OPTIMIZATION
https://doi.org/10.1080/02331934.2026.2628933

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Estimating error bounds for differential systems involving coupled variational-hemivariational inequalities with applications to quasistatic contact problems

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ABSTRACT
The main goal of this study is to explore new error bounds for a complex class of differential nonlinear systems involving coupled variational-hemivariational inequalities with the nesting structure. We first construct suitable regularized gap functions of the corresponding control system consisting of coupled variational-hemivariational inequalities, and provide their relevant properties. Then, new error bounds for this class of differential nonlinear system are established based on the computational technologies involving coupled gap functions. Consequently, a result of error bounds for the history-dependent problem derived by coupled variational-hemivariational inequalities is also derived. Lastly, we apply the obtained error bound results to the quasistatic contact problems of two elastic bodies.

ARTICLE HISTORY
Received 30 March 2025
Accepted 31 January 2026

KEYWORDS
Differential variational-hemivariational inequality; history-dependent operator; gap function; error bound; quasistatic contact problems

2020 MATHEMATICS SUBJECT CLASSIFICATIONS
35M86; 35R45; 47J20; 65M15

1. Introduction

In order to describe the target problem, we first introduce the below mappings:

$$\begin{array}{ll}
A:\mathcal{D}(A)\subset\mathbf{U}\to\mathbf{U}, & H:[0,T]\times\mathbf{X}_1\times\mathbf{U}\to\mathbf{U},\\
F:[0,T]\times\mathbf{X}_2\times\mathbf{X}_1\to\mathbf{X}_1^*, & Q:[0,T]\times\mathbf{V}_1\times\mathbf{U}\to\mathbf{X}_1^*,\\
\Upsilon:\mathbf{V}_2\times\mathbf{X}_1\times\mathbf{X}_1\to\mathbb{R}, & J_1:\mathbf{W}_2\times\mathbf{W}_1\to\mathbb{R},\\
J_2:\mathbf{W}_1\times\mathbf{W}_2\to\mathbb{R}, & \mathcal{S}_1:C([0,T];\mathbf{X}_1)\to C([0,T];\mathbf{V}_1),\\
\mathcal{S}_2:C([0,T];\mathbf{X}_1)\to C([0,T];\mathbf{V}_2), & G:[0,T]\times\mathbf{X}_1\times\mathbf{X}_2\to\mathbf{X}_2^*,\\
\varphi:\mathbf{X}_2\to\mathbb{R}, & \mathbf{M}:\mathbf{X}_1\to\mathbf{W}_1,\\
\mathbf{N}:\mathbf{X}_2\to\mathbf{W}_2, & f\in C\left([0,T];\mathbf{X}_2^*\right),
\end{array}$$

where $\mathcal{D}(A)$ is the domain of A, $0<T<\infty$, $(\mathbf{U},\|\cdot\|_{\mathbf{U}})$, $(\mathbf{X}_i,\|\cdot\|_{\mathbf{X}_i})$ and $(\mathbf{W}_i,\|\cdot\|_{\mathbf{W}_i})$ are Banach spaces. In addition, each $(\mathbf{V}_i,\|\cdot\|_{\mathbf{V}_i})$ is a normed space and P_i is nonempty subset $\mathbf{X}_i$. As traditional notations, $\langle\cdot,\cdot\rangle_{\mathbf{X}_i^*\times\mathbf{X}_i}$ represents the duality pairing between $\mathbf{X}_i$ and $\mathbf{X}_i^*$, for $i=1,2$.

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Very recently, an application to quasistatic contact models with body pressure contact made between the two elastic bodies is investigated in [1], which considered a class of nonlinear differential variational-hemivariational inequalities employing history-dependent operators, stated as below.

Problem 1.1: Find $u \in C([0,T];\mathbf{U})$, $x \in C([0,T];P_1)$ and $z \in C([0,T];P_2)$ such that for all $t \in [0,T]$,

$$\begin{cases} u'(t) = Au(t) + H(t,u(t),x(t)) \\ u(0) = u_0, \end{cases} \tag{1}$$

$$\begin{aligned} &\big\langle F(t,z(t),x(t)) + Q\left(t,\left(\mathcal{S}_1 x\right)(t),u(t)\right), y - x(t)\big\rangle_{\mathbf{X}_1^* \times \mathbf{X}_1} + \Upsilon\left(\left(\mathcal{S}_2 x\right)(t), x(t), y\right) \\ &\quad - \Upsilon\left(\left(\mathcal{S}_2 x\right)(t), x(t), x(t)\right) + J_1^\circ\left(\mathbf{N}z(t), \mathbf{M}x(t); \mathbf{M}y - \mathbf{M}x(t)\right) \geqslant 0, \quad \forall y \in P_1, \end{aligned} \tag{2}$$

$$\begin{aligned} &\langle G(t,x(t),z(t)), v - z(t)\rangle_{\mathbf{X}_2^* \times \mathbf{X}_2} + J_2^\circ\left(\mathbf{M}x(t), \mathbf{N}z(t); \mathbf{N}v - \mathbf{N}z(t)\right) \\ &\quad + \varphi(v) - \varphi(z(t)) \geqslant \big\langle f(t), v - z(t)\big\rangle_{\mathbf{X}_2^* \times \mathbf{X}_2}, \quad \forall v \in P_2, \end{aligned} \tag{3}$$

where for each $i = 1, 2$,

$$C([0,T];P_i) := \{x \in C([0,T];\mathbf{X}_i) : x(t) \in P_i \quad \text{for a.e. } t \in [0,T]\}$$

and $J_i^\circ(e_i, w_i; v_i)$ stands for Clarke's generalized directional derivative of locally Lipschitz function J_i at $w_i \in \mathbf{W}_i$ in the direction $v_i \in \mathbf{W}_i$, where $e_1 \in \mathbf{W}_2$ and $e_2 \in \mathbf{W}_1$ (see Definition 2.4).

Hao-Wang-Han [1] examined both the unique solvability and convergence of Problem 1.1 using a penalty method. Indeed, Problem 1.1 represents a generalization of differential variational inequalities (DVIs), which are nonlinear dynamic systems that combine ordinary differential equations (DEs) with time-dependent variational inequalities. DVIs are a valuable tool for studying various areas of applications, including dynamic traffic networks, spatial price equilibrium control models, dynamic Nash equilibrium problems and frictional contact problems, see e.g. [2–4]. The DVIs were initially introduced by Liu-Migórski-Zeng [5, 6] and Liu-Zeng [7] in infinite-dimensional Banach spaces. Migórski-Liu-Zeng [8] developed and analysed DVIs with history-dependent operators, a notable mathematical model.

Additionally, a class of differential hemivariational inequalities (DHVIs), which combines a differential equation and a time-dependent hemivariational inequality, was also explored by Liu-Zeng-Migórski in [9]. The theory of hemivariational inequalities extends the concept of variational inequalities, which originally introduced by Panagiotopoulos [10, 11]. This theory addresses various mechanical problems involving nonconvex and nonsmooth energy potentials and bases on the Clarke generalized gradient for locally Lipschitz functions. Furthermore, the variational-hemivariational inequality generalizes the hemivariational inequality to include both convex and nonconvex potentials, see e.g. [12, 13].

Recently, Migórski-Cai [14] introduced a new class of differential variational-hemivariational inequalities (DVHVIs) that combine ordinary differential equations with variational-hemivariational inequalities. In [14], Migórski-Cai investigated the well-posedness and regularity of solutions for (DVHVIs) under suitable conditions. Furthermore, the DVHVIs have been extended to novel differential systems by Migórski-Cai [15] and Hao-Wang-Han [1]. These systems involve a differential equation combined with coupled variational-hemivariational inequalities, incorporating history-dependent operators within a nested structure. Especially, Hao-Wang-Han [1] gave applications to a novel category of quasistatic contact problems involving two elastic bodies in mechanics by using the form of Problem 1.1.

On the other hand, the concept of a gap function for a class of variational inequality problems was initially proposed by Auslender [16]. Using appropriate gap functions, variational inequalities can be transformed into equivalent minimization problems, which allows for the application of descent algorithms for solving them. To address the issue that the Auslender gap function is nondifferentiable in general, Fukushima [17] introduced the regularized gap function, which smooths the gap function for better handling to solve variational inequalities. After that, Yamashita-Fukushima [18] derived global error bounds for a class of variational inequalities via regularized gap functions. These error bounds are crucial for assessing the convergence rates of iterative algorithms and for determining the difference between approximate and exact solutions a variety of challenging problems. Consequently, many researchers have expanded upon the interesting topic on regularized gap functions and error bounds for a range of problems such as variational inequalities, equilibrium problems, hemivariational inequalities and variational-hemivariational inequalities, see e.g. [19–27] and therein references.

