\mathcal{L}(z)-\mathcal{L}(x)]\,[\mathcal{L}(z)-\mathcal{L}(y)]\\ \succ\ & [\mathcal{L}(z)-\mathcal{L}(x)-\mathcal{L}(y)]^2\,.\end{aligned}\tag{17}$$

Moreover, the implication remains true when "$\succ$" is replaced by "$\succeq$".

Proof. By [3, Theorem V.3.7], it suffices to prove this result for the following algebras:

(i) The algebra $\mathcal{S}_n$ of $n\times n$ real symmetric matrices;
(ii) The algebra $\mathcal{H}_n$ of all $n\times n$ complex Hermitian matrices;
(iii) The algebra $\mathcal{Q}_n$ of all $n\times n$ quaternionic Hermitian matrices;
(iv) The algebra $\mathcal{O}_3$ of all 3×3 octonionic Hermitian matrices.

Among others, the four classes of matrix algebras are equipped with the Jordan product $x\circ y:=\frac{1}{2}(xy+yx)$ and the trace inner product $\langle x,y\rangle_{\mathrm{T}}:=\Re\mathrm{Tr}(xy^*)$, where the notation "$*$" means the conjugate transpose, $\mathrm{Tr}(xy)$ denotes the trace of xy which is the multiplication of matrices x and y, and $\Re a$ means the real part of a.

Let $\mathbb{C},\mathbb{Q}$ and $\mathbb{O}$ denote the complex number field, the quaternion field and the octonion field, respectively. Let $\mathbb{W}$ be the algebra of $n\times n$ matrices with entries in $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Q}$, or the algebra of 3×3 matrices with entries in $\mathbb{O}$, equipped with the inner product $\langle\cdot,\cdot\rangle_{\mathrm{T}}$ and the norm $\|\cdot\|_{\mathrm{T}}$ induced by $\langle\cdot,\cdot\rangle_{\mathrm{T}}$. By [3, Propositions V.1.2, V.1.5 and V.2.1], it is not difficult to verify that for any $u,v,w\in\mathbb{W}$,

$$\begin{aligned}\Re\mathrm{Tr}[(wu)(vw)]&=\Re\mathrm{Tr}[w(uvw)]=\Re\mathrm{Tr}[wuvw]\\ &=\Re\mathrm{Tr}[w(uv)w],\end{aligned}\tag{18}$$

$$\Re\mathrm{Tr}[w(uv)w]=\Re\mathrm{Tr}[w(vu)w]\quad\text{if } u,v,w \text{ are Hermitian,}\tag{19}$$

and

$$\Re\mathrm{Tr}(uv)=\Re\mathrm{Tr}(uv^*)\quad\text{if } u \text{ is Hermitian.}\tag{20}$$

Also, by [3, Prop. V.2.1] we may verify that $\langle\mathcal{L}(x)y,z\rangle_{\mathrm{T}}=\langle y,\mathcal{L}(x)z\rangle_{\mathrm{T}}$ for all x, y and z from the space $\mathcal{S}_n$, or $\mathcal{H}_n$, or $\mathcal{Q}_n$, or $\mathcal{O}_3$. Fix any x and y from $\mathcal{S}_n$, or $\mathcal{H}_n$, or $\mathcal{Q}_n$, or $\mathcal{O}_3$. Since $z^2\succ_{\mathcal{K}}x^2+y^2$, from the Löwner–Heinz inequality in [13] it follows that

$$z\succ_{\mathcal{K}}x\quad\text{and}\quad z\succ_{\mathcal{K}}y.\tag{21}$$

Fix any $0\neq a$ from the same space as x and y. From the above discussions, we have

