AN INVERSE PROBLEM FOR A CLASS OF QUASI-HEMIVARIATIONAL INEQUALITIES ON CONSTANT CURVATURE HADAMARD MANIFOLDS

VO MINH TAM AND JEIN-SHAN CHEN*

ABSTRACT. In this paper, we investigate a class of quasi-hemivariational inequalities involving the generalized subdifferentials in the sense of Clarke and the set-valued constraint in the setting of constant curvature Hadamard manifolds. Using the Kakutani-Fan-Glicksberg type fixed point theorem for multi-valued maps on Hadamard manifolds, we prove the nonemptiness and compactness of the solution set of such problems under suitable assumptions. The other goal of the paper is to consider a nonlinear inverse problem, which is described as a regularized optimal control problem for the quasi-hemivariational inequality on Hadamard manifolds and provide the existence result.

1. INTRODUCTION

Assume that $\mathfrak{M}$ is a constant curvature Hadamard manifold (see Section 2 for more details) and $\mathfrak{M}_1$, $\mathfrak{M}_2$ are submanifolds of $\mathfrak{M}$. Given three nonempty sets $K \subset \mathfrak{M}$, $\Xi \subset \mathfrak{M}_1$ and $\mathcal{E} \subset \mathfrak{M}_2$, in this paper, we aim to the study of the nonlinear inverse problem on Hadamard manifolds as follows:

Problem 1.1. Find $(\xi^*, \eta^*) \in \Xi \times \mathcal{E}$ such that

$$\inf_{(\xi,\eta)\in\Xi\times\mathcal{E}} \mho_\beta(\xi,\eta) = \mho_\beta(\xi^*,\eta^*), \tag{1.1}$$

where the cost functional $\mho\colon \Xi \times \mathcal{E} \to \mathbb{R}$ is given by

$$\mho_\beta(\xi,\eta) := \min_{u\in\mathbf{U}(\xi,\eta)} Q(u) + \beta W(\xi,\eta). \tag{1.2}$$

Here, $\mathbf{U}(\xi,\eta)$ stands for the solution set of the following quasi-hemivariational inequality with respect to $(\xi,\eta) \in \Xi \times \mathcal{E}$ (denoted as $(\text{QHVI})_{\xi,\eta}$ for short), which consists of finding $u = u(\xi,\eta) \in K$ with $u \in A(u)$ such that

$$\left\langle F(u,\xi), \exp_u^{-1} v\right\rangle_{\mathcal{R}} + \Psi^0\left(u; \exp_u^{-1} v\right) \geq \left\langle G(\eta), \exp_u^{-1} v\right\rangle_{\mathcal{R}}, \ \forall v \in A(u),$$

where $\Psi^0(u;z)$ denotes the Clarke's generalized directional derivative at the point $u \in K$ in the direction $z \in T_u\mathfrak{M}$, $\beta > 0$ is a given regularized parameter. Here, nonlinear functions $Q\colon \mathfrak{M} \to \mathbb{R}$, $W\colon \Xi\times\mathcal{E} \to \mathbb{R}$, $F\colon K \times \mathfrak{M}_1 \to T\mathfrak{M}$, $G\colon \mathfrak{M}_2 \to T\mathfrak{M}$ and a set-valued mapping $A\colon K \rightrightarrows K$ will be specialized in Section 3.

We list out some special cases of $(\text{QHVI})_{\xi,\eta}$ as below.

1991 *Mathematics Subject Classification.* 47J20, 49J27, 49J40, 49N45.

Key words and phrases. Quasi-hemivariational inequality, inverse problem, Hadamard manifold, constant curvature

* Corresponding author.

(a) If the function Ψ is constant, i.e., $\Psi^0(u;\cdot)=0$ for all $u\in\mathfrak{M}$, then the problem $(\text{QHVI})_{\xi,\eta}$ reduces to the quasi-variational inequality under perturbed parameters on Hadamard manifolds, denoted by $(\text{QVI})_{\xi,\eta}$, which consists of finding $u=u(\xi,\eta)\in K$ with $u\in A(u)$ such that

$$\left\langle F(u,\xi),\exp_u^{-1}v\right\rangle_{\mathcal{R}}\geq\left\langle G(\eta),\exp_u^{-1}v\right\rangle_{\mathcal{R}},\ \forall v\in A(u).$$

Note that this problem has not been considered in previous literature.

(b) In addition, if $A(u)\equiv K$, $F(u,\xi)=F(u)$ and $G\equiv 0$ for all $\xi\in\Xi$, $u\in\mathfrak{M}$, then the problem $(\text{QVI})_{\xi,\eta}$ reduces to the variational inequality problem in the setting of Hadamard manifolds in [16], which consists of finding $u\in K$ such that

$$\left\langle F(u),\exp_u^{-1}v\right\rangle_{\mathcal{R}}\geq 0,\quad\forall v\in K.$$

(c) If $A(u)\equiv K$, $F(u,\xi)=F(u)$ and $G\equiv 0$ for all $\xi\in\Xi$, $u\in\mathfrak{M}$, then the problem $(\text{QHVI})_{\xi,\eta}$ reduces to the hemivariational inequality problem in the setting of Hadamard manifolds in [23], denoted by (HVI), which consists of finding $u\in K$ such that

$$\left\langle F(u),\exp_u^{-1}v\right\rangle_{\mathcal{R}}+\Psi^0\left(u;\exp_u^{-1}v\right)\geq 0,\ \forall v\in K.$$

Over the past few years, many concepts and techniques of optimization and nonlinear analysis on linear spaces have been extended to the setting of Riemannian manifolds, particularly on Hadamard manifolds. This natural extension has important advantages, for instance, some nonsmooth and nonconvex problems can be transformed into smooth and convex ones on Riemannian manifolds or Hadamard manifolds by choosing an appropriate Riemannian metric, see [6, 13, 19, 24] and references therein.

On the other hand, in the early 1980s, Panagiotopoulos [17, 18] developed the idea of hemivariational inequalities to solve nonmontone and nonsmooth problems occurring in mechanics. The theory of hemivariational inequalities can be regarded as an extension of variational inequalities based on the generalized subdifferentials in the sense of Clarke of locally Lipschitz functions to the situation involving both the convex and nonconvex potentials. It has been applied in various fields such as economics, engineering sciences and mechanics. We refer the reader to the monographs [7, 15, 21] and the references therein. Recently, Tang et al. [23] introduced a class of hemivariational inequalities on Hadamard manifolds (HVI) and studied its existence of solutions using KKM-technique and coercivity conditions. Hung et al. [10] developed a class of quasi-hemivariational inequalities involving the set-valued constraint in the setting of Hadamard manifolds. Some results of gap functions and error bounds for this class of problems were established in [10]. Very recently, Tam et al. [22] investigated the Levitin-Polyak well-posedness for hemivariational inequalities of the split type on Hadamard manifolds. However, in [10, 22] the authors have not considered the existence of solutions to quasi-hemivariational inequalities and hemivariational inequalities of the split type.

Motivated by the aforementioned works, in this paper, we introduce the general class of quasi-hemivariational inequalities involving the generalized subdifferentials in the sense of Clarke and the set-valued constraint under perturbed parameters $(\text{QHVI})_{\xi,\eta}$ and develop a nonlinear inverse problem driven by $(\text{QHVI})_{\xi,\eta}$ in the setting of constant curvature Hadamard manifolds. The purposes of the work are twofold. The first is to show the nonemptiness and compactness of the solution set of the problem $(\mathrm{QHVI})_{\xi,\eta}$ by using the Kakutani-Fan-Glicksberg type fixed point theorem for multi-valued maps on Hadamard manifolds without KKM-technique and coercivity conditions considered in [23]. The second aim is to consider a nonlinear inverse problem (Problem 1.1) to identify parameters for $(\mathrm{QHVI})_{\xi,\eta}$ and establish the existence result for Problem 1.1

The rest of the paper is structured as follows. Section 2 gives some preliminary materials on Hadamard manifolds, which will be used in Section 3. Our main results in this paper are stated and proved in Section 3 including the nonemptiness and compactness for $(\mathrm{QHVI})_{\xi,\eta}$ and Problem 1.1.

2. Preliminaries

The definitions and results about Riemannian manifolds that will be used throughout this work are introduced in this section, most of them can be found in [3, 4, 12, 20, 25].