However, up to now, there are only a few papers devoted to regularized gap functions and error bounds for classes of DVIs and DVHVIs. In [28], Cen-Min-Nguyen-Yao initially developed and examined this topic for a specific class of DVHVIs. Their work significantly advanced the study of error bounds for systems involving nonlinear differential equations combined with time-dependent variational-hemivariational inequalities. Very recently, Tam-Wu [29] developed the error bound results for DVHVIs involving fractional order derivative operators and history-dependent operators using the estimate technologies via regularized gap functions.

The present work is a further development and extension of [28, 29]. The novelty features of this paper are described in twofold.

- First, we derive new results on error bounds under suitable hypotheses for Problem 1.1. This novelty approach relies on advanced computational technologies involving regularized gap functions for a general class of differential systems described by the differential Equation (1) combined with ‘coupled variational-hemivariational inequalities’ with the nesting structure (2)–(3). Meanwhile, the authors in [28, 29] focussed on error bounds for classes of differential variational-hemivariational inequalities involving only ‘a single variational-hemivariational inequality’. Besides, the main results suggest the new research direction of examining error bounds for the general classes of systems constructed by coupled variational-hemivariational inequalities with the nested structure introduced in Bai-Costea-Zeng [30], Costea [31], and Migórski-Ogorzały-Dudek [32], which are based on regularized gap functions.
- Secondly, we apply the obtained error bound results to the quasistatic contact problems involving two elastic bodies in mechanics. For readers’ convenience, we summarize

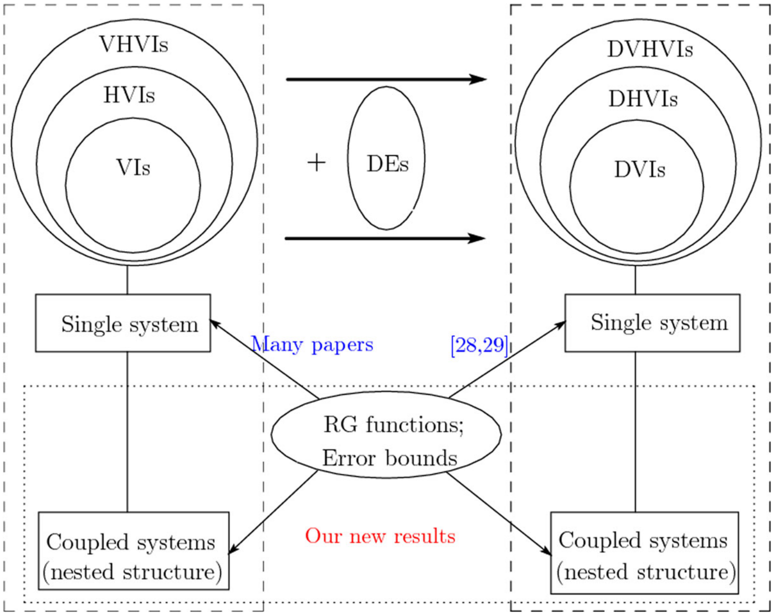

Figure 1. Illustration of the developments regarding regularized gap (RG) functions and error bounds to different kinds of problems.

and illustrate the contribution of this work and its connection to previous research in Figure 1.

The remainder of the paper is structured as follows. Section 2 presents some essential concepts and results necessary for the subsequent discussion, along with the assumptions required for Problem 1.1 and the results concerning uniqueness and existence. Section 3 focuses on establishing new error bounds to Problem 1.1, based on computational techniques involving coupled gap functions. The established error bounds constitutes the main result of the paper. Finally, Section 4 provides an application of the obtained error bound results to the quasistatic contact problems of two elastic bodies.

2. Preliminaries and hypotheses

In this section, we present essential concepts and build up background materials for subsequent discussion. Additionally, the assumptions required for Problem 1.1 and the existence of solutions to Problem 1.1 are derived and explained.

Let $\mathbf{X}$ be a Banach space and denote by $\mathbf{X}^*$ the topological dual of $\mathbf{X}$. The norm on $\mathbf{X}$ and duality pairing of $\mathbf{X}$ and $\mathbf{X}^*$ are denoted by $\|\cdot\|_{\mathbf{X}}$ and $\langle\cdot,\cdot\rangle_{\mathbf{X}^*\times\mathbf{X}}$, respectively.

Definition 2.1 (see [33, 34]): Let $\mathcal{F}:\mathbf{X}\to\overline{\mathbb{R}}:=(-\infty,+\infty]$ be a function. Then, $\mathcal{F}$ is said to be

(a) proper, if $\mathcal{F}\not\equiv+\infty$;

(b) lower semicontinuous (resp., upper semicontinuous) at $\bar{w} \in \mathbf{X}$, if for any $\{w_n\} \subset \mathbf{X}$ such that $w_n \to \bar{w}$, it holds $\mathcal{F}(\bar{w}) \le \liminf_{n\to\infty} \mathcal{F}(w_n)$ (resp., $\limsup_{n\to\infty} \mathcal{F}(w_n) \le \mathcal{F}(\bar{w})$);

(c) lower semicontinuous (resp., upper semicontinuous) on $\mathbf{X}$, if $\mathcal{F}$ is lower semicontinuous (resp., upper semicontinuous) at every $\bar{w} \in \mathbf{X}$;

(d) convex, if $\mathcal{F}(\alpha w + (1-\alpha)v) \le \alpha\mathcal{F}(w) + (1-\alpha)\mathcal{F}(v)$ for all $w, v \in \mathbf{X}$ and $\alpha \in [0,1]$.

Definition 2.2 (see [34]): An operator $G\colon \mathbf{X} \to \mathbf{X}^*$ is said to be demicontinuous, if $w_n \to \bar{w}$ in $\mathbf{X}$ implies $G(w_n) \to G(\bar{w})$ weakly in $\mathbf{X}^*$.

Definition 2.3 (see [34]): Let $\mathcal{G}\colon \mathbf{X} \to \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous function. The convex subdifferential $\partial^c\mathcal{G}\colon \mathbf{X} \rightrightarrows \mathbf{X}^*$ of $\mathcal{G}$ is defined by

$$\partial^c\mathcal{G}(w) = \left\{\eta^* \in \mathbf{X}^* \mid \langle \eta^*, v - w\rangle_{\mathbf{X}^*\times\mathbf{X}} \le \mathcal{G}(v) - \mathcal{G}(w) \text{ for all } v \in \mathbf{X}\right\} \quad \text{for all } w \in \mathbf{X}.$$

An element $\eta^* \in \partial^c\mathcal{G}(w)$ is called a subgradient of $\mathcal{G}$ at $w \in \mathbf{X}$.

Definition 2.4 (see [33, 34]): $\mathcal{F} : \mathbf{X} \to \mathbb{R}$ is called a locally Lipschitz function, if for every $w \in \mathbf{X}$, there exist a constant $l_w > 0$ and a neighbourhood $\mathcal{N}_w$ of w satisfying the following inequality:

$$|\mathcal{F}(w_1) - \mathcal{F}(w_2)| \le l_w\|w_1 - w_2\|_{\mathbf{X}} \quad \text{for all } w_1, w_2 \in \mathcal{N}_w.$$

Let $\mathcal{F} : \mathbf{X} \to \mathbb{R}$ be a locally Lipschitz function, $\mathcal{F}^0(w; z)$ is called Clarke's generalized directional derivative of $\mathcal{F}$ at $w \in \mathbf{X}$ in the direction $z \in \mathbf{X}$ given by

$$\mathcal{F}^0(w; z) := \limsup_{x\to w,\ t\to 0^+} \frac{\mathcal{F}(x + tz) - \mathcal{F}(x)}{t}.$$

We also denote by $\partial\mathcal{F}(w)$ the generalized gradient (in the sense of Clarke) of $\mathcal{F}$ at $w \in \mathbf{X}$ and it is defined by

$$\partial\mathcal{F}(x) = \left\{\zeta^* \in \mathbf{X}^* \mid \mathcal{F}^0(w; z) \ge \langle \zeta^*, z\rangle_{\mathbf{X}^*\times\mathbf{X}} \quad \text{for all } z \in \mathbf{X}\right\}.$$

The following proposition gathers some useful properties of Clarke's generalized directional derivative and generalized gradient for locally Lipschitz functions.

Proposition 2.5 (see [34, Proposition 3.23]): *Given a locally Lipschitz function* $\mathcal{F} : \mathbf{X} \to \mathbb{R}$, *the following statements are true:*

(i) *For every* $w \in \mathbf{X}$, *the function* $\mathbf{X} \ni z \mapsto \mathcal{F}^0(w; z) \in \mathbb{R}$ *is positively homogeneous, i.e.* $\mathcal{F}^0(w; \alpha z) = \alpha\mathcal{F}^0(w; z)$ *for all* $\alpha \ge 0, z \in \mathbf{X}$ *and subadditive, i.e.* $\mathcal{F}^0(w; z_1 + z_2) \le \mathcal{F}^0(w; z_1) + \mathcal{F}^0(w; z_2)$ *for all* $z_1, z_2 \in \mathbf{X}$.