$$\begin{aligned}&4\,\langle a,[\mathcal{L}(z)-\mathcal{L}(x)]\,[\mathcal{L}(z)-\mathcal{L}(y)]\,a\rangle_{\mathrm{T}}\\ &\quad=4\,\langle(z-x)\circ a,(z-y)\circ a\rangle_{\mathrm{T}}\\ &\quad=\langle a(z-x)+(z-x)a,a(z-y)+(z-y)a\rangle_{\mathrm{T}}\\ &\quad=2\Re\mathrm{Tr}\,[(a(z-x)+(z-x)a)((z-y)a)]\\ &\quad=2\Re\mathrm{Tr}\,[a(z-x)(z-y)a+(z-x)a(z-y)a]\\ &\quad=2\Re\mathrm{Tr}\left[a(z^2-zy-xz+xy)a\right]\\ &\qquad+2\Re\mathrm{Tr}\left[(z-x)^{1/2}(z-x)^{1/2}a(z-y)^{1/2}(z-y)^{1/2}a\right]\\ &\quad>\Re\mathrm{Tr}\left[a(2xy-2zx-2zy+z^2+x^2+y^2)a\right]\\ &\qquad+2\Re\mathrm{Tr}\left[(z-x)^{1/2}a(z-y)^{1/2}(z-y)^{1/2}a(z-x)^{1/2}\right]\\ &\quad=\Re\mathrm{Tr}[a(z-x-y)^2a]+2\left\|(z-x)^{1/2}a(z-y)^{1/2}\right\|_{\mathrm{T}}^2\\ &\quad\geq\Re\mathrm{Tr}[a(z-x-y)^2a]\\ &\quad=\Re\mathrm{Tr}[(a(z-x-y))((z-x-y)a)]\\ &\quad=\Re\mathrm{Tr}[a(z-x-y)(a(z-x-y))^*]\\ &\quad=\|(z-x-y)a\|_{\mathrm{T}}^2,\end{aligned}$$

where the first equality is by the symmetry of $\mathcal{L}(\cdot)$ with respect to $\langle\cdot,\cdot\rangle_{\mathrm{T}}$, the third is due to (20) and the fact that $a(z-x)+(z-x)a$ is Hermitian, the fourth is by (18), the fifth is by (18) and (21), and the first inequality is using $z^2\succ_{\mathcal{K}}x^2+y^2$. On the other hand,

$$\begin{aligned}&\left\langle a,[\mathcal{L}(z)-\mathcal{L}(x)-\mathcal{L}(y)]^2\,a\right\rangle_{\mathrm{T}}\\ &\quad=\langle(z-x-y)\circ a,(z-x-y)\circ a\rangle_{\mathrm{T}}\\ &\quad=\Re\mathrm{Tr}\,[((z-x-y)\circ a)((z-x-y)\circ a)]\\ &\quad\leq\Re\mathrm{Tr}\left[((z-x-y)\circ a)((z-x-y)\circ a)^*\right]\\ &\quad=\|(z-x-y)\circ a\|_{\mathrm{T}}^2\\ &\quad=\frac{\|(z-x-y)a+a(z-x-y)\|_{\mathrm{T}}^2}{4}\\ &\quad\leq\|(z-x-y)a\|_{\mathrm{T}}^2,\end{aligned}$$

where the first inequality is since $\Re\mathrm{Tr}(uu) \leq \Re\mathrm{Tr}(uu^*)$, and the last one is due to Cauchy–Schwarz inequality. From the last two equations, we get the desired result. □

Comparing with Corollary 3.1, we only obtain a weaker implication for the case $r \geq 3$ by using the condition $z^2 \succ_{\mathcal{K}} x^2 + y^2$. We are not clear whether the result of Corollary 3.1 holds for the case $r \geq 3$. Nonetheless, we would view it as a positive conjecture because numerical simulations support that it is true for the real symmetric matrix algebra $\mathcal{S}_n$.

Now, by Corollary 3.1 and Proposition 3.2, we can establish the following property of $\nabla\psi_{\rm FB}$, which is an extension of [2, Lemma 6] and [18, Lemma 6.3(c)].

Proposition 3.3. *Let $\psi_{\rm FB}$ be defined by (4). For any $x, y \in \mathbb{V}$, the following results hold.*

(a) $\langle \nabla_x\psi_{\rm FB}(x, y), \nabla_y\psi_{\rm FB}(x, y)\rangle \geq 0$, *with the equality holding if and only if* $\phi_{\rm FB}(x, y) = 0$,
(b) $\psi_{\rm FB}(x, y) = 0 \iff \nabla_x\psi_{\rm FB}(x, y) = 0 \iff \nabla_y\psi_{\rm FB}(x, y) = 0$.