Given a p-dimensional differentiable manifold $\mathfrak{M}$, we will denote by $T_w\mathfrak{M}$ the tangent space of $\mathfrak{M}$ at w and $T\mathfrak{M} = \bigcup_{w\in\mathfrak{M}} T_w\mathfrak{M}$ the tangent bundle of $\mathfrak{M}$. A *Riemannian metric* on $T_w\mathfrak{M}$ is an inner product $\langle\cdot,\cdot\rangle_{\mathcal{R}_w}$ on $T_w\mathfrak{M}$. A tensor field $\langle\cdot,\cdot\rangle_{\mathcal{R}}$ is said to be a Riemannian metric on $\mathfrak{M}$ if for every $w \in \mathfrak{M}$, the tensor $\langle\cdot,\cdot\rangle_{\mathcal{R}_w}$ is a Riemannian metric on $T_w\mathfrak{M}$, where the subscript w can be omitted if no confusion occurs. A *Riemannian manifold*, denoted by $(\mathfrak{M}, \langle\cdot,\cdot\rangle_{\mathcal{R}})$, is a differentiable manifold $\mathfrak{M}$ endowed with a Riemannian metric $\langle\cdot,\cdot\rangle_{\mathcal{R}}$.

Let $\gamma : [a, b] \to \mathfrak{M}$ be a piecewise smooth curve joining w to z, that is, $\gamma(a) = w$ and $\gamma(b) = z$, $l_{\mathcal{R}}(\gamma) := \int_a^b \|\gamma'(t)\|dt$ defines the length of γ. For any $w, z \in \mathfrak{M}$, the Riemannian distance $d_{\mathcal{R}}(w, z)$, which induces the original topology on $\mathfrak{M}$, is defined by minimizing this length over the set of all such curves joining w to z.

A vector field Z is said to be parallel along γ if $\nabla_{\gamma'}Z = \mathbf{0}$, where γ is a smooth curve in $\mathfrak{M}$, $\mathbf{0}$ denotes the zero tangent vector and ∇ is the Levi-Civita connection associated with the Riemannian metric. We say that γ is a geodesic if γ' itself is parallel along γ. If the length of a geodesic joining w to z in $\mathfrak{M}$ equals $d_{\mathcal{R}}(w, z)$, then it is said to be *minimal*. A Riemannian manifold is said to be *complete* if for any $w \in \mathfrak{M}$, all geodesics emanating from w are defined for all $t \in \mathbb{R}$. If $\mathfrak{M}$ is complete then any point in $\mathfrak{M}$ can be joined by a minimal geodesic. Furthermore, $(\mathfrak{M}, d_{\mathcal{R}})$ is a complete metric space, and so bounded closed subsets of $\mathfrak{M}$ are compact (Hopf-Rinow Theorem). $\mathfrak{M}$ is said to be a *Hadamard manifold* if it is a complete simply connected Riemannian manifold of nonpositive sectional curvature.

Given a Hadamard manifold $\mathfrak{M}$, the *exponential map* $\exp_w : T_w\mathfrak{M} \to \mathfrak{M}$ at $w \in \mathfrak{M}$ is defined by $\exp_w(z) = \gamma_z(1, w)$ for each $z \in T_w\mathfrak{M}$, where $\gamma(\cdot) = \gamma_z(\cdot, w)$ is the geodesic starting at w with velocity z, that is, $\gamma(0) = w$ and $\gamma'(0) = z$. It is easy to see that $\exp_w(tz) = \gamma_z(w)$ for each real number t. Moreover, the exponential map $\exp_w : T_w\mathfrak{M} \to \mathfrak{M}$ is a diffeomorphism for all $w \in \mathfrak{M}$. For $w \in \mathfrak{M}$, $\exp_w^{-1} : \mathfrak{M} \to$

$T_w\mathfrak{M}$ is the inverse of the exponential map. For any $w, u \in \mathfrak{M}$, we have

$$\| \exp_w^{-1}(u)\|_{\mathcal{R}} = d_{\mathcal{R}}(w, u). \tag{2.1}$$

For any two distinct points $w, u \in \mathfrak{M}$, there exists a unique normalized geodesic γ joining w to u such that $\gamma(t) = \exp_w(t \exp_w^{-1} u)$ for all $t \in [0, 1]$. Note that the exponential map and its inverse are continuous in the setting of Hadamard manifolds.

In the rest of the section, unless otherwise specified, $\mathfrak{M}$ is a constant curvature Hadamard manifold.

Definition 2.1 (see [24])**.** A subset $D \subset \mathfrak{M}$ is said to be geodesic convex if, for any points w and z in D, the geodesic joining w to z is contained in D, that is, if $\gamma : [0, 1] \to \mathfrak{M}$ is a geodesic such that $w = \gamma(0)$ and $z = \gamma(1)$, then $\gamma(t) = \exp_w \left(t \exp_w^{-1} z\right) \in D$ for all $t \in [0, 1]$.

Lemma 2.2 (see [6], p. 136)**.** *Let $\mathfrak{M}$ be a constant curvature Hadamard manifold and $w(r)$ be the geodesic segment starting at w_1 and ending at w_2 in $\mathfrak{M}$ for all $r \in [0, 1]$. Then, there are some $\alpha, \beta \geq 0$ such that $\exp_u^{-1} w(r) = \alpha \exp_u^{-1} w_1 + \beta \exp_u^{-1} w_2$ for all $u \in \mathfrak{M}$, where $\alpha^2 + \beta^2 \neq 0$.*

Lemma 2.3 (see [13])**.** *Let $w_0 \in \mathfrak{M}$ and $\{w_n\}$ be a sequence in $\mathfrak{M}$ such that $w_n \to w_0$. Then, the following assertions hold:*

(i) *For any $u \in \mathfrak{M}$, $\exp_{w_n}^{-1} u \to \exp_{w_0}^{-1} u$ and $\exp_u^{-1} w_n \to \exp_u^{-1} w_0$;*
(ii) *If $\{z_n\}$ is a sequence such that $z_n \in T_{w_n}\mathfrak{M}$ and $z_n \to z_0$, then $z_0 \in T_{w_0}\mathfrak{M}$;*
(iii) *Given sequences $\{z_n\}$ and $\{v_n\}$ satisfying $z_n, v_n \in T_{w_n}\mathfrak{M}$, if $z_n \to z_0$ and $v_n \to v_0$, then $\langle z_n, v_n\rangle_{\mathcal{R}} \to \langle z_0, v_0\rangle_{\mathcal{R}}$.*

Definition 2.4 (see [8])**.** Let $\mathfrak{M}$ be a Riemannian manifold. A real-valued function $\theta : \mathfrak{M} \to \mathbb{R}$ is said to be

(a) Lipschitz of rank L on a given subset $\mathcal{D}$ of $\mathfrak{M}$ if
$$|\theta(u) - \theta(w)| \leq L d_{\mathcal{R}}(u, w), \quad \forall u, w \in \mathcal{D}.$$
(b) Lipschitz near $w \in \mathfrak{M}$ if it is Lipschitz of some rank on an open neighborhood of w.
(c) locally Lipschitz on $\mathfrak{M}$ if it is Lipschitz near w, for every $w \in \mathfrak{M}$.

Definition 2.5 (see [8])**.** Let $\theta : \mathfrak{M} \to \mathbb{R}$ be a locally Lipschitz function on a Riemannian manifold $\mathfrak{M}$. The Clarke's generalized directional derivative of θ at $w \in \mathfrak{M}$ in direction $z \in T_w\mathfrak{M}$, denoted by $\theta^0(w; z)$, is defined as

$$\theta^0(w; z) := \limsup_{u \to w, t\downarrow 0} \frac{\theta \circ \varphi^{-1} (\varphi(u) + t\mathrm{d}\varphi(w)(z)) - \theta \circ \varphi^{-1}(\varphi(u))}{t}, \tag{2.2}$$

where (φ, U) is a chart at w. Indeed, $\theta^0(w; z) = \left(\theta \circ \varphi^{-1}\right)^0 (\varphi(w); \mathrm{d}\varphi(w)(z))$. Note that this definition is not dependent on charts. Taking into account $0_w \in T_w\mathfrak{M}$, one has

$$\theta^0(w; z) = (\theta \circ \exp_w)^0 (0_w; z).$$

In next lemma, we recall some primary properties of the Clarke's generalized directional derivative on Riemannian manifolds.