(ii) *For every* $z \in \mathbf{X}$, *we obtain* $\mathcal{F}^0(w; z) = \max\{\langle \zeta, z\rangle_{\mathbf{X}^*\times\mathbf{X}} \mid \zeta \in \partial\mathcal{F}(w)\}$.

(iii) $\mathbf{X} \times \mathbf{X} \ni (w, z) \mapsto \mathcal{F}^0(w; z) \in \mathbb{R}$ *is an upper semicontinuous function.*

(iv) *For every* $w \in \mathbf{X}$, $\partial\mathcal{F}(w) \subset \mathbf{X}^*$ *is a nonempty, convex, weakly∗ compact set of* $\mathbf{X}^*$.

For further study and analysis of Problem 1.1, we need the following hypotheses:

$\underline{\mathfrak{h}(A)}$: The mapping $A : \mathcal{D}(A) \subset \mathbf{U} \to \mathbf{U}$ is the infinitesimal generator of a C_0-semigroup $\{\mathcal{T}(t)\}_{t>0}$ in $\mathbf{U}$.

$\underline{\mathfrak{h}(P)}$: For each $i = 1, 2$,

(a) P_i is a nonempty, closed, and convex subset of $\mathbf{X}_i$;
(b) P_i is a bounded subset of $\mathbf{X}_i$.

$\underline{\mathfrak{h}(H)\ :}$ The mapping $H : [0, T] \times \mathbf{U} \times \mathbf{X}_1 \to \mathbf{U}$ satisfies the following conditions.

(a) $H(\cdot, u, x)$ is strongly measurable on $[0, T]$ for all $(u, x) \in \mathbf{U} \times \mathbf{X}_1$;
(b) $H(\cdot, 0_{\mathbf{U}}, 0_{\mathbf{X}_1}) \in L^1([0, T]; \mathbf{U})$;
(c) There exists a constant $a_H > 0$ such that

$$\|H(t, u_1, x_1) - H(t, u_2, x_2)\|_{\mathbf{U}} \leqslant a_H \left(\|u_1 - u_2\|_{\mathbf{U}} + \|x_1 - x_2\|_{\mathbf{X}_1}\right)$$

for all $t \in [0, T]$ and $(u_i, x_i) \in \mathbf{U} \times \mathbf{X}_1 (i = 1, 2)$.

$\underline{\mathfrak{h}(F)}$: The mapping $F : [0, T] \times \mathbf{X}_2 \times \mathbf{X}_1 \to \mathbf{X}_1^*$ satisfies the following conditions.

(a) The mapping $x \mapsto F(t, z, x)$ is demicontinuous for all $z \in \mathbf{X}_2$ and $t \in [0, T]$;
(b) There exists a constant $a_F > 0$ such that

$$\langle F(t, z, x_1) - F(t, z, x_2), x_1 - x_2 \rangle_{\mathbf{X}_1^* \times \mathbf{X}_1} \geqslant a_F \|x_1 - x_2\|_{\mathbf{X}_1}^2$$

for all $x_i \in \mathbf{X}_1 (i = 1, 2), z \in \mathbf{X}_2$, and $t \in [0, T]$;
(c) The mapping $t \mapsto F(t, z, x)$ is continuous for all $(x, z) \in \mathbf{X}_1 \times \mathbf{X}_2$ and there exists a constant $L_F > 0$ such that

$$\|F(t, z_1, x) - F(t, z_2, x)\|_{\mathbf{X}_1^*} \leqslant L_F \|z_1 - z_2\|_{\mathbf{X}_2},$$

for all $t \in [0, T], z_i \in \mathbf{X}_2 (i = 1, 2)$ and $x \in \mathbf{X}_1$;
(d) $\sup_{t \in [0,T]} \|F(t, 0_{\mathbf{X}_2}, x)\|_{\mathbf{X}_1^*} < \infty$, for all $x \in \mathbf{X}_1$.

$\underline{\mathfrak{h}(Q)}$: The mapping $Q : [0, T] \times \mathbf{V}_1 \times \mathbf{U} \to \mathbf{X}_1^*$ satisfies the following conditions.

(a) The mapping $t \mapsto Q(t, v, u)$ is continuous for all $(u, v) \in \mathbf{U} \times \mathbf{V}_1$ and there exists a constant $L_Q > 0$ such that

$$\|Q(t, v_1, u_1) - Q(t, v_2, u_2)\|_{\mathbf{X}_1^*} \leqslant L_Q \left(\|v_1 - v_2\|_{\mathbf{V}_1} + \|u_1 - u_2\|_{\mathbf{U}}\right)$$

for all $t \in [0, T], v_i \in \mathbf{V}_1, u_i \in \mathbf{U} (i = 1, 2)$;
(b) $\sup_{t \in [0,T]} \|Q(t, \mathcal{S}_1(0_{\mathbf{X}_1}), u)\|_{\mathbf{X}_1^*} < \infty$, for all $u \in \mathbf{U}$.

$\underline{\mathfrak{h}(J_1)}$: The mapping $J_1 : \mathbf{W}_2 \times \mathbf{W}_1 \to \mathbb{R}$ satisfies the following conditions.

(a) For each $w \in \mathbf{W}_2, x \mapsto J_1(w, x)$ is a locally Lipschitz function;
(b) $\|\zeta\|_{\mathbf{W}_1^*} \leqslant a_{J_1} + b_{J_1}(\|x\|_{\mathbf{W}_1} + \|w\|_{\mathbf{W}_2})$ for all $x \in \mathbf{W}_1, w \in \mathbf{W}_2$ with $\zeta \in \partial J_1(w, x)$ and $a_{J_1}, b_{J_1} > 0$;

(c) There exists a constant $m_{J_1} > 0$ such that

$$\langle \zeta_1 - \zeta_2, x_1 - x_2 \rangle_{\mathbf{W}_1^* \times \mathbf{W}_1} \geqslant -m_{J_1} \|x_1 - x_2\|_{\mathbf{W}_1} \left(\|x_1 - x_2\|_{\mathbf{W}_1} + \|w_1 - w_2\|_{\mathbf{W}_2} \right)$$

for all $x_i \in \mathbf{W}_1, w_i \in \mathbf{W}_2, \zeta_i \in \partial J_1(w_i, x_i) (i = 1, 2)$;

(d) It holds $\limsup_{n\to\infty} J_1^\circ(v_n, \zeta; \zeta - \mu_n) \leq J_1^\circ(v, \zeta; \zeta - \mu)$, whenever $\zeta \in \mathbf{W}_1, v_n \to v$ in $\mathbf{W}_2$ and $\mu_n \to \mu$ in $\mathbf{W}_1$.

$\underline{\mathfrak{h}(J_2) :}$ The mapping $J_2 \colon \mathbf{W}_1 \times \mathbf{W}_2 \to \mathbb{R}$ satisfies the following conditions.

(a) For each $x \in \mathbf{W}_1$, $w \mapsto J_2(x, w)$ is a locally Lipschitz function;

(b) $\|\eta\|_{\mathbf{W}_2^*} \leqslant a_{J_2} + b_{J_2}(\|w\|_{\mathbf{W}_2} + \|x\|_{\mathbf{W}_1})$ for all $x \in \mathbf{W}_1, w \in \mathbf{W}_2$ with $\eta \in \partial J_2(x, w)$ with $a_{J_2}, b_{J_2} > 0$;

(c) There exists a constant $m_{J_2} > 0$ such that

$$\langle \eta_1 - \eta_2, w_1 - w_2 \rangle_{\mathbf{W}_2^* \times \mathbf{W}_2} \geqslant -m_{J_2} \|w_1 - w_2\|_{\mathbf{W}_2} \left(\|x_1 - x_2\|_{\mathbf{W}_1} + \|w_1 - w_2\|_{\mathbf{W}_2} \right)$$

for all $x_i \in \mathbf{W}_1, w_i \in \mathbf{W}_2, \eta_i \in \partial J_2(x_i, w_i) (i = 1, 2)$;

(d) It holds $\limsup_{n\to\infty} J_2^\circ(\zeta_n, v; v - z_n) \leq J_2^\circ(\zeta, v; v - z)$, whenever $v \in \mathbf{W}_2, \zeta_n \to \zeta$ in $\mathbf{W}_1$ and $z_n \to z$ in $\mathbf{W}_2$.

$\underline{\mathfrak{h}(\Upsilon) :}$ The mapping $\Upsilon : \mathbf{V}_2 \times \mathbf{X}_1 \times \mathbf{X}_1 \to \mathbb{R}$ satisfies the following conditions.