Proof. (a) Fix any $x, y \in \mathbb{V}$. If $(x, y) = (0, 0)$, the result is immediate by Lemma 2.2. We next assume that $(x, y) \neq (0, 0)$ and let $z = (x^2 + y^2)^{1/2}$. For simplicity, we abbreviate $\phi_{\rm FB}(x, y)$ as $\phi_{\rm FB}$. Applying Lemma 2.2, Corollary 3.1 and Proposition 3.2, we have that

$$\begin{aligned}
&\langle \nabla_x\psi_{\rm FB}(x, y), \nabla_y\psi_{\rm FB}(x, y)\rangle \\
&= \left\langle \left[\mathcal{L}(x)\mathcal{L}^{-1}(z) - \mathcal{I}\right]\phi_{\rm FB},\ \left[\mathcal{L}(y)\mathcal{L}^{-1}(z) - \mathcal{I}\right]\phi_{\rm FB}\right\rangle \\
&= \left\langle [\mathcal{L}(y) - \mathcal{L}(z)]\,[\mathcal{L}(x) - \mathcal{L}(z)]\,\mathcal{L}^{-1}(z)\phi_{\rm FB},\ \mathcal{L}^{-1}(z)\phi_{\rm FB}\right\rangle \\
&\geq c^{-1}\left\langle [\mathcal{L}(z) - \mathcal{L}(x) - \mathcal{L}(y)]^2\,\mathcal{L}^{-1}(z)\phi_{\rm FB},\ \mathcal{L}^{-1}(z)\phi_{\rm FB}\right\rangle \\
&= c^{-1}\left\|[\mathcal{L}(z) - \mathcal{L}(x) - \mathcal{L}(y)]\,\mathcal{L}^{-1}(z)\phi_{\rm FB}\right\|^2 \\
&= c^{-1}\left\|\mathcal{L}(\phi_{\rm FB})\mathcal{L}^{-1}(z)\phi_{\rm FB}\right\|^2 \geq 0
\end{aligned} \tag{22}$$

where $c = 2$ if $r = 2$, and otherwise $c = 4$, and the last equality is due to

$$\mathcal{L}(z) - \mathcal{L}(x) - \mathcal{L}(y) = \mathcal{L}(z - x - y) = \mathcal{L}(\phi_{\rm FB}).$$

This proves the first part. If $\langle \nabla_x\psi_{\rm FB}(x, y), \nabla_y\psi_{\rm FB}(x, y)\rangle = 0$, then from (22) we obtain $\mathcal{L}(\phi_{\rm FB})\mathcal{L}^{-1}(z)\phi_{\rm FB} = 0$. From the definition of inner product in (7), we get

$$\begin{aligned}
0 &= \mathrm{tr}\left(\mathcal{L}(\phi_{\rm FB})\mathcal{L}^{-1}(z)\phi_{\rm FB}\right) = \langle e, \mathcal{L}(\phi_{\rm FB})\mathcal{L}^{-1}(z)\phi_{\rm FB}\rangle \\
&= \langle \phi_{\rm FB}, \mathcal{L}^{-1}(z)\phi_{\rm FB}\rangle,
\end{aligned}$$

which by [14] implies $\phi_{\rm FB} = 0$. Conversely, if $\phi_{\rm FB} = 0$, then from Lemma 2.2 we readily obtain $\langle \nabla_x\psi_{\rm FB}(x, y), \nabla_y\psi_{\rm FB}(x, y)\rangle = 0$. Thus, we complete the proof of part (a).
(b) The result is immediate by part (a) and the formulas of $\nabla_x\psi_{\rm FB}$ and $\nabla_y\psi_{\rm FB}$. □