Lemma 2.6 (see [8], Theorem 2.4)**.** *Let $\mathfrak{M}$ be a Riemannian manifold, $w \in \mathfrak{M}$ and $\theta : \mathfrak{M} \to \mathbb{R}$ be Lipschitz of rank L on an open neighbourhood $\mathcal{N}_w$ of w. Then, the following assertions hold.*

(i) *For each $u \in \mathcal{N}_w$, the function $T_u\mathfrak{M} \ni z \mapsto \theta^0(u; z)$ is finite, positively homogeneous and subadditive on $T_u\mathfrak{M}$, and satisfies*

$$\left|\theta^0(u; z)\right| \leq L\|z\|_{\mathcal{R}};$$

(ii) *$\theta^0(\cdot;\cdot)$ is upper semicontinuous on $\mathcal{N}_w \times T_u\mathfrak{M}$ as a function of (u, z) and, as a function of z alone, is Lipschitz of rank L on $T_u\mathfrak{M}$, for each $u \in \mathcal{N}_w$.*

Definition 2.7 (see [24])**.** Let $\mathfrak{M}$ be a Hadamard manifold. A real-valued function $h\colon \mathfrak{M} \to \mathbb{R}$ is said to be geodesic convex if, for any $w_1, w_2 \in \mathfrak{M}$ and $s \in [0, 1]$,

$$h\left(\exp_{w_1}\left(s \exp_{w_1}^{-1} w_2\right)\right) \leq (1 - s)h(w_1) + sh(w_2).$$

Note that $h\colon \mathfrak{M} \to \mathbb{R}$ is called geodesic concave if $-h$ is geodesic convex, i.e., for any $w_1, w_2 \in \mathfrak{M}$ and $s \in [0, 1]$,

$$h\left(\exp_{w_1}\left(s \exp_{w_1}^{-1} w_2\right)\right) \geq (1 - s)h(w_1) + sh(w_2).$$

Next, we recall some basic definitions of set-valued mappings and their properties on Hadamard manifolds.

Definition 2.8 (see [14])**.** Let $\mathfrak{M}$ be a Hadamard manifold, $\Gamma : \mathfrak{M} \rightrightarrows \mathfrak{M}$ be a set-valued mapping and $w_0 \in \mathfrak{M}$. Then, Γ is said to be

(a) lower semicontinuous at w_0 if, for any open set $O \subset \mathfrak{M}$ satisfying $\Gamma(w_0) \cap O \neq \emptyset$, there exists an open neighborhood $N(w_0)$ of w_0 such that $\Gamma(w) \cap O \neq \emptyset$ for all $w \in N(w_0)$;

(b) upper semicontinuous at w_0 if, for any open set $O \subset \mathfrak{M}$ satisfying $\Gamma(w_0) \subset O$, there exists an open neighborhood $N(w_0)$ of w_0 such that $\Gamma(w) \subset O$ for all $w \in N(w_0)$;

(c) upper Kuratowski semicontinuous at w_0 if, for any sequences $\{w_k\}, \{v_k\} \subset \mathfrak{M}$ with each $v_k \in \Gamma(w_k)$, the relations $\lim\limits_{k\to\infty} w_k = w_0$ and $\lim\limits_{k\to\infty} v_k = v_0$ imply $v_0 \in \Gamma(w_0)$;

(d) lower semicontinuous (resp. upper semicontinuous, upper Kuratowski semicontinuous) on $\mathfrak{M}$ if Γ is lower semicontinuous (resp. upper semicontinuous, upper Kuratowski semicontinuous) at every point $w \in \mathfrak{M}$;

(e) continuous on $\mathfrak{M}$ if Γ is lower semicontinuous and upper semicontinuous at every point $w \in \mathfrak{M}$.

It is known that a mapping satisfies a property on A if it holds true at each point of a set $A \subset \mathfrak{M}$. If $A = \mathfrak{M}$, we omit "on $\mathfrak{M}$" in the statement.

Lemma 2.9 (see [1], Corollary 2.12)**.** *Let $\mathfrak{M}$ be a Hadamard manifold, C be a geodesic convex and compact subset of $\mathfrak{M}$ and $\Phi : C \rightrightarrows C$ be a set-valued map such that*

(i) *for each $w \in C$, $\Phi(w)$ is nonempty and geodesic convex;*

(ii) *for each $w \in C$, $\Phi^{-1}(v) = \{w \in C : v \in \Phi(w)\}$ is open on C.*

Then, there exists $x^* \in K$ such that $x^* \in \Phi(x^*)$.

Using Lemma 2.9 and [1, Theorem 2.14] yields the following result.

Lemma 2.10. *Let $\mathfrak{M}$ be a Hadamard manifold, C be a geodesic convex and compact subset of $\mathfrak{M}$ and $\mathcal{A}$ be a subset of $C \times C$ such that*

(i) *for all $w \in C$, $(w, w) \in \mathcal{A}$;*
(ii) *for all $v \in C$, the set $\{w \in C : (w, v) \in \mathcal{A}\}$ is closed on C;*
(iii) *for all $w \in C$, the set $\{v \in C : (w, v) \notin \mathcal{A}\}$ is geodesic convex.*

Then, there exists $w^* \in C$ such that $(w^*, v) \in \mathcal{A}$ for all $v \in C$.

To end this section, we derive the following lemma which provides the Kakutani-Fan-Glicksberg type fixed point theorem on Hadamard manifolds.

Lemma 2.11 (see [5, 14])**.** *Let $\mathfrak{M}$ be a Hadamard manifold, C be a compact and geodesic convex subset of $\mathfrak{M}$ and $\Gamma : C \rightrightarrows C$ be a upper Kuratowski semicontinuous mapping. If $\Gamma(u)$ is closed and geodesic convex for any $w \in C$, then there exists $w^* \in C$ such that $w^* \in \Gamma(w^*)$.*

3. Main results

In this section, we first prove that the solution set of the problem $(\mathrm{QHVI})_{\xi,\eta}$ is nonempty and compact by using the Kakutani-Fan-Glicksberg type fixed point theorem on Hadamard manifolds under some suitable conditions. Then, Problem 1.1 in the form of a regularized optimal control problem for $(\mathrm{QHVI})_{\xi,\eta}$ is investigated to establish the existence result.

To proceed, we now present the geodesic quasiconcavity-like of real functions in the setting of Hadamard manifolds.

Definition 3.1. Let $K \subset \mathfrak{M}$ be a nonempty geodesic convex subset of a Hadamard manifold $\mathfrak{M}$. Then the function $h\colon K \to \mathbb{R}$ is said to be *geodesic quasiconcave-like* if, for any $w_1, w_2 \in K$ and any $s \in [0, 1]$ such that

$$[h(w_1) \geq 0 \text{ and } h(w_2) \geq 0] \Rightarrow h\left(\exp_{w_1}\left(s \exp_{w_1}^{-1} w_2\right)\right) \geq 0.$$

Remark 3.2. We point out a few facts.

(i) The geodesic quasiconcavity-like in Definition 3.1 is a special case of the quasiconcavities-like with respect to cones in [9, Definition 3.1] when set-valued mappings reduce to real functions and cones are equivalent to 0.
(ii) Note that, from Definition 2.7 and Definition 3.1, every geodesic quasiconcave-like function is geodesic concave.

Our first main result in this work is presented in the below theorem for the nonemptiness and compactness for $(\mathrm{QHVI})_{\xi,\eta}$.

Theorem 3.3. *Let $\mathfrak{M}$ be a Hadamard manifold with constant curvature and $K \subset \mathfrak{M}$ be a nonempty compact and geodesic convex set. Suppose that the following conditions hold:*

(i) *$A\colon K \rightrightarrows K$ is a continuous set-valued mapping such that $A(u)$ is nonempty closed and geodesic convex for all $u \in K$;*
(ii) *for all $\xi \in \Xi$, $F(\cdot, \xi)\colon K \to T\mathfrak{M}$ is a continuous vector field;*
(iii) *$\Psi\colon K \to \mathbb{R}$ is a locally Lipschitz function;*

(iv) *for all $v \in K$ and $(\xi,\eta) \in \Xi \times \mathcal{E}$, the function*

$$K \ni w \mapsto \left\langle F(w,\xi), \exp_w^{-1} v \right\rangle_{\mathcal{R}} + \Psi^0\left(w; \exp_w^{-1} v\right) - \left\langle G(\eta), \exp_w^{-1} v \right\rangle_{\mathcal{R}}$$

is geodesic quasiconcave-like, where $G\colon \mathcal{E} \to T\mathfrak{M}$ is a vector field.

Then, for each $(\xi,\eta) \in \Xi \times \mathcal{E}$ fixed, the solution set of (QHVI)$_{\xi,\eta}$, i.e., $\mathbf{U}(\xi,\eta)$, is nonempty and compact.