(a) $\Upsilon(s, x, \cdot)$ is convex and lower semicontinuous, for all $(s, x) \in \mathbf{V}_2 \times \mathbf{X}_1$;

(b) There exist constants $a_\Upsilon, b_\Upsilon > 0$ such that

$$\begin{aligned} &\Upsilon(s_1, x_1, u_2) - \Upsilon(s_1, x_1, u_1) + \Upsilon(s_2, x_2, u_1) - \Upsilon(s_2, x_2, u_2) \\ &\quad \leqslant a_\Upsilon \|x_1 - x_2\|_{\mathbf{X}_1} \|u_1 - u_2\|_{\mathbf{X}_1} + b_\Upsilon \|u_1 - u_2\|_{\mathbf{X}_1} \|s_1 - s_2\|_{\mathbf{V}_2} \end{aligned}$$

for all $(s_i, x_i, u_i) \in \mathbf{V}_2 \times \mathbf{X}_1 \times \mathbf{X}_1 (i = 1, 2)$;

(c) It holds $\Upsilon(s, x, u_1) - \Upsilon(s, x, u_2) \leq \varrho_\Upsilon(\|s\|_{\mathbf{V}_2}, \|x\|_{\mathbf{X}_1}) \|u_1 - u_2\|_{\mathbf{X}_1}$, for all $(s, x) \in \mathbf{V}_2 \times \mathbf{X}_1, u_1, u_2 \in \mathbf{X}_1$, where the function $\varrho_\Upsilon \colon \mathbb{R}^2 \to [0, \infty)$ is continuous;

(d) The inequality holds

$$\liminf_{n\to\infty} \left[\Upsilon(s_n, x_n, x_n) - \Upsilon(s_n, x_n, u) \right] \geq \Upsilon(s, x, x) - \Upsilon(s, x, u),$$

whenever $u \in \mathbf{X}_1, x_n \to x$ in $\mathbf{X}_1, s_n \to s$ in $\mathbf{V}_2$.

$\underline{\mathfrak{h}(\mathcal{S}) :}$ The mappings $\mathcal{S}_1 : C([0, T]; \mathbf{X}_1) \to C([0, T]; \mathbf{V}_1)$ and $\mathcal{S}_2 : C([0, T]; \mathbf{X}_1) \to C([0, T]; \mathbf{V}_2)$ are history-dependent operators, i.e. there exist constants $a_{\mathcal{S}_1}, a_{\mathcal{S}_2} > 0$ such that

$$\|(\mathcal{S}_j x_1)(t) - (\mathcal{S}_j x_2)(t)\|_{\mathbf{V}_j} \leq a_{\mathcal{S}_j} \int_0^t \|x_1(s) - x_2(s)\|_{\mathbf{X}_1} \, ds \quad \text{for a.e. } t \in [0, T],$$

for all $j = 1, 2$ and $x_1, x_2 \in C([0, T]; \mathbf{X}_1)$.

$\underline{\mathfrak{h}(G) :}$ The mapping $G : [0, T] \times \mathbf{X}_1 \times \mathbf{X}_2 \to \mathbf{X}_2^*$ satisfies the following conditions.

(a) The mapping $z \mapsto G(t, x, z)$ is demicontinuous for all $x \in \mathbf{X}_1$ and $t \in [0, T]$;

(b) There exists a constant $a_G > 0$ such that

$$\langle G(t,x,z_1) - G(t,x,z_2), z_1 - z_2\rangle_{\mathbf{X}_2^*\times\mathbf{X}_2} \geqslant a_G \|z_1 - z_2\|_{\mathbf{X}_2}^2$$

for all $z_i \in \mathbf{X}_2 (i = 1,2), x \in \mathbf{X}_1, t \in [0,T]$;

(c) The mapping $t \mapsto G(t,x,z)$ is continuous for all $(x,z) \in \mathbf{X}_1 \times \mathbf{X}_2$ and there exists a constant $L_G > 0$ such that

$$\|G(t,x_1,z) - G(t,x_2,z)\|_{\mathbf{X}_2^*} \le L_G\|x_1 - x_2\|_{\mathbf{X}_1},$$

for all $t \in [0,T]$ $x_i (i = 1,2) \in \mathbf{X}_1$ and $z \in \mathbf{X}_2$;

(d) $\sup_{t\in[0,T]} \|G(t,0_{\mathbf{X}_1},x)\|_{\mathbf{X}_2^*} < \infty$, for all $x \in \mathbf{X}_2$.

$\underline{\mathfrak{h}(M)}$: The operator $\mathbf{M} : \mathbf{X}_1 \to \mathbf{W}_1$ is linear and compact.
$\underline{\mathfrak{h}(N)}$: The operator $\mathbf{N} : \mathbf{X}_2 \to \mathbf{W}_2$ is linear and compact.
$\underline{\mathfrak{h}(\varphi)}$: The mapping $\varphi : \mathbf{X}_2 \to \mathbb{R}$ is convex and continuous.
$\underline{\mathfrak{h}(0)}$: (a) $a_F > m_{J_1}\|\mathbf{M}\|^2 + a_\Upsilon, \quad a_G > m_{J_2}\|\mathbf{N}\|^2$; (b) The following inequality holds:

$$\frac{\left(m_{J_2}\|\mathbf{N}\|\,\|\mathbf{M}\| + L_G\right)\left(m_{J_1}\|\mathbf{M}\|\,\|\mathbf{N}\| + L_F\right)}{\left(a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon\right)\left(a_G - m_{J_2}\|\mathbf{N}\|^2\right)} < 1.$$

We elaborate a bit more about the aforementioned assumptions imposed on the target problem.

Remark 2.1: (i) In order to study the existence and uniqueness of solutions to Problem 1.1, Hao-Wang-Han [1] introduced the $\mathfrak{h}(A)$, $\mathfrak{h}(P)$(a), $\mathfrak{h}(H)$, $\mathfrak{h}(F)(a,b,c)$, $\mathfrak{h}(Q)(a)$, $\mathfrak{h}(J_1)(a,b,c)$, $\mathfrak{h}(G)(a,b,c)$, $\mathfrak{h}(J_2)(a,b,c)$, $\mathfrak{h}(\Upsilon)(a,b)$, $\mathfrak{h}(\mathcal{S})$, $\mathfrak{h}(\mathbf{M})$, $\mathfrak{h}(\mathbf{N})$, $\mathfrak{h}(\varphi)$ and $\mathfrak{h}(0)$. Note that the assumptions $\mathfrak{h}(F)(c)$ and $\mathfrak{h}(G)(c)$ imply the continuity of $F(\cdot,\cdot,x)$ and $G(\cdot,\cdot,z)$, respectively for all $x \in \mathbf{X}_1$ and $z \in \mathbf{X}_2$.

(ii) Besides, we have added further assumptions $\mathfrak{h}(P)$(b), $\mathfrak{h}(F)(d)$, $\mathfrak{h}(G)(d)$, $\mathfrak{h}(Q)(b)$, $\mathfrak{h}(J_1)(d)$, $\mathfrak{h}(J_2)(d)$ and $\mathfrak{h}(\Upsilon)(c,d)$, which serve as the essential conditions for deriving some crucial properties of regularized gap functions to exploring error bounds for Problem 1.1 in the next section.

(iii) In assumptions $\mathfrak{h}(J_1)(c)$ and $\mathfrak{h}(J_2)(c)$, the notation $\partial J_j(e_j,z_j)$ denotes the generalized gradient of J_j with respect to the variable $z_j \in \mathbf{W}_j$ for $j = 1,2$, where $e_1 \in \mathbf{W}_2$ and $e_2 \in \mathbf{W}_1$.
Using the relation between Clarke's generalized directional derivative and the generalized gradient, one obtains that $\mathfrak{h}(J_1)(c)$, which is known as the relaxed monotonicity condition, is equivalent to

$$\begin{aligned} &J_1^\circ(w_1,x_1;x_2 - x_1) + J_1^\circ(w_2,x_2;x_1 - x_2) \\ &\quad\le m_{J_1}\|x_1 - x_2\|_{\mathbf{W}_1}\left(\|x_1 - x_2\|_{\mathbf{W}_1} + \|w_1 - w_2\|_{\mathbf{W}_2}\right), \end{aligned} \tag{4}$$

for all $x_i \in \mathbf{W}_1, w_i \in \mathbf{W}_2$ $(i = 1,2)$.