In what follows, we assume $\mathbb{A}$ is a direct product of simple Euclidean Jordan algebras

$$\mathbb{A} = \mathbb{A}_1 \times \mathbb{A}_2 \times \cdots \times \mathbb{A}_m,$$

where each $\mathbb{A}_i = (\mathbb{V}_i, \circ, \langle\cdot,\cdot\rangle_{\mathbb{V}_i})$ is a simple Euclidean Jordan algebra with $\dim(\mathbb{V}_i) = n_i$ and $\sum_{i=1}^m n_i = n$. For any $x, y \in \mathbb{V}$, we write $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m)$ with $x_i, y_i \in \mathbb{V}_i$. Then, $x \circ y = (x_1 \circ y_1, \ldots, x_m \circ y_m)$ and $\langle x, y\rangle = \langle x_1, y_1\rangle + \cdots + \langle x_m, y_m\rangle$, and consequently, the SCCP (1) is equivalent to finding a vector $\zeta \in \mathbb{V}$ such that

$$F_i(\zeta) \in \mathcal{K}^i, \qquad G_i(\zeta) \in \mathcal{K}^i, \qquad \langle F_i(\zeta), G_i(\zeta)\rangle = 0, \quad i = 1, 2, \ldots, m \tag{23}$$

where $\mathcal{K}^i$ is a symmetric cone in $\mathbb{V}_i$, and $F = (F_1, \ldots, F_m)$ and $G = (G_1, \ldots, G_m)$ with $F_i, G_i : \mathbb{V} \to \mathbb{V}_i$. Note that $\mathcal{K} = \mathcal{K}^1 \times \cdots \times \mathcal{K}^m$, and $\psi_{\rm FB}(x, y)$ and $\phi_{\rm FB}(x, y)$ are written as

$$\psi_{\rm FB}(x, y) = \left(\psi_{\rm FB}(x_1, y_1), \ldots, \psi_{\rm FB}(x_m, y_m)\right) \quad \text{and}$$
$$\phi_{\rm FB}(x, y) = \left(\phi_{\rm FB}(x_1, y_1), \ldots, \phi_{\rm FB}(x_m, y_m)\right).$$

To establish the main result of this paper, we also need the definition of the Cartesian P_0-property for a linear transformation from $\mathbb{V}$ to $\mathbb{V}$. Specifically, a linear transformation $\mathcal{G} : \mathbb{V} \to \mathbb{V}$ is said to have *the Cartesian P_0-property* if for any $0 \neq \zeta = (\zeta_1, \ldots, \zeta_m) \in \mathbb{V}$, there exists an index $\nu \in \{1, 2, \ldots, m\}$ such that $\zeta_\nu \neq 0$ and $\langle \zeta_\nu, (\mathcal{G}\zeta)_\nu\rangle \geq 0$. This is a direct extension of the Cartesian P_0-property introduced by Chen and Qi [1] for SDCPs.

Theorem 3.1. *Let F and G be differentiable mappings from $\mathbb{V}$ to $\mathbb{V}$. Then, every stationary point of (6) is a solution of (1) under one of the following conditions:*

(a) *For every $\zeta \in \mathbb{V}$, $\nabla F(\zeta)$ and $-\nabla G(\zeta)$ are column monotone, i.e., for any $u, v \in \mathbb{V}$,*

$$\nabla F(\zeta)u - \nabla G(\zeta)v = 0 \Longrightarrow \langle u, v\rangle \geq 0.$$

(b) *For every $\zeta \in \mathbb{V}$, $\nabla G(\zeta)$ is invertible and $\nabla G(\zeta)^{-1}\nabla F(\zeta)$ has the Cartesian P_0-property.*