Proof. For each $(\xi,\eta) \in \Xi \times \mathcal{E}$ fixed, we consider the set-valued mapping $\mathcal{S}_{\xi,\eta} : K \rightrightarrows K$ given by

$$\begin{aligned}\mathcal{S}_{\xi,\eta}(u) = \big\{w \in A(u)\colon\ & \left\langle F(w,\xi), \exp_w^{-1} v \right\rangle_{\mathcal{R}} + \Psi^0\left(w; \exp_w^{-1} v\right) \\ & \geq \left\langle G(\eta), \exp_w^{-1} v \right\rangle_{\mathcal{R}},\ \forall v \in A(u)\big\}.\end{aligned}$$

Claim 1: We verify that $\mathcal{S}_{\xi,\eta}(u)$ is nonempty for any $u \in K$.

In fact, for each $u \in K$, consider the following set

$$\begin{aligned}\mathcal{P}_{\xi,\eta} = \big\{(w,v) \in A(u) \times A(u) : & \left\langle F(w,\xi), \exp_w^{-1} v \right\rangle_{\mathcal{R}} + \Psi^0\left(w; \exp_w^{-1} v\right) \\ & \geq \left\langle G(\eta), \exp_w^{-1} v \right\rangle_{\mathcal{R}}\big\}.\end{aligned}$$

It is easy to see that for any $w \in A(u)$, there holds $(w,w) \in \mathcal{P}_{\xi,\eta}$. We now prove that for any $w \in A(u)$, the set $V_u^w := \{v \in A(u) : (w,v) \notin \mathcal{P}_{\xi,\eta}\}$ is geodesic convex on K. Indeed, V_u^w can be rewritten as follows:

$$V_u^w = \left\{v \in A(u) : \left\langle F(w,\xi), \exp_w^{-1} v \right\rangle_{\mathcal{R}} + \Psi^0\left(w; \exp_w^{-1} v\right) < \left\langle G(\eta), \exp_w^{-1} v \right\rangle_{\mathcal{R}}\right\}.$$

For any $v_1, v_2 \in V_u^w$ and $r \in [0,1]$, we set $v(r) = \exp_{v_1}\left(r \exp_{v_1}^{-1} v_2\right)$. Then, we have $v_i \in A(u)$ and

$$\left\langle F(w,\xi), \exp_w^{-1} v_i \right\rangle_{\mathcal{R}} + \Psi^0\left(w; \exp_w^{-1} v_i\right) < \left\langle G(\eta), \exp_w^{-1} v_i \right\rangle_{\mathcal{R}} \tag{3.1}$$

for all $i = 1, 2$. It follows from the geodesic convexity of $A(u)$ that $v(r) \in A(u)$. Since $v(r)$ is the geodesic segment starting at v_1 and ending at v_2, by Lemma 2.2, there are some $\alpha, \beta \geq 0$ such that $\exp_w^{-1} v(r) = \alpha \exp_w^{-1} v_1 + \beta \exp_w^{-1} v_2$, where $\alpha^2 + \beta^2 \neq 0$. Using the linearity of the inner product $\langle \cdot, \cdot \rangle_{\mathcal{R}}$ defined on the tangent space $T_w\mathfrak{M}$, one has

$$\left\langle F(w,\xi), \exp_w^{-1} v(r) \right\rangle_{\mathcal{R}} = \alpha \left\langle F(w,\xi), \exp_w^{-1} v_1 \right\rangle_{\mathcal{R}} + \beta \left\langle F(w,\xi), \exp_w^{-1} v_2 \right\rangle_{\mathcal{R}}. \tag{3.2}$$

Note that the function $z \mapsto \Psi^0(w; z)$ is positively homogeneous and subadditive on $T_w\mathfrak{M}$. Consequently, we have

$$\Psi^0\left(w; \exp_w^{-1} v(r)\right) \leq \alpha \Psi^0\left(w; \exp_w^{-1} v_1\right) + \beta \Psi^0\left(w; \exp_w^{-1} v_2\right). \tag{3.3}$$

From (3.1)–(3.3), we see that

$$\begin{aligned} & \left\langle F(w,\xi), \exp_w^{-1} v(r) \right\rangle_{\mathcal{R}} + \Psi^0\left(w; \exp_w^{-1} v(r)\right) \\ \leq\ & \alpha \left(\left\langle F(w,\xi), \exp_w^{-1} v_1 \right\rangle_{\mathcal{R}} + \Psi^0\left(w; \exp_w^{-1} v_1\right)\right) \\ & +\beta \left(\left\langle F(w,\xi), \exp_w^{-1} v_2 \right\rangle_{\mathcal{R}} + \Psi^0\left(w; \exp_w^{-1} v_2\right)\right) \\ <\ & \alpha \left\langle G(\eta), \exp_w^{-1} v_1 \right\rangle_{\mathcal{R}} + \left\langle G(\eta), \beta \exp_w^{-1} v_2 \right\rangle_{\mathcal{R}} \\ =\ & \left\langle G(\eta), \alpha \exp_w^{-1} v_1 + \beta \exp_w^{-1} v_2 \right\rangle_{\mathcal{R}} \\ =\ & \left\langle G(\eta), \exp_w^{-1} v(r) \right\rangle_{\mathcal{R}}. \end{aligned}$$

Hence, $v(r) \in V_u^w$, i.e., V_u^w is geodesic convex on K. Moreover, for any $v \in A(u)$, the set $\widetilde{V}_u^v := \{w \in A(u) : (w, v) \in \mathcal{P}_{\xi,\eta}\}$ is closed on K. To see this, let $\{w_n\} \subset \widetilde{V}_u^v$ and $w_n \to w_0$. We shall argue that $w_0 \in \widetilde{V}_u^v$. For each $n \in \mathbb{N}$, since $w_n \in \widetilde{V}_u^v$, we have $w_n \in A(u)$ and $(w_n, v) \in \mathcal{P}_{\xi,\eta}$, i.e.,

$$\left\langle F(w_n, \xi), \exp_{w_n}^{-1} v \right\rangle_{\mathcal{R}} + \Psi^0\left(w_n; \exp_{w_n}^{-1} v\right) \geq \left\langle G(\eta), \exp_{w_n}^{-1} v \right\rangle_{\mathcal{R}}, \tag{3.4}$$

for all $v \in A(u)$. By the continuity of $F(\cdot, \xi)$, $\langle \cdot, \cdot \rangle_{\mathcal{R}}$ and the upper semicontinuity of $\Psi^0(\cdot; \cdot)$, passing to the upper limit in (3.4), as $n \to \infty$, we obtain

$$\begin{aligned} & \left\langle F(w_0, \xi), \exp_{w_0}^{-1} v \right\rangle_{\mathcal{R}} + \Psi^0\left(w_0; \exp_{w_0}^{-1} v\right) \\ \geq\ & \limsup_{n\to\infty} \left\langle F(w_n, \xi), \exp_{w_n}^{-1} v \right\rangle_{\mathcal{R}} + \limsup_{n\to\infty} \Psi^0\left(w_n; \exp_{w_n}^{-1} v\right) \\ \geq\ & \limsup_{n\to\infty} \left(\left\langle F(w_n, \xi), \exp_{w_n}^{-1} v \right\rangle_{\mathcal{R}} + \Psi^0\left(w_n; \exp_{w_n}^{-1} v\right)\right) \\ \geq\ & \limsup_{n\to\infty} \left\langle G(\eta), \exp_{w_n}^{-1} v \right\rangle_{\mathcal{R}} \\ =\ & \left\langle G(\eta), \exp_{w_0}^{-1} v \right\rangle_{\mathcal{R}}, \end{aligned}$$

for all $v \in A(u)$. This implies that $w_0 \in \widetilde{V}_u^v$, and hence $\widetilde{V}_u^v$ is closed on K.

Now, by applying Lemma 2.10, there exists $w^* \in A(u)$ such that $(w^*, v) \in \mathcal{P}_{\xi,\eta}$, for all $v \in A(u)$, i.e.,

$$\left\langle F(w^*, \xi), \exp_{w^*}^{-1} v \right\rangle_{\mathcal{R}} + \Psi^0\left(w^*; \exp_{w^*}^{-1} v\right) \geq \left\langle G(\eta), \exp_{w^*}^{-1} v \right\rangle_{\mathcal{R}}, \ \forall v \in A(u),$$

which says that $\mathcal{S}_{\xi,\eta}(u)$ is nonempty for any $u \in K$.