Indeed, assume that (4) holds. Let $(x_i, w_i) \in \mathbf{W}_1 \times \mathbf{W}_2, \zeta_i \in \partial J_1(w_i, x_i)$ $(i = 1, 2)$. By Proposition 2.5(ii), we have

$$\langle \zeta_i, x \rangle_{\mathbf{W}_1^* \times \mathbf{W}_1} \le J_1^\circ(w_i, x_i; x), \quad \forall\, x \in \mathbf{W}_1.$$

Then, it follows from (4) that

$$\begin{aligned}\langle \zeta_1 - \zeta_2, x_2 - x_1 \rangle_{\mathbf{W}_1^* \times \mathbf{W}_1} &= \langle \zeta_1, x_2 - x_1 \rangle_{\mathbf{W}_1^* \times \mathbf{W}_1} + \langle \zeta_2, x_1 - x_2 \rangle_{\mathbf{W}_1^* \times \mathbf{W}_1} \\ &\le J_1^\circ(w_1, x_1; x_2 - x_1) + J_1^\circ(w_2, x_2; x_1 - x_2) \\ &\le m_{J_1} \|x_1 - x_2\|_{\mathbf{W}_1} \left(\|x_1 - x_2\|_{\mathbf{W}_1} + \|w_1 - w_2\|_{\mathbf{W}_2} \right),\end{aligned}$$

which implies that $\mathfrak{h}(J_1)(c)$ holds.
Conversely, suppose that $\mathfrak{h}(J_1)(c)$ is satisfied. Then, one has

$$\begin{aligned}&\langle \zeta_1 - \zeta_2, x_2 - x_1 \rangle_{\mathbf{W}_1^* \times \mathbf{W}_1} \\ &\quad \le m_{J_1} \|x_1 - x_2\|_{\mathbf{W}_1} \left(\|x_1 - x_2\|_{\mathbf{W}_1} + \|w_1 - w_2\|_{\mathbf{W}_2} \right),\end{aligned} \tag{5}$$

for all $(x_i, w_i) \in \mathbf{W}_1 \times \mathbf{W}_2, \zeta_i \in \partial J_1(w_i, x_i)$ $(i = 1, 2)$.
Using Proposition 2.5(ii), we see that there exist $\zeta_i^* \in \partial J_1(w_i, x_i)$ $(i = 1, 2)$ such that

$$\begin{aligned}J_1^\circ(w_1, x_1; x_2 - x_1) &= \langle \zeta_1^*, x_2 - x_1 \rangle_{\mathbf{W}_1^* \times \mathbf{W}_1}, \\ J_1^\circ(w_2, x_2; x_1 - x_2) &= \langle \zeta_2^*, x_1 - x_2 \rangle_{\mathbf{W}_1^* \times \mathbf{W}_1}.\end{aligned}$$

Adding two equalities and using (5), we get

$$\begin{aligned}&J_1^\circ(w_1, x_1; x_2 - x_1) + J_1^\circ(w_2, x_2; x_1 - x_2) \\ &\quad = \left\langle \zeta_1^* - \zeta_2^*, x_2 - x_1 \right\rangle_{\mathbf{W}_1^* \times \mathbf{W}_1} \\ &\quad \le m_{J_1} \|x_1 - x_2\|_{\mathbf{W}_1} \left(\|x_1 - x_2\|_{\mathbf{W}_1} + \|w_1 - w_2\|_{\mathbf{W}_2} \right),\end{aligned}$$

which completes the proof.
Similarly, we also obtain that $\mathfrak{h}(J_2)(c)$ is equivalent to

$$\begin{aligned}&J_2^\circ(x_1, w_1; w_2 - w_1) + J_2^\circ(x_2, w_2; w_1 - w_2) \\ &\quad \le m_{J_2} \|w_1 - w_2\|_{\mathbf{W}_2} \left(\|x_1 - x_2\|_{\mathbf{W}_1} + \|w_1 - w_2\|_{\mathbf{W}_2} \right),\end{aligned}$$

for all $x_i \in \mathbf{W}_1, w_i \in \mathbf{W}_2$ $(i = 1, 2)$.

(iv) The compatibility inequalities in $\mathfrak{h}(0)$ are typically referred to as smallness conditions related to the data of Problem 1.1.

In order to explain the solution existence, we recall the definition of the mild solution of Problem 1.1 which is provided in [1].

Definition 2.6: A triple of functions $(u, x, z) \in C([0, T]; \mathbf{U}) \times C([0, T]; P_1) \times C([0, T]; P_2)$ is said to be a mild solution to Problem 1.1 if such that $u(0) = u_0$, for all $t \in [0, T]$, system (2), system (3), and the following system hold:

$$u(t) = \mathcal{T}(t)u_0 + \int_0^t \mathcal{T}(t - s)H(s, u(s), x(s))\, \mathrm{d}s.$$

In fact, from [35], there exists a constant $m_H > 0$ such that $\|\mathcal{T}(\cdot)\| \leq m_H$.

We end this section by presenting the following result, which provides a convenient framework for mentioning the existence and uniqueness of solutions to Problem 1.1 in the rest of the paper.

Theorem 2.7 (see [1, Theorem 2.2]): *Suppose that the assumptions* $\mathfrak{h}(A)$, $\mathfrak{h}(P)(a)$, $\mathfrak{h}(H)$, $\mathfrak{h}(F)(a,b,c)$, $\mathfrak{h}(Q)(a)$, $\mathfrak{h}(J_1)(a,b,c)$, $\mathfrak{h}(G)(a,b,c)$, $\mathfrak{h}(J_2)(a,b,c)$, $\mathfrak{h}(\Upsilon)(a,b)$, $\mathfrak{h}(\mathcal{S})$, $\mathfrak{h}(\mathbf{M})$, $\mathfrak{h}(\mathbf{N})$, $\mathfrak{h}(\varphi)$ *and* $\mathfrak{h}(0)$ *hold. Then, for each* $u_0 \in \mathbf{U}$, *Problem 1.1 has a unique solution* $(u,x,z) \in C([0,T];\mathbf{U}) \times C([0,T];P_1) \times C([0,T];P_2)$.

Remark 2.2: The full proof of this theorem has been provided in Section 3 of Hao-Wang-Han [1]. Here, we give a sketch of the proof of Theorem 2.7, which is divided into three steps corresponding to the proof of [1, Theorem 2.2].

Step 1. For every $(u,z) \in C([0,T];\mathbf{U}) \times C([0,T];P_2)$, by employing the auxiliary problem technique together with the unique fixed-point method, we obtain that problem (2) admits a unique solution $x_{uz} \in C([0,T];P_1)$.

Step 2. For each $x \in C([0,T];P_1)$, by applying the unique fixed-point method and Gronwall's inequality, we conclude that the system consisting of (1) and (3) admits a unique solution $(u_x, z_x) \in C([0,T];\mathbf{U}) \times C([0,T];P_2)$.

Step 3. Define operators $\mathcal{Z}_1 : C([0,T];P_2) \to C([0,T];P_1)$ and $\mathcal{Z}_2 : C([0,T];P_1) \to C([0,T];P_2)$ by $\mathcal{Z}_1 v = x_v$, for all $v \in C([0,T];P_2)$ and $\mathcal{Z}_2 y = z_y$, for all $y \in C([0,T];P_1)$, respectively. By the assumptions of this theorem, together with Gronwall's inequality and Banach's fixed-point theorem, we get that the composition $\mathcal{Z}_2 \circ \mathcal{Z}_1 : C([0,T];P_2) \to C([0,T];P_2)$ has a unique fixed point.

Then, the proof of Theorem 2.7 is completed by combining Steps 1–3.

3. Main results

This section provides global error bounds for Problem 1.1 based on the computational technologies involving coupled gap functions of the Fukushima type. Then, a corollary to the history-dependent system given by coupled variational-hemivariational inequalities with the nesting structure is derived. For convenience, from now on we omit the subscript $\mathbf{X}_i^* \times \mathbf{X}_i$ in the duality pairing $\langle \cdot, \cdot \rangle_{\mathbf{X}_i^* \times \mathbf{X}_i}$ $(i = 1, 2)$ whenever no confusion arises.

Let $\omega_i > 0$ with $i = 1, 2$ and $u \in C([0,T];\mathbf{U})$ be fixed. Let us consider two functions $\Delta_{\omega_1} \colon [0,T] \times \mathbf{U} \times C([0,T];P_1) \times C([0,T];P_2) \to \mathbb{R}$ and $\Theta_{\omega_2} \colon [0,T] \times C([0,T];P_2) \times C([0,T];P_1) \to \mathbb{R}$ given by

$$\begin{aligned}
&\Delta_{\omega_1}(t, u(t), x, z) \\
&\quad = \sup_{y \in P_1} \Bigg(\big\langle F(t, z(t), x(t)) + Q\left(t, (\mathcal{S}_1 x)(t), u(t)\right), x(t) - y \big\rangle - \Upsilon\left((\mathcal{S}_2 x)(t), x(t), y \right) \\
&\qquad + \Upsilon\left((\mathcal{S}_2 x)(t), x(t), x(t)\right) - J_1^{\circ}\left(\mathbf{N}z(t), \mathbf{M}x(t); \mathbf{M}y - \mathbf{M}x(t) \right) \\
&\qquad - \frac{1}{2\omega_1} \|x(t) - y\|_{\mathbf{X}_1}^2 \Bigg)
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
&\Theta_{\omega_2}(t,z,x)\\
&\quad=\sup_{v\in P_2}\Big(\langle G(t,x(t),z(t))-f(t),z(t)-v\rangle-J_2^\circ\left(\mathbf{M}x(t),\mathbf{N}z(t);\mathbf{N}v-\mathbf{N}z(t)\right)\\
&\qquad-\varphi(v)+\varphi(z(t))-\frac{1}{2\omega_2}\|z(t)-v\|_{\mathbf{X}_2}^2\Big),
\end{aligned}\tag{7}$$

for all $(x,z)\in C([0,T];P_1)\times C([0,T];P_2)$ and $t\in[0,T]$.