Proof. Let ζ be an arbitrary stationary point of the minimization problem (6). Then,

$$\begin{aligned}
0 = \nabla\Psi_{\rm FB}(\zeta) &= \nabla F(\zeta)\nabla_x\psi_{\rm FB}(F(\zeta), G(\zeta)) \\
&\quad + \nabla G(\zeta)\nabla_y\psi_{\rm FB}(F(\zeta), G(\zeta))
\end{aligned} \tag{24}$$

where for any $u = (u_1, \ldots, u_m)$, $v = (v_1, \ldots, v_m) \in \mathbb{V}$ with $u_i, v_i \in \mathbb{V}_i$, we write

$$\nabla_x\psi_{\rm FB}(u, v) = \left(\nabla_{x_1}\psi_{\rm FB}(u_1, v_1), \ldots, \nabla_{x_m}\psi_{\rm FB}(u_m, v_m)\right),$$
$$\nabla_y\psi_{\rm FB}(u, v) = \left(\nabla_{y_1}\psi_{\rm FB}(u_1, v_1), \ldots, \nabla_{y_m}\psi_{\rm FB}(u_m, v_m)\right).$$

If condition (a) is satisfied, then from the column monotonicity of ∇F and ∇G we have

$$\begin{aligned}
0 &\geq \left\langle \nabla_x\psi_{\rm FB}(F(\zeta), G(\zeta)), \nabla_y\psi_{\rm FB}(F(\zeta), G(\zeta))\right\rangle \\
&= \sum_{i=1}^m \left\langle \nabla_{x_i}\psi_{\rm FB}(F_i(\zeta), G_i(\zeta)), \nabla_{y_i}\psi_{\rm FB}(F_i(\zeta), G_i(\zeta))\right\rangle.
\end{aligned}$$

Together with Proposition 3.3(a), we get $\phi_{\rm FB}(F_i(\zeta), G_i(\zeta)) = 0$ for all i. By [5, Prop. 6] and Eq. (23), this shows that ζ is a solution of (1).

Suppose that condition (b) is satisfied. From Eq. (24), it follows that

$$0 = \nabla G(\zeta)^{-1}\nabla F(\zeta)\nabla_x\psi_{\rm FB}(F(\zeta), G(\zeta)) + \nabla_y\psi_{\rm FB}(F(\zeta), G(\zeta)). \tag{25}$$

Assume on the contrary that ζ is not a solution of (1). Then, by Proposition 3.3(b),

$$\nabla_x\psi_{\rm FB}(F(\zeta), G(\zeta)) \neq 0 \quad \text{and} \quad \nabla_y\psi_{\rm FB}(F(\zeta), G(\zeta)) \neq 0.$$

Using the Cartesian P_0-property of $\nabla G(\zeta)^{-1}\nabla F(\zeta)$, there exists an index $\nu \in \{1, 2, \ldots, m\}$ such that $\nabla_{x_\nu}\psi_{\rm FB}(F_\nu(\zeta), G_\nu(\zeta)) \neq 0$ and

$$\left\langle \nabla_{x_\nu}\psi_{\rm FB}(F_\nu(\zeta), G_\nu(\zeta)), \left[\nabla G(\zeta)^{-1}\nabla F(\zeta)\nabla_x\psi_{\rm FB}(F(\zeta), G(\zeta))\right]_\nu\right\rangle \geq 0.$$

On the other hand, from Eq. (25) it follows that

$$\begin{aligned}
&\left\langle \nabla_{x_\nu}\psi_{\rm FB}(F_\nu(\zeta), G_\nu(\zeta)), \left[\nabla G(\zeta)^{-1}\nabla F(\zeta)\nabla_x\psi_{\rm FB}(F(\zeta), G(\zeta))\right]_\nu\right\rangle \\
&\quad = -\langle \nabla_{x_\nu}\psi_{\rm FB}(F_\nu(\zeta), G_\nu(\zeta)), \nabla_{y_\nu}\psi_{\rm FB}(F_\nu(\zeta), G_\nu(\zeta))\rangle.
\end{aligned}$$

Combining the last two equations with the inequality in Proposition 3.3(a) yields that

$$\langle \nabla_{x_\nu}\psi_{\rm FB}(F_\nu(\zeta), G_\nu(\zeta)), \nabla_{y_\nu}\psi_{\rm FB}(F_\nu(\zeta), G_\nu(\zeta))\rangle = 0,$$

and consequently $\phi_{FB}(F_\nu(\zeta), G_\nu(\zeta)) = 0$. This contradicts $\nabla_{x_\nu}\psi_{FB}(F_\nu(\zeta), G_\nu(\zeta)) \neq 0$. Therefore, ζ must be a solution of (1). The proof is complete. □