Claim 2: We prove that $\mathcal{S}_{\xi,\eta}(u)$ is geodesic convex for any $u \in K$.

Indeed, let $w_1, w_2 \in \mathcal{S}_{\xi,\eta}(u)$ and $s \in [0, 1]$. Since $w_1, w_2 \in A(u)$ and $A(u)$ is a geodesic convex set, we have $w(s) = \exp_{w_1}(s \exp_{w_1}^{-1} w_2) \in A(u)$. It follows from $w_1, w_2 \in \mathcal{S}_{\xi,\eta}(u)$ that

$$\left\langle F(w_i, \xi), \exp_{w_i}^{-1} v \right\rangle_{\mathcal{R}} + \Psi^0\left(w_i; \exp_{w_i}^{-1} v\right) - \left\langle G(\eta), \exp_{w_i}^{-1} v \right\rangle_{\mathcal{R}} \geq 0,$$

for all $v \in A(u)$. By condition (iv), we know

$$\left\langle F(w(s), \xi), \exp_{w(s)}^{-1} v \right\rangle_{\mathcal{R}} + \Psi^0\left(w(s); \exp_{w(s)}^{-1} v\right) - \left\langle G(\eta), \exp_{w(s)}^{-1} v \right\rangle_{\mathcal{R}} \geq 0,$$

for all $v \in A(u)$. This implies that

$$\left\langle F(w(s), \xi), \exp_{w(s)}^{-1} v \right\rangle_{\mathcal{R}} + \Psi^0\left(w(s); \exp_{w(s)}^{-1} v\right) \geq \left\langle G(\eta), \exp_{w(s)}^{-1} v \right\rangle_{\mathcal{R}},$$

for all $v \in A(u)$, i.e., $w(s) \in \mathcal{S}_{\xi,\eta}(u)$. Thus, $\mathcal{S}_{\xi,\eta}(u)$ is geodesic convex for any $u \in K$.

Claim 3: We show that $\mathcal{S}_{\xi,\eta}(u)$ is compact for any $u \in K$.

Since $\mathcal{S}_{\xi,\eta}(u) \subset K$ and K is compact, we only need to prove that $\mathcal{S}_{\xi,\eta}(u)$ is closed for any $u \in K$. Let $\{w_n\}_{n\in\mathbb{N}} \subset \mathcal{S}_{\xi,\eta}(u)$ and $w_n \to w_0$ as $n \to \infty$. We verify that $w_0 \in \mathcal{S}_{\xi,\eta}(u)$. In fact, due to $\{w_n\}_{n\in\mathbb{N}} \subset \mathcal{S}_{\xi,\eta}(u)$, one has $w_n \in A(u)$ and

$$\left\langle F(w_n, \xi), \exp_{w_n}^{-1} v \right\rangle_{\mathcal{R}} + \Psi^0\left(w_n; \exp_{w_n}^{-1} v\right) - \left\langle G(\eta), \exp_{w_n}^{-1} v \right\rangle_{\mathcal{R}} \geq 0,$$

for all $v \in A(u)$. Because $A(u)$ is closed and $w_n \to w_0$, there holds $w_0 \in A(u)$. Moreover, passing to the upper limit in the above inequality, as $n \to \infty$, it leads to

$$\left\langle F(w_0,\xi), \exp_{w_0}^{-1} v\right\rangle_{\mathcal{R}} + \Psi^0\left(w_0; \exp_{w_0}^{-1} v\right) - \left\langle G(\eta), \exp_{w_0}^{-1} v\right\rangle_{\mathcal{R}} \geq 0,$$

for all $v \in A(u)$. Thus, $w_0 \in \mathcal{S}_{\xi,\eta}(u)$, which says $\mathcal{S}_{\xi,\eta}(u)$ is compact for any $u \in K$.

Claim 4: We prove that $\mathcal{S}_{\xi,\eta}(\cdot)$ is upper Kuratowski semicontinuous on K.

In fact, let $\{u_n\}_{n\in\mathbb{N}} \subset K$ such that $u_n \to u_0$ with $w_n \in \mathcal{S}_{\xi,\eta}(u_n)$ such that $w_n \to w_0 \in K$. Since $w_n \in A(u_n)$ and A is continuous and closed valued on K, we know that $w_0 \in A(u_0)$ and for every $v_0 \in A(u_0)$, there exists $v_n \in A(u_n)$ such that $v_n \to v_0$. Thus, we have $w_0 \in \mathcal{S}_{\xi,\eta}(u_0)$. On the contrary, we suppose that $w_0 \notin \mathcal{S}_{\xi,\eta}(u_0)$. Then, there exists $v_0 \in A(u_0)$ such that

$$\left\langle F(w_0,\xi), \exp_{w_0}^{-1} v_0\right\rangle_{\mathcal{R}} + \Psi^0\left(w_0; \exp_{w_0}^{-1} v_0\right) - \left\langle G(\eta), \exp_{w_0}^{-1} v_0\right\rangle_{\mathcal{R}} < 0. \tag{3.5}$$

Consequently, it follows from $w_n \in \mathcal{S}_{\xi,\eta}(u_n)$ that

$$\left\langle F(w_n,\xi), \exp_{w_n}^{-1} v_n\right\rangle_{\mathcal{R}} + \Psi^0\left(w_n; \exp_{w_n}^{-1} v_n\right) - \left\langle G(\eta), \exp_{w_n}^{-1} v_n\right\rangle_{\mathcal{R}} \geq 0. \tag{3.6}$$

Passing to the upper limit in the inequality (3.6), as $n \to \infty$, one has

$$\left\langle F(w_0,\xi), \exp_{w_0}^{-1} v_0\right\rangle_{\mathcal{R}} + \Psi^0\left(w_0; \exp_{w_0}^{-1} v_0\right) - \left\langle G(\eta), \exp_{w_0}^{-1} v_0\right\rangle_{\mathcal{R}} \geq 0. \tag{3.7}$$

This is the contradiction between (3.5) and (3.7). Hence, $w_0 \in \mathcal{S}_{\xi,\eta}(u_0)$, which shows that $\mathcal{S}_{\xi,\eta}(\cdot)$ is upper Kuratowski semicontinuous on K.

Claim 5: We need to prove the solution set of $(\text{QHVI})_{\xi,\eta}$, i.e., $\mathbf{U}(\xi,\eta)$, is nonempty and compact.

Applying the aforementioned results of Claim 1 to Claim 4, it is clear to see that $\mathcal{S}_{\xi,\eta}(\cdot)$ is upper Kuratowski semicontinuous on K and $\mathcal{S}_{\xi,\eta}(u)$ is a nonempty compact and geodesic convex subset of K for all $u \in K$. By Lemma 2.11, there exists a point $u^* \in K$ such that $u^* \in \mathcal{S}_{\xi,\eta}(u^*)$. This indicates that $u^* \in A(u^*)$ satisfies

$$\left\langle F(u^*,\xi), \exp_{u^*}^{-1} v\right\rangle_{\mathcal{R}} + \Psi^0\left(u^*; \exp_{u^*}^{-1} v\right) \geq \left\langle G(\eta), \exp_{u^*}^{-1} v\right\rangle_{\mathcal{R}}, \ \forall v \in A(u^*),$$

which implies that $\mathbf{U}(\xi,\eta)$ is nonempty.

Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbf{U}(\xi,\eta)$ and $u_n \to u_0$, as $n \to \infty$. It can be verified that $u_0 \in \mathbf{U}(\xi,\eta)$. Indeed, since $\{u_n\}_{n\in\mathbb{N}} \subset \mathbf{U}(\xi,\eta)$, one has $u_n \in A(u_n)$ and

$$\left\langle F(u_n,\xi), \exp_{u_n}^{-1} v\right\rangle_{\mathcal{R}} + \Psi^0\left(u_n; \exp_{u_n}^{-1} v\right) - \left\langle G(\eta), \exp_{u_n}^{-1} v\right\rangle_{\mathcal{R}} \geq 0, \tag{3.8}$$

for all $v \in A(u_n)$. Because A is upper Kuratowski semicontinuous and closed-valued, we know that $u_0 \in A(u_0)$. For every $v_0 \in A(u_0)$, by the lower semicontinuity of A, there exist $v_n \in A(u_n)$ such that $v_n \to v_0$. Applying inequality (3.8) yields

$$\left\langle F(u_n,\xi), \exp_{u_n}^{-1} v_n\right\rangle_{\mathcal{R}} + \Psi^0\left(u_n; \exp_{u_n}^{-1} v_n\right) - \left\langle G(\eta), \exp_{u_n}^{-1} v_n\right\rangle_{\mathcal{R}} \geq 0.$$

Then, passing to the upper limit as $n \to \infty$ in the above inequality, we have

$$\left\langle F(u_0,\xi), \exp_{u_0}^{-1} v_0\right\rangle_{\mathcal{R}} + \Psi^0\left(u_0; \exp_{u_0}^{-1} v_0\right) - \left\langle G(\eta), \exp_{u_0}^{-1} v_0\right\rangle_{\mathcal{R}} \geq 0.$$

This implies $u_0 \in \mathbf{U}(\xi,\eta)$, i.e., $\mathbf{U}(\xi,\eta)$ is a closed set. Furthermore, since $\mathbf{U}(\xi,\eta) \subset K$ and K is compact, we conclude that $\mathbf{U}(\xi,\eta)$ is a compact set in K. □

Remark 3.4. The problem $(\mathrm{QHVI})_{\xi,\eta}$ is a generalization of hemivariational inequality problems on Hadamard manifolds presented in Tang et al. [23] to the quasi-hemivariational inequality involving the generalized subdifferentials in the sense of Clarke and the set-valued constraint under perturbed parameters. Besides, our proof techniques in Theorem 3.3 for the existence result of $(\mathrm{QHVI})_{\xi,\eta}$ are different from those for Theorem 3.1, Theorem 3.2 and Theorem 3.3 in [23]. In particular, we adopt the Kakutani-Fan-Glicksberg fixed point theorem (Lemma 2.11) as a main tool for proving the existence result of solution set in Theorem 3.3, while Tang et al. [23] studied existence conditions by using KKM technique and coercivity conditions. Furthermore, Theorem 3.3 additionally studies the compactness of the solution set of $(\mathrm{QHVI})_{\xi,\eta}$.

We now give an example to illustrate our first main result in Theorem 3.3.

Example 3.5. Let $\mathfrak{M} = \mathbb{R}_{++} := \{w \in \mathbb{R} : w > 0\}$ be a Riemannian manifold. Here $\mathfrak{M}$ is endowed with the Riemannian metric $\langle \cdot, \cdot \rangle_{\mathcal{R}}$ defined by

$$\langle u, z\rangle_{\mathcal{R}} = \frac{1}{w^2} uz, \quad \forall w \in \mathfrak{M}, \quad \forall (u, z) \in T_w\mathfrak{M} \times T_w\mathfrak{M},$$

where $T_w\mathfrak{M}$ denotes the tangent plane at $w \in \mathfrak{M}$. For all $w \in \mathfrak{M}$, the tangent plane $T_w\mathfrak{M}$ equals to $\mathbb{R}$. It is well known that $\mathfrak{M}$ is a Hadamard manifold with sectional curvature 0, see [2, 14].

The Riemannian distance between the points w and z in $\mathfrak{M}$ is described by

$$d_{\mathcal{R}}(w, z) = \left| \ln \left(\frac{w}{z} \right) \right|.$$

The exponential map $\exp_w$ and the inverse exponential map $\exp_w^{-1}$ are given by

$$\exp_w(tv) = we^{(\frac{v}{w})t} \quad \text{and} \quad \exp_w^{-1}(z) = w \ln \left(\frac{z}{w} \right)$$

for all $w, z \in \mathfrak{M}$ and $v \in T_w\mathfrak{M}$.

Now, consider $\mathfrak{M}_1 = \mathfrak{M}_2 = \mathfrak{M}$, $K = \Xi = \mathcal{E} = \{u \in \mathbb{R} : u = e^{\frac{5}{3}s}, s \in [0, 1]\}$. For any $u \in K, \xi \in \Xi$ and $\eta \in \mathcal{E}$, we define

$$A(u) = \left\{ w \in \mathbb{R} : w = e^{5m},\, m \in \left[\frac{1}{5} \ln(u), \frac{1}{3} \right] \right\},$$

$$F(u, \xi) = \xi u(1 + \ln u), \quad G(\eta) = -3 \ln^2 \eta \quad \text{and} \quad \Psi(u) = \frac{3}{2} \ln u + 2.$$

It is not difficult to see that K is a nonempty compact and geodesic convex set and the assumptions (i)–(iii) are satisfied.

Moreover, the Clarke's generalized directional derivative of Ψ at $u \in K$ in the direction $z \in T_u\mathfrak{M}$ is computed as $\Psi^0(u; z) = \dfrac{3z}{2u}$. Then, for all $v \in K$ and

$(\xi,\eta)\in\Xi\times\mathcal{E}$, we have

$$\begin{aligned} h(w) &:= \langle F(w,\xi),\exp_w^{-1}v\rangle_{\mathcal{R}}+\Psi^0\left(w;\exp_w^{-1}v\right)-\langle G(\eta),\exp_w^{-1}v\rangle_{\mathcal{R}}\\ &= \xi(1+\ln w)\ln\left(\frac{v}{w}\right)+\frac{3}{2}\ln\left(\frac{v}{w}\right)+3\frac{\ln^2\eta}{w}\ln\left(\frac{v}{w}\right)\\ &= \left[\xi(1+\ln w)+\frac{3}{2}+3\frac{\ln^2\eta}{w}\right](\ln v-\ln w),\ \text{for al } w\in K. \end{aligned}$$

For any $w_1,w_2\in K$ such that $h(w_1)\geq 0$ and $h(w_2)\geq 0$, using K being geodesic convex for all $s\in[0,1]$, it yields $w(s)=\exp_{w_1}\left(s\exp_{w_1}^{-1}w_2\right)\in K$. It follows from $h(w_1)\geq 0$ and $h(w_2)\geq 0$ that $\ln w_1\leq\ln v$ and $\ln w_2\leq\ln v$, respectively. Hence, $(1-s)\ln(w_1)+s\ln(w_2)\leq\ln v$ and so, $\ln v-\ln(w_1^{1-s}w_2^s)\geq 0$. Note that $w(s)=\exp_{w_1}\left(s\exp_{w_1}^{-1}w_2\right)=w_1^{1-s}w_2^s$. Therefore, $\ln v-\ln(w(s))\geq 0$, which implies that

$$0\leq\left[\xi(1+\ln w(s))+\frac{3}{2}+3\frac{\ln^2\eta}{w(s)}\right](\ln v-\ln w(s))=h(w(s))$$

for all $v\in K$ and $(\xi,\eta)\in\Xi\times\mathcal{E}$. To sum up, we show that the mapping $w\mapsto h(w)$ is geodesic quasiconcave-like.

Now, all conditions in Theorem 3.3 hold, hence we conclude that the solution set of $(\mathrm{QHVI})_{\xi,\eta}$ is nonempty and compact. In fact, for each $(\xi,\eta)\in\Xi\times\mathcal{E}$ fixed, we compute

$$\begin{aligned} &\mathbf{U}(\xi,\eta)\\ =\ &\big\{u\in K: u\in A(u),\ \langle F(u,\xi),\exp_u^{-1}v\rangle_{\mathcal{R}}+\Psi^0\left(u;\exp_u^{-1}v\right)\\ &\qquad\geq\langle G(\eta),\exp_u^{-1}v\rangle_{\mathcal{R}},\ \forall v\in A(u)\big\}\\ =\ &\left\{u\in K: u\in A(u),\ \left[\xi(1+\ln u)+\frac{3}{2}+3\frac{\ln^2\eta}{u}\right](\ln v-\ln u)\geq 0,\ \forall v\in A(u)\right\}\\ =\ &\{1\}. \end{aligned}$$

To establish the other main result, we need to employ some assumptions. Given two nonempty, closed and convex sets $\Xi\subset\mathfrak{M}_1$ and $\mathcal{E}\subset\mathfrak{M}_2$, we assume the following hypotheses to Problem 1.1.

- $\mathbf{A}(Q)$: $Q\colon\mathfrak{M}\to\mathbb{R}$ is a lower semicontinuous and bounded from below function, i.e.,

$$Q(u)\leq\liminf_{n\to\infty}Q(u_n)\ \text{and}\ m_Q\leq Q(w)\ \text{for all}\ w\in\mathfrak{M},$$

  whenever $\{u_n\}_{n\in\mathbb{N}}\subset\mathfrak{M}$ and $u\in\mathfrak{M}$ are such that $u_n\to u$ for some $m_Q\in\mathbb{R}$.
- $\mathbf{A}(W)$: $W\colon\Xi\times\mathcal{E}\to\mathbb{R}$ is lower semicontinuous and bounded from below.