Proposition 3.1: *Let $(u,z)\in C([0,T];\mathbf{U})\times C([0,T];P_2)$ be given as fixed. Assume that hypotheses $\mathfrak{h}(P)(a)$, $\mathfrak{h}(\Upsilon)(a)$, $\mathfrak{h}(J_1)(a)$ and $\mathfrak{h}(\mathbf{M})$ are satisfied. Then, for each $\omega_1>0$, the function Δ_{ω_1} defined by (6) satisfies two following properties: for each $t\in[0,T]$,*

(i) $\Delta_{\omega_1}(t,u(t),x,z)\geq 0$, *for all $x\in C([0,T];P_1)$;*
(ii) $x^*\in C([0,T];P_1)$ *satisfies $\Delta_{\omega_1}(t,u(t),x^*,z)=0$ if and only if x^* is the solution of the variational-hemivariational inequality corresponding to (u,z) given by the system (2), i.e.*

$$\begin{aligned}
&\left\langle F(t,z(t),x^*(t))+Q\left(t,\left(\mathcal{S}_1x^*\right)(t),u(t)\right),y-x^*(t)\right\rangle\\
&\quad+\Upsilon\left(\left(\mathcal{S}_2x^*\right)(t),x^*(t),y\right)-\Upsilon\left(\left(\mathcal{S}_2x^*\right)(t),x^*(t),x^*(t)\right)\\
&\quad+J_1^\circ\left(\mathbf{N}z(t),\mathbf{M}x^*(t);\mathbf{M}y-\mathbf{M}x^*(t)\right)\geqslant 0,\quad\forall\, y\in P_1.
\end{aligned}$$

Proof: Let the parameter $\omega_1>0, t\in[0,T]$ and $(u,z)\in C([0,T];\mathbf{U})\times C([0,T];P_2)$ be fixed, we prove the two above properties of the function Δ_{ω_1}.

(i) It follows from the definition of Δ_{ω_1} in (6) that for any $x\in C([0,T];P_1)$,

$$\begin{aligned}
&\Delta_{\omega_1}(t,u(t),x,z)\\
&\quad\geq\langle F(t,z(t),x(t))+Q\left(t,\left(\mathcal{S}_1x\right)(t),u(t)\right),x(t)-x(t)\rangle-\Upsilon\left(\left(\mathcal{S}_2x\right)(t),x(t),x(t)\right)\\
&\qquad+\Upsilon\left(\left(\mathcal{S}_2x\right)(t),x(t),x(t)\right)-J_1^\circ\left(\mathbf{N}z(t),\mathbf{M}x(t);\mathbf{M}x(t)-\mathbf{M}x(t)\right)\\
&\qquad-\frac{1}{2\omega_1}\|x(t)-x(t)\|_{\mathbf{X}_1}^2\\
&\quad=-J_1^\circ\left(\mathbf{N}z(t),\mathbf{M}x(t);\mathbf{0}_{\mathbf{W}_1}\right)=0.
\end{aligned}$$

Thus, $\Delta_{\omega_1}(t,u(t),x,z)\geq 0$ for all $x\in C([0,T];P_1)$.

(ii) ($\Longleftarrow$) If $x^*(t)$ satisfies the following

$$\begin{aligned}
&\left\langle F(t,z(t),x^*(t))+Q\left(t,\left(\mathcal{S}_1x^*\right)(t),u(t)\right),y-x^*(t)\right\rangle+\Upsilon\left(\left(\mathcal{S}_2x^*\right)(t),x^*(t),y\right)\\
&\quad-\Upsilon\left(\left(\mathcal{S}_2x^*\right)(t),x^*(t),x^*(t)\right)+J_1^\circ\left(\mathbf{N}z(t),\mathbf{M}x^*(t);\mathbf{M}y-\mathbf{M}x^*(t)\right)\geqslant 0,\quad\forall\, y\in P_1.
\end{aligned}$$

This implies that

$$\begin{aligned}0 \geq \sup_{y\in P_1} \Big(&\big\langle F(t,z(t),x^*(t)) + Q\left(t,\left(\mathcal{S}_1x^*\right)(t),u(t)\right), x^*(t)-y\big\rangle \\ &- \Upsilon\left(\left(\mathcal{S}_2x^*\right)(t),x^*(t),y\right) + \Upsilon\left(\left(\mathcal{S}_2x^*\right)(t),x^*(t),x^*(t)\right) \\ &-J_1^\circ\left(\mathbf{N}z(t),\mathbf{M}x^*(t);\mathbf{M}y-\mathbf{M}x^*(t)\right) - \frac{1}{2\omega_1}\|x^*(t)-y\|^2_{\mathbf{X}_1}\Big) \\ = \Delta_{\omega_1}&(t,u(t),x^*,z).\end{aligned}$$

By the fact (i), we can conclude that $\Delta_{\omega_1}(t,u(t),x^*,z)=0$.

($\Longrightarrow$) Let $x^*\in C([0,T];P_1)$ satisfies $\Delta_{\omega_1}(t,u(t),x^*,z)=0$. It follows from (6) that

$$\begin{aligned}&\big\langle F(t,z(t),x^*(t)) + Q\left(t,\left(\mathcal{S}_1x^*\right)(t),u(t)\right), x^*(t)-y\big\rangle - \Upsilon\left(\left(\mathcal{S}_2x^*\right)(t),x^*(t),y\right) \\ &\quad + \Upsilon\left(\left(\mathcal{S}_2x^*\right)(t),x^*(t),x^*(t)\right) - J_1^\circ\left(\mathbf{N}z(t),\mathbf{M}x^*(t);\mathbf{M}y-\mathbf{M}x^*(t)\right) \\ &\leq \frac{1}{2\omega_1}\|x^*(t)-y\|^2_{\mathbf{X}_1},\end{aligned}$$

for all $y\in P_1$. For any $y\in P_1$ fixed and $r\in(0,1)$, by the convexity of P_1, we obtain that $y_r := (1-r)x^*(t)+ry\in P_1, \forall t\in[0,T]$. Taking $y=y_r$ into the above inequality and using the linearity of $\mathbf{M}$, the positive homogeneity of $\beta\mapsto J_1^\circ(v,w;\beta)$ and the convexity of $\beta\mapsto\Upsilon(\xi,w,\beta)$ for all $\xi\in\mathbf{V}_1$, $w\in\mathbf{W}_1$ and $v\in\mathbf{W}_2$, it gives

$$\begin{aligned}&r\big\langle F(t,z(t),x^*(t)) + Q\left(t,\left(\mathcal{S}_1x^*\right)(t),u(t)\right), x^*(t)-y\big\rangle - r\Upsilon\left(\left(\mathcal{S}_2x^*\right)(t),x^*(t),y\right) \\ &\quad + r\Upsilon\left(\left(\mathcal{S}_2x^*\right)(t),x^*(t),x^*(t)\right) - rJ_1^\circ\left(\mathbf{N}z(t),\mathbf{M}x^*(t);\mathbf{M}y-\mathbf{M}x^*(t)\right) \\ &\leq \frac{r^2}{2\omega_1}\|x^*(t)-y\|^2_{\mathbf{X}_1},\end{aligned}$$

that is,

$$\begin{aligned}&\big\langle F(t,z(t),x^*(t)) + Q\left(t,\left(\mathcal{S}_1x^*\right)(t),u(t)\right), x^*(t)-y\big\rangle - \Upsilon\left(\left(\mathcal{S}_2x^*\right)(t),x^*(t),y\right) \\ &\quad + \Upsilon\left(\left(\mathcal{S}_2x^*\right)(t),x^*(t),x^*(t)\right) - J_1^\circ\left(\mathbf{N}z(t),\mathbf{M}x^*(t);\mathbf{M}y-\mathbf{M}x^*(t)\right) \\ &\leq \frac{r}{2\omega_1}\|x^*(t)-y\|^2_{\mathbf{X}_1},\end{aligned}$$

for all $y\in P_1$. Letting $r\to 0^+$ for the above inequality, we have

$$\begin{aligned}&\big\langle F(t,z(t),x^*(t)) + Q\left(t,\left(\mathcal{S}_1x^*\right)(t),u(t)\right), y-x^*(t)\big\rangle + \Upsilon\left(\left(\mathcal{S}_2x\right)(t),x^*(t),y\right) \\ &- \Upsilon\left(\left(\mathcal{S}_2x^*\right)(t),x(t),x^*(t)\right) + J_1^\circ\left(\mathbf{N}z(t),\mathbf{M}x^*(t);\mathbf{M}y-\mathbf{M}x^*(t)\right) \geqslant 0\end{aligned}$$

for all $y\in P_1$. So, this gives that x^* is the solution of the variational-hemivariational inequality corresponding to (u,z) given by the system (2). ■

Remark 3.1: Given $(u,z)\in C([0,T];\mathbf{U})\times C([0,T];P_2)$ and $\omega_1>0$, the function Δ_{ω_1} satisfying two properties (i) and (ii) provided in Proposition 3.1 is known as 'a regularized gap function' of the variational-hemivariational inequality corresponding to (u,z) given by the system (2), where $-\frac{1}{2\omega_1}\|x(t)-y\|^2_{\mathbf{X}_1}$ is the regularized term.