Theorem 3.1 unifies and extends the results of [2, Prop. 3] and [18, Prop. 6.1]. When ∇G is invertible, the column monotonicity of $\nabla F(\zeta)$ and $-\nabla G(\zeta)$ is equivalent to the positive semidefiniteness of $\nabla G(\zeta)^{-1}\nabla F(\zeta)$, which implies the Cartesian P_0-property. This means that condition (b) is weaker than condition (a) for invertible ∇G.

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References

[1] X. Chen, H.D. Qi, Cartesian *P*-property and its applications to the semidefinite linear complementarity problem, Mathematical Programming 106 (2006) 177–201.
[2] J.-S. Chen, P. Tseng, An unconstrained smooth minimization reformulation of the second-order cone complementarity problem, Mathematical Programming 104 (2005) 293–327.
[3] J. Faraut, A. Korányi, Analysis on Symmetric Cones, in: Oxford Mathematical Monographs, Oxford University Press, New York, 1994.
[4] M. Fukushima, Z.-Q. Luo, P. Tseng, Smoothing functions for second-order cone complementarity problems, SIAM Journal on Optimization 12 (2002) 436–460.
[5] M.S. Gowda, R. Sznajder, J. Tao, Some *P*-properties for linear transformations on Euclidean Jordan algebras, Linear Algebra and its Applications 393 (2004) 203–232.
[6] Z.H. Huang, T. Ni, Smoothing algorithms for complementarity problems over symmetric cones, Computational Optimization and Applications 45 (2010) 557–579.
[7] C. Kanzow, H. Kleinmichel, A new class of semismooth Newton-type methods for nonlinear complementarity problems, Computational Optimization and Applications 11 (1998) 227–251.
[8] C. Kanzow, N. Yamashita, M. Fukushima, New NCP functions and their properties, Journal of Optimization Theory and Applications 97 (1997) 115–135.
[9] M. Koecher, in: A. Brieg, S. Walcher (Eds.), The Minnesota Notes on Jordan Algebras and their Applications, Springer, Berlin, 1999.
[10] L.C. Kong, J. Sun, N.H. Xiu, A regularized smoothing Newton method for symmetric cone complementarity problems, Pacific Journal of Optimization 19 (2008) 1028–1047.
[11] L.C. Kong, L. Tuncel, N.H. Xiu, Vector-valued implicit Lagrangian for symmetric cone complementarity problems, Asia-Pacific Journal of Operational Research 26 (2009) 199–233.
[12] L.C. Kong, N.H. Xiu, The Fischer–Burmeister complementarity function on Euclidean Jordan algebras, Pacific Journal of Optimization (2009) (published online).
[13] Y. Lim, Applications of geometric means on symmetric cones, Mathematische Annalen 319 (2001) 457–468.
[14] Y. Liu, L. Zhang, Y. Wang, Some propeties of a class of merit functions for symmetric cone complementarity problems, Asia-Pacific Journal of Operational Research 23 (2006) 473–496.
[15] S.-H. Pan, J.-S. Chen, A one-parametric class of merit functions for the symmetric cone complementarity problem, Journal of Mathematical Analysis and Applications 355 (2009) 195–215.
[16] S.H. Schmieta, F. Alizadeh, Extension of primal–dual interior point algorithms for symmetric cones, Mathematical Programming 96 (2003) 409–438.
[17] D. Sun, J. Sun, Löwner's operator and spectral functions on Euclidean Jordan algebras, Mathematics of Operations Research 33 (2008) 421–445.
[18] P. Tseng, Merit function for semidefinite complementarity problems, Mathematical Programming 83 (1998) 159–185.
[19] A. Yoshise, Interior point trajectories and a homogenous model for nonlinear complementarity problems over symmetric cones, SIAM Journal on Optimization 17 (2006) 1129–1153.