We now provide the existence result for Problem 1.1 of the regularized optimization type.

Theorem 3.6. *Assume that all conditions shown as in Theorem 3.3 hold. If, in addition, $\mathbf{A}(Q)$ and $\mathbf{A}(W)$ are satisfied, then for each $\beta>0$ the solution set of Problem 1.1 is nonempty.*

Proof. We shall prove this theorem by five steps.

Step 1: The functional $\mho$ defined in equation (1.1) is well-defined.

We emphasize that it suffices to demonstrate that for any fixed $(\xi, \eta) \in \Xi \times \mathcal{E}$, the following optimization problem

$$\inf_{u \in \mathbf{U}(\xi,\eta)} Q(u) \tag{3.9}$$

is solvable, that is, there exists $u^* \in \mathbf{U}(\xi, \eta)$ such that the following equality

$$\inf_{u \in \mathbf{U}(\xi,\eta)} Q(u) = Q(u^*) \tag{3.10}$$

holds. Indeed, since the function Q is bounded from below, there exists $\{u_n\}_{n \in \mathbb{N}} \subset \mathbf{U}(\xi, \eta)$ being a minimizing sequence to problem (3.9), i.e.,

$$\inf_{u \in \mathbf{U}(\xi,\eta)} Q(u) = \lim_{n \to \infty} Q(u_n).$$

Thanks to Theorem 3.3, we obtain that $\mathbf{U}(\xi, \eta)$ is a compact set. Therefore, without any loss of generality, we may assume that $u_n \to u^*$, as $n \to \infty$ with $u^* \in \mathbf{U}(\xi, \eta)$. Then, it follows from the weakly lower semicontinuity of Q that

$$\inf_{u \in \mathbf{U}(\xi,\eta)} Q(u) = \lim_{n \to \infty} Q(u_n) = \liminf_{n \to \infty} Q(u_n) \geq Q(u^*) \geq \inf_{u \in \mathbf{U}(\xi,\eta)} Q(u).$$

This shows that for every $(\xi, \eta) \in \Xi \times \mathcal{E}$ there exists $u^* \in \mathbf{U}(\xi, \eta)$ such that equality equation (3.10) is valid. Thus, $\mho$ is well-defined.

Step 2: The function $\mathbf{U}$ maps bounded sets of $\Xi \times \mathcal{E}$ to bounded sets of K.

For any fixed $(\xi, \eta) \in \Xi \times \mathcal{E}$ and $u \in \mathbf{U}(\xi, \eta)$, it follows from Claim 5 in the proof of Theorem 3.3 that $\mathbf{U}(\xi, \eta)$ is a bounded set for all $(\xi, \eta) \in \Xi \times \mathcal{E}$. This gives that $\mathbf{U}$ maps bounded sets of $\Xi \times \mathcal{E}$ into bounded sets of K.

Step 3: If $\{(\xi_n, \eta_n)\}_{n \in \mathbb{N}} \subset \Xi \times \mathcal{E}$ is a sequence such that $\xi_n \to \xi$ in Ξ and $\eta_n \to \eta$ in $\mathcal{E}$ for some $(\xi, \eta) \in \Xi \times \mathcal{E}$, then there holds

$$\emptyset \neq \limsup_{n \to \infty} \mathbf{U}(\xi_n, \eta_n) \subset \mathbf{U}(\xi, \eta). \tag{3.11}$$

Using Step 2, we conclude that $\bigcup_{n \geq 1} \mathbf{U}(\xi_n, \eta_n)$ is bounded in $\mathfrak{M}$ and so the set $\limsup\limits_{n \to \infty} \mathbf{U}(\xi_n, \eta_n) \neq \emptyset$ is nonempty. Let $u \in \limsup\limits_{n \to \infty} \mathbf{U}(\xi_n, \eta_n)$ be arbitrary. Then, we can find a sequence $\{u_n\}_{n \in \mathbb{N}} \subset \mathfrak{M}$ (by taking a subsequence if necessary) such that

$$u_n \in \mathbf{U}(\xi_n, \eta_n) \text{ and } u_n \to u \text{ in } \mathfrak{M}.$$

Thus, for every $n \in \mathbb{N}$, we have $u_n \in A(u_n)$ and

$$\left\langle F(u_n, \xi_n), \exp_{u_n}^{-1} v \right\rangle_{\mathcal{R}} + \Psi^0 \left(u_n; \exp_{u_n}^{-1} v\right) - \left\langle G(\eta_u), \exp_{u_n}^{-1} v \right\rangle_{\mathcal{R}} \geq 0, \tag{3.12}$$

for all $v \in A(u_n)$. Since A is upper semicontinuous and closed-valued and $u_n \to u$, we achieve that $u \in A(u)$. Moreover, for every $v \in A(u)$, by the lower semicontinuity of A, there exist $v_n \in A(u_n)$ for all $n \in \mathbb{N}$ such that $v_n \to v$. Plugging $v = v_n$ into equation (3.12) yields

$$\left\langle F(u_n, \xi_n), \exp_{u_n}^{-1} v_n \right\rangle_{\mathcal{R}} + \Psi^0 \left(u_n; \exp_{u_n}^{-1} v_n\right) - \left\langle G(\eta_n), \exp_{u_n}^{-1} v_n \right\rangle_{\mathcal{R}} \geq 0.$$

Then, taking the upper limit as $n \to \infty$ in the above inequality leads to

$$\left\langle F(u,\xi), \exp_u^{-1} v \right\rangle_{\mathcal{R}} + \Psi^0\left(u; \exp_u^{-1} v\right) - \left\langle G(\eta), \exp_u^{-1} v \right\rangle_{\mathcal{R}} \geq 0.$$

This implies that $u \in \mathbf{U}(\xi,\eta)$ and so $\limsup_{n\to\infty} \mathbf{U}(\xi_n,\eta_n) \subset \mathbf{U}(\xi,\eta)$. Thus, equation (3.11) is proved.

Step 4: If $\{(\xi_n,\eta_n)\}_{n\in\mathbb{N}} \subset \Xi \times \mathcal{E}$ is such that $\xi_n \to \xi$ in Ξ and $\eta_n \to \eta$ in $\mathcal{E}$ for some $(\xi,\eta) \in \Xi \times \mathcal{E}$, then the inequality

$$\mho_\beta(\xi,\eta) \leq \liminf_{n\to\infty} \mho_\beta(\xi_n,\eta_n) \tag{3.13}$$

holds.

Let $\{(\xi_n,\eta_n)\}_{n\in\mathbb{N}} \subset \Xi \times \mathcal{E}$ be such that $\xi_n \to \xi$ in Ξ and $\eta_n \to \eta$ in $\mathcal{E}$ for some $(\xi,\eta) \in \Xi \times \mathcal{E}$. In addition, let $\{u_n\}_{n\in\mathbb{N}} \subset \mathfrak{M}$ be a sequence satisfying

$$u_n \in \mathbf{U}(\xi_n,\eta_n) \quad \text{and} \quad \inf_{u\in\mathbf{U}(\xi_n,\eta_n)} Q(u) = Q(u_n) \tag{3.14}$$

for each $n \in \mathbb{N}$. It follows from Step 2 that $\bigcup_{n\in\mathbb{N}} \mathbf{U}(\xi_n,\eta_n)$ is bounded. Hence, without loss of generality, we may assume that $u_n \to u^*$ in $\mathfrak{M}$ for some $u^* \in \mathfrak{M}$. Then, by Step 3, we obtain that $u^* \in \limsup_{n\to\infty} \mathbf{U}(\xi_n,\eta_n) \subset \mathbf{U}(\xi,\eta)$. Using the lower semicontinuity of Q and W, we have

$$\begin{aligned}
\liminf_{n\to\infty} \mho_\beta(\xi_n,\eta_n) &= \liminf_{n\to\infty} \left[Q(u_n) + \beta W(\xi_n,\eta_n)\right] \\
&\geq \liminf_{n\to\infty} Q(u_n) + \liminf_{n\to\infty} \beta W(\xi_n,\eta_n) \\
&\geq Q(u^*) + \beta W(\xi,\eta) \\
&\geq \min_{u\in\mathbf{U}(\xi,\eta)} Q(u) + \beta W(\xi,\eta) \\
&= \mho_\beta(\xi,\eta).
\end{aligned}$$

Thus, the inequality (3.13) holds.

Step 5: The solution set of Problem 1.1 is nonempty.