Similarly to Proposition 3.1, we also show that the function Θ_{ω_2} defined by (7) is a regularized gap function of the variational-hemivariational inequality corresponding to $x \in C([0,T];P_1)$ given by the system (3).

Proposition 3.2: *Let $x \in C([0,T];P_1)$ be fixed. Assume that hypotheses $\mathfrak{h}(P)(a)$, $\mathfrak{h}(\varphi)$, $\mathfrak{h}(J_2)(a)$ and $\mathfrak{h}(\mathbf{N})$ hold. Then, for each $\omega_2 > 0$, the function Θ_{ω_2} defined by (7) satisfies the following properties: for each $t \in [0,T]$,*

(i) $\Theta_{\omega_2}(t,z,x) \geq 0$, *for all $z \in C([0,T];P_2)$;*
(ii) $z^* \in C([0,T];P_2)$ *satisfies $\Theta_{\omega_2}(t,z^*,x) = 0$ if and only if z^* is such that*

$$\begin{aligned}&\langle G(t,x(t),z^*(t)), v - z^*(t)\rangle + J_2^\circ\left(\mathbf{M}x(t),\mathbf{N}z^*(t);\mathbf{N}v - \mathbf{N}z^*(t)\right)\\&\quad + \varphi(v) - \varphi(z^*(t)) \geqslant \langle f(t), v - z^*(t)\rangle, \quad \forall\, v \in P_2.\end{aligned}$$

The regularized gap functions Δ_{ω_1} and Θ_{ω_2} have some useful properties involving to various convergence results and analysing the behaviour of functions in terms of their limits and integrals. The following propositions elucidate some of these useful properties.

Proposition 3.3: *Suppose that assumptions $\mathfrak{h}(P)$, $\mathfrak{h}(F)$, $\mathfrak{h}(Q)$, $\mathfrak{h}(\Upsilon)$, $\mathfrak{h}(J_1)$, $\mathfrak{h}(\mathcal{S})$, $\mathfrak{h}(\mathbf{N})$ and $\mathfrak{h}(\mathbf{M})$ are satisfied. Then, for any parameter $\omega_1 > 0$ and $u \in C([0,T];\mathbf{U})$ fixed, we obtain the following conclusions for the regularized gap function Δ_{ω_1} defined by (6):*

(i) *For each $t \in [0,T]$, $(x,z) \mapsto \Delta_{\omega_1}(t,u(t),x,z)$ is a lower semicontinuous function.*
(ii) *For each $(x,z) \in C([0,T];P_1) \times C([0,T];P_2)$ fixed, the function $t \mapsto \Delta_{\omega_1}(t,u(t),x,z)$ belongs to $L_+^\infty(0,T)$.*

Proof: (i) For each parameter $\omega_1 > 0$ and $u \in C([0,T];\mathbf{U})$ fixed, let the function $\Xi_{\omega_1} : [0,T] \times \mathbf{U} \times C([0,T];P_1) \times C([0,T];P_2) \times P_1 \to \mathbb{R}$ be given by

$$\begin{aligned}&\Xi_{\omega_1}(t,u(t),x,z,y)\\&\quad = \left\langle F(t,z(t),x(t)) + Q\left(t,(\mathcal{S}_1x)(t),u(t)\right), x(t) - y\right\rangle - \Upsilon\left((\mathcal{S}_2x)(t),x(t),y\right)\\&\qquad + \Upsilon\left((\mathcal{S}_2x)(t),x(t),x(t)\right) - J_1^\circ\left(\mathbf{N}z(t),\mathbf{M}x(t);\mathbf{M}y - \mathbf{M}x(t)\right)\\&\qquad - \frac{1}{2\omega_1}\|x(t) - y\|_{\mathbf{X}_1}^2\end{aligned}$$

for all $(x,z) \in C([0,T];P_1) \times C([0,T];P_2)$, $y \in P_1$ and $t \in [0,T]$. Then the regularized gap function Δ_{ω_1} can be rewritten that

$$\Delta_{\omega_1}(t,u(t),x,z) = \sup_{y \in P_1} \Xi_{\omega_1}(t,u(t),x,z,y) \tag{8}$$

for all $(x,z) \in C([0,T];P_1) \times C([0,T];P_2)$ and $t \in [0,T]$. Let $x_n(t) \to x(t)$ and $z_n(t) \to z(t)$ for al $t \in [0,T]$ as $n \to \infty$. Thanks to hypotheses $\mathfrak{h}(F)(a,c)$, $\mathfrak{h}(Q)(b)$, $\mathfrak{h}(\Upsilon)(d)$,

$\mathfrak{h}(J_1)(d)$, $\mathfrak{h}(\mathcal{S})$, $\mathfrak{h}(\mathbf{M})$ and the upper semicontinuity of $J_1^\circ(v, \cdot; \cdot)$ for all $v \in \mathbf{W}_2$, it gives

$$\begin{aligned}
\liminf_{n\to\infty} \Delta_{\omega_1}(t, u(t), x_n, z_n) &= \liminf_{n\to\infty} \sup_{y\in P_1} \Xi_{\omega_1}(t, u(t), x_n, z_n, x) \\
&\geq \liminf_{n\to\infty} \Xi_{\omega_1}(t, u(t), x_n, z_n, w) \\
&\geq \liminf_{n\to\infty} \langle F(t, z_n(t), x_n(t)) + Q\left(t, \left(\mathcal{S}_1 x_n\right)(t), u(t)\right), x_n(t) - w\rangle \\
&\quad + \liminf_{n\to\infty} \left(\Upsilon\left(\left(\mathcal{S}_2 x_n\right)(t), x_n(t), x_n(t)\right) - \Upsilon\left(\left(\mathcal{S}_2 x_n\right)(t), x_n(t), w\right)\right) \\
&\quad - \limsup_{n\to\infty} J_1^\circ\left(\mathbf{N} z_n(t), \mathbf{M} x_n(t); \mathbf{M} w - \mathbf{M} x_n(t)\right) \\
&\quad - \limsup_{n\to\infty} \left(\frac{1}{2\omega_1}\|x_n(t) - w\|_{\mathbf{X}_1}^2\right) \\
&\geq \langle F(t, z(t), x(t)) + Q\left(t, \left(\mathcal{S}_1 x\right)(t), u(t)\right), x(t) - w\rangle - \Upsilon\left(\left(\mathcal{S}_2 x\right)(t), x(t), w\right) \\
&\quad + \Upsilon\left(\left(\mathcal{S}_2 x\right)(t), x(t), x(t)\right) - J_1^\circ\left(\mathbf{N} z(t), \mathbf{M} x(t); \mathbf{M} w - \mathbf{M} x(t)\right) \\
&\quad - \frac{1}{2\omega_1}\|x(t) - w\|_{\mathbf{X}_1}^2 \\
&= \Xi_{\omega_1}(t, u(t), x, z, w), \quad \text{for all} w \in P_1.
\end{aligned} \tag{9}$$

It follows from (8)–(9) that

$$\Delta_{\omega_1}(t, u(t), x, z) = \sup_{w\in P_1} \Xi_{\omega_1}(t, u(t), x, z, w) \leq \liminf_{n\to\infty} \Delta_{\omega_1}(t, u(t), x_n, z_n),$$

for all $t \in [0, T]$. This proves that given any parameter $\omega_1 > 0$, $u \in C([0, T]; \mathbf{U})$ fixed and for each $t \in [0, T]$, $(x, z) \mapsto \Delta_{\omega_1}(t, u(t), x, z)$ is a lower semicontinuous function.

(ii) Let $(u, x, z) \in C([0, T]; \mathbf{U}) \times C([0, T]; P_1) \times C([0, T]; P_2)$ be arbitrarily fixed. We shall verify that the function $t \mapsto \Delta_{\omega_1}(t, u(t), x, z)$ is measurable and essentially bounded. Indeed, we consider the set

$$\mathbf{D}_\alpha^{\Delta_{\omega_1}} := \left\{t \in [0, T] \mid \Delta_{\omega_1}(t, u(t), x, z) \leq \alpha\right\},$$

where $\alpha \in \mathbb{R}$ satisfies $\mathbf{D}_\alpha^{\Delta_{\omega_1}} \neq \emptyset$.