Using assumptions $\mathbf{A}(Q)$ and $\mathbf{A}(W)$ and from the definition of the function $\mho$, we can conclude that $\mho$ is bounded from below. Thus, there exists a minimizing sequence $\{(\xi_n,\eta_n)\}_{n\in\mathbb{N}} \subset \Xi \times \mathcal{E}$ of equation (1.1) satisfying

$$\inf_{(\xi,\eta)\in\Xi\times\mathcal{E}} \mho_\beta(\xi,\eta) = \lim_{n\to\infty} \mho_\beta(\xi_n,\eta_n). \tag{3.15}$$

By the definition of $\mho_\beta$, it is not difficult to see that the sequence $\{(\xi_n,\eta_n)\}_{n\in\mathbb{N}} \subset \Xi \times \mathcal{E}$ are bounded in $\mathfrak{M}_1 \times \mathfrak{M}_2$. Passing to a subsequence if necessary, we have

$$\xi_n \to \xi^* \text{ in } \mathfrak{M}_1 \text{ and } \eta_n \to \eta^* \text{ in } \mathfrak{M}_2 \tag{3.16}$$

for some $(\xi^*,\eta^*) \in \Xi \times \mathcal{E}$.

Given a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathfrak{M}$ satisfying equation (3.14). Using the convergence $\xi_n \to \xi$, $\eta_n \to \eta$ and the boundedness of $\mathbf{U}$ (see Step 2), we obtain the boundedness of the sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathfrak{M}$. Without loss of generality, we can assume that

$u_n \to u^*$ in $\mathfrak{M}$ for some $u^* \in \mathfrak{M}$. From Step 3, it is clear that $u^* \in \mathbf{U}(\xi^*, \eta^*)$. Thus, we achieve

$$\begin{aligned}
\liminf_{n\to\infty} \mho_\beta(\xi_n, \eta_n) &= \liminf_{n\to\infty} [Q(u_n) + \beta W(\xi_n, \eta_n)] \\
&\geq \liminf_{n\to\infty} Q(u_n) + \liminf_{n\to\infty} \beta W(\xi_n, \eta_n) \\
&\geq Q(u^*) + \beta W(\xi^*, \eta^*) \\
&\geq \min_{u\in \mathbf{U}(\xi^*,\eta^*)} Q(u) + \beta W(\xi^*, \eta^*) \\
&= \mho_\beta(\xi^*, \eta^*) \\
&\geq \inf_{(\xi,\eta)\in \Xi\times\mathcal{E}} \mho_\beta(\xi, \eta). \qquad (3.17)
\end{aligned}$$

Combining relations (3.15) and (3.17), it indicates that $(\xi^*, \eta^*) \in \Xi \times \mathcal{E}$ is a solution to Problem 1.1. □

4. Conclusions

In this paper, we studied a class of quasi-hemivariational inequalities $(\text{QHVI})_{\xi,\eta}$ on constant curvature Hadamard manifolds. Our main contribution lies on providing the properties of solution sets including the nonemptiness and compactness for $(\text{QHVI})_{\xi,\eta}$ and the existence of nonlinear inverse problem driven by a quasi-hemivariational inequality in the setting of Hadamard manifolds, see Theorem 3.3 and Theorem 3.6.

Very recently, Kumari and Ahmad [11] considered a penalty function method for a class of variational inequalitires on Hadamard manifolds. This approach induced an interesting result that the sequence of a solution to the penalized variational inequality has at least one limit point which belongs to the feasible region. In view of this, a possible future direction will be on developing the algorithm by applying penalty function methods for quasi-hemivariational inequalities on Hadamard manifolds.

Acknowledgements

It is the authors' great honor to contribute this article to the special issue in honor of the 70th birthday of Prof. Do Sang Kim. We value the friendship and hereby express our heartfelt wishes. The authors are also grateful to the anonymous referees for their valuable remarks which help improving the presentation of the paper.

References

[1] Q.H. Ansari, M. Islam and J.-C. Yao, *Nonsmooth variational inequalities on Hadamard manifolds*, Appl. Anal. **99** (2020), 340–358.
[2] G.C. Bento, O.P. Ferreira and P.R. Oliveira, *Proximal point method for a special class of nonconvex functions on Hadamard manifolds*, Optimization **64** (2015), 289–319.
[3] M.P. do Carmo, *Riemannian Geometry*, Birkhäuser, Boston, 1992.
[4] I. Chavel, *Riemannian Geometry – a Modern Introduction*, second edition, Cambridge University Press, New York, 2006.
[5] V. Colao, G. López, G. Marino and V. Martín-Márquez, *Equilibrium problems in Hadamard manifolds*, J. Math Anal Appl. **388** (2012), 61–77.

[6] O.P. Ferreira, L.R. Lucambio Pérez and S.Z. Németh, *Singularities of monotone vector fields and an extragradient-type algorithm*, J. Global Optim. **31** (2005), 133–151.

[7] W. Han, S. Migórski and M. Sofonea (eds), *Advances in Variational and Hemivariational Inequalities: Theory, Numerical Analysis, and Applications*, Springer, 2015.

[8] S. Hosseini and M.R. Pouryayevali, *Generalized gradients and characterization of epi-Lipschitz sets in Riemannian manifolds*, Nonlinear Anal. **74** (2011), 3884–3895.

[9] N.V. Hung, E. Köbis and V.M. Tam, *Existence conditions for set-valued vector quasi-equilibrium problems on Hadamard manifolds with variable domination structure and applications*, J. Nonlinear Convex Anal. **20** (2019), 2597–2612.

[10] N.V. Hung, V.M. Tam and A. Pitea, *Global error bounds for mixed Quasi-Hemivariational inequality problems on Hadamard manifolds*, Optimization **69** (2020), 2033–2052.

[11] B. Kumari and I. Ahmad, *Penalty function method for a variational inequality on Hadamard manifolds*, OPSEARCH **60** (2023), 527–538.

[12] J.M. Lee, *Riemannian Manifolds: An Introduction to Curvature*, Springer-Verlag, New York, 1997.

[13] C. Li, G. López and V. Martín-Márquez, *Monotone vector fields and the proximal point algorithm on Hadamard manifolds*, J. Lond. Math. Soc. **72** (2009), 663–683.

[14] X. Li and N.J. Huang, *Generalized vector quasi-equilibrium problems on Hadamard manifolds*, Optim. Lett. **9** (2015), 155–170.

[15] S. Migórski, A. Ochal and M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities, Models and Analysis of Contact Problems*, Springer, New York, 2013.

[16] S.Z. Németh, *Variational inequalities on Hadamard manifolds*, Nonlinear Anal. **52** (2003), 1491–1498.

[17] P.D. Panagiotopoulos, *Nonconvex energy functions, hemivariational inequalities and substationarity principles*, Acta Mech. **42** (1983), 160–183.

[18] P.D. Panagiotopoulos, *Hemivariational Inequalities, Applications to Mechanics and Engineering*, Springer, Berlin, 1993.

[19] T. Rapcsák, *Geodesic convexity in nonlinear optimization*, J. Optim. Theory Appl. **69** (1991), 169–183.

[20] T. Sakai, *Riemannian Geometry, in: Translations of Mathematical Monographs*, **149**, American Mathematical Society, Providence, RI, 1996.

[21] M. Sofonea and S. Migórski, *Variational–Hemivariational Inequalities with Applications*, Pure and Applied Mathematics, Chapman & Hall/CRC Press, Boca Raton, 2018.

[22] V.M. Tam, N.V. Hung, Z.H. Liu and J.C. Yao, *Levitin-Polyak well-posedness by perturbations for the split hemivariational inequality problem on Hadamard manifolds*, J. Optim. Theory Appl. **195** (2022), 684–706.

[23] G.J. Tang, L. Zhou and N.J. Huang, *Existence results for a class of hemivariational inequality problems on Hadamard manifolds*, Optimization **65** (2016), 1451–1461.

[24] C. Udriste, *Convex Functions and Optimization Methods on Riemannian Manifolds*, Kluwer Academic Publishers, Dordrecht, 1994.

[25] J.A. Wolf, *Spaces of Constant Curvature*, third ed., Publish or Perish, Boston, 1972.

(V. M. Tam) Department of Mathematics, Dong Thap University, Cao Lanh City 870000, Dong Thap Province, Vietnam; and Department of Mathematics, National Taiwan Normal University, Taipei 116059, Taiwan

Email address: `vmtam@dthu.edu.vn`

(J.-S. Chen) Department of Mathematics, National Taiwan Normal University, Taipei 116059, Taiwan

Email address: `jschen@math.ntnu.edu.tw`