First, we show that the set $\mathbf{D}_\alpha^{\Delta_{\omega_1}}$ is closed for all $\alpha \in \mathbb{R}$, then $t \mapsto \Delta_{\omega_1}(t, u(t), x, z)$ is measurable. Indeed, let sequence $\{t_n\} \subset \mathbf{D}_\alpha^{\Delta_{\omega_1}}$ satisfy $t_n \to t$ for some $t \in [0, T]$. Then, for each $n \in \mathbb{N}$,

$$\begin{aligned}
\alpha &\geq \Delta_{\omega_1}(t_n, u(t), x, z) \\
&\geq \langle F(t_n, z(t_n), x(t_n)) + Q\left(t_n, \left(\mathcal{S}_1 x\right)(t_n), u(t)\right), x(t_n) - w\rangle \\
&\quad - \Upsilon\left(\left(\mathcal{S}_2 x\right)(t_n), x(t_n), w\right) + \Upsilon\left(\left(\mathcal{S}_2 x\right)(t_n), x(t_n), x(t_n)\right) \\
&\quad - J_1^\circ\left(\mathbf{N} z(t_n), \mathbf{M} x(t_n); \mathbf{M} w - \mathbf{M} x(t_n)\right) - \frac{1}{2\omega_1}\|x(t_n) - w\|_{\mathbf{X}_1}^2
\end{aligned} \tag{10}$$

for all $w \in P_1$. Recall that $t_n \to t$ and u, x, z are continuous, so we obtain $u(t_n) \to u(t)$ in $\mathbf{U}$, $x(t_n) \to x(t)$ in $\mathbf{X}_1$ and $z(t_n) \to z(t)$ in $\mathbf{X}_2$ as $n \to \infty$. Passing to the lower limit as

$n \to \infty$ in (10) and under hypotheses $\mathfrak{h}(F)(a,c)$, $\mathfrak{h}(Q)(b)$, $\mathfrak{h}(\Upsilon)(d)$, $\mathfrak{h}(J_1)(d)$, $\mathfrak{h}(\mathcal{S})$, $\mathfrak{h}(\mathbf{M})$ and the upper semicontinuity of $J_1^\circ(v,\cdot;\cdot)$ for all $v \in \mathbf{W}_2$, one has

$$\begin{aligned}
\alpha &\geq \liminf_{n\to\infty} \langle F(t_n, z(t_n), x(t_n)) + Q\left(t_n, \left(\mathcal{S}_1 x\right)(t_n), u(t_n)\right), x(t_n) - w\rangle \\
&\quad + \liminf_{n\to\infty} \left(\Upsilon\left(\left(\mathcal{S}_2 x\right)(t_n), x(t_n), x(t_n)\right) - \Upsilon\left(\left(\mathcal{S}_2 x\right)(t_n), x(t_n), w\right)\right) \\
&\quad - \limsup_{n\to\infty} J_1^\circ\left(\mathbf{N}z(t_n), \mathbf{M}x(t_n); \mathbf{M}w - \mathbf{M}x(t_n)\right) \\
&\quad - \limsup_{n\to\infty} \left(\frac{1}{2\omega_1}\|x(t_n) - w\|_{\mathbf{X}_1}^2\right) \\
&\geq \langle F(t, z(t), x(t)) + Q\left(t, \left(\mathcal{S}_1 x\right)(t), u(t)\right), x(t) - w\rangle - \Upsilon\left(\left(\mathcal{S}_2 x\right)(t), x(t), w\right) \\
&\quad + \Upsilon\left(\left(\mathcal{S}_2 x\right)(t), x(t), x(t)\right) - J_1^\circ\left(\mathbf{N}z(t), \mathbf{M}x(t); \mathbf{M}w - \mathbf{M}x(t)\right) \\
&\quad - \frac{1}{2\omega_1}\|x(t) - w\|_{\mathbf{X}_1}^2 \\
&= \Xi_{\omega_1}(t, u(t), x, z, w)
\end{aligned}$$

for all $w \in P_1$. Passing the supremum with $w \in P_1$ in the inequalities above, one has

$$\Delta_{\omega_1}(t, u(t), x, z) = \sup_{w\in P_1} \Xi_{\omega_1}(t, u(t), x, z, w) \leq \alpha.$$

This implies that $t \in \mathbf{D}_\alpha^{\Delta_{\omega_1}}$, namely, $\mathbf{D}_\alpha^{\Delta_{\omega_1}}$ is closed. Therefore, we obtain that the function $t \mapsto \Delta_{\omega_1}(t, u(t), x, z)$ is measurable on the interval $[0, T]$.

Next, we will show that $t \mapsto \Delta_{\omega_1}(t, u(t), x, z)$ is uniformly bounded. Using assumptions $\mathfrak{h}(F)(c)$, $\mathfrak{h}(Q)(a)$, $\mathfrak{h}(\Upsilon)(c)$, $\mathfrak{h}(\mathcal{S})$, $\forall y \in P_1$, we obtain

$$\begin{aligned}
&\langle F(t, z(t), x(t)) + Q\left(t, \left(\mathcal{S}_1 x\right)(t), u(t)\right), x(t) - w\rangle \\
&\quad = \langle F(t, z(t), x(t)) - F(t, 0_{\mathbf{X}_2}, x(t), x(t) - w\rangle \\
&\qquad + \langle Q\left(t, \left(\mathcal{S}_1 x\right)(t), u(t)\right) - Q\left(t, \mathcal{S}_1(0_{\mathbf{X}_1}), u(t)\right), x(t) - w\rangle \\
&\qquad + \langle F(t, 0_{\mathbf{X}_2}, x(t), x(t) - w\rangle + \langle Q\left(t, \mathcal{S}_1(0_{\mathbf{X}_1}), u(t)\right), x(t) - w\rangle \\
&\quad \leq \left(L_F\|z(t)\|_{\mathbf{X}_2} + L_Q\|\left(\mathcal{S}_1 x\right)(t) - \mathcal{S}_1(0_{\mathbf{X}_1})\|_{\mathbf{V}_1}\right)\|x(t) - w\|_{\mathbf{X}_1} \\
&\qquad + (\|F(t, 0_{\mathbf{X}_2}, x(t))\|_{\mathbf{X}_1^*} + \|Q\left(t, \mathcal{S}_1(0_{\mathbf{X}_1}), u(t)\right)\|_{\mathbf{X}_1^*})\|x(t) - w\|_{\mathbf{X}_1} \\
&\quad \leq \left(L_F\|z(t)\|_{\mathbf{X}_2} + L_Q a_{S_1} T\|x\|_{L^1(0,T,\mathbf{X}_1)}\right)\left(\|x(t)\|_{\mathbf{X}_1} + \|w\|_{\mathbf{X}_1}\right) \\
&\qquad + (\|F(t, 0_{\mathbf{X}_2}, x(t))\|_{\mathbf{X}_1^*} + \|Q\left(t, \mathcal{S}_1(0_{\mathbf{X}_1}), u(t)\right)\|_{\mathbf{X}_1^*})(\|x(t)\|_{\mathbf{X}_1} + \|w\|_{\mathbf{X}_1})
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
&\Upsilon\left((\mathcal{S}_2 x)(t), x(t), x(t)\right) - \Upsilon\left((\mathcal{S}_2 x)(t), x(t), w\right) \\
&\quad \leq \varrho_\Upsilon\left(\|(\mathcal{S}_2 x)(t)\|_{\mathbf{V}_2}, \|x(t)\|_{\mathbf{X}_1}\right)\|x(t) - w\|_{\mathbf{X}_1} \\
&\quad \leq \varrho_\Upsilon\left(\|(\mathcal{S}_2 x)(t)\|_{\mathbf{V}_2}, \|x(t)\|_{\mathbf{X}_1}\right)(\|x(t)\|_{\mathbf{X}_1} + \|w\|_{\mathbf{X}_1}),
\end{aligned} \tag{12}$$

where the function $\varrho_\Upsilon\colon \mathbb{R}^2 \to [0,\infty)$ is continuous. It follows from conditions $\mathfrak{h}(J_1)(b)$, $\mathfrak{h}(\mathbf{M})$ and $\mathfrak{h}(\mathbf{N})$ that

$$\begin{aligned}
&-J_1^\circ\left(\mathbf{N}z(t),\mathbf{M}x(t);\mathbf{M}w-\mathbf{M}x(t)\right)\\
&\quad\le -\left\langle \xi_{\mathbf{M}x(t)},\mathbf{M}w-\mathbf{M}x(t)\right\rangle\\
&\quad\le \left\|\xi_{\mathbf{M}x(t)}\right\|_{\mathbf{W}_1{}^*}\|\mathbf{M}\|\|w-x(t)\|_{\mathbf{X}_1}\\
&\quad\le \left(a_{J_1}+b_{J_1}\left(\|\mathbf{M}\|\|x(t)\|_{\mathbf{X}_1}+\|\mathbf{N}\|\|z(t)\|_{\mathbf{X}_2}\right)\right)\|\mathbf{M}\|(\|x(t)\|_{\mathbf{X}_1}+\|w\|_{\mathbf{X}_1})
\end{aligned}\tag{13}$$

for all $\xi_{\mathbf{M}x(t)}\in\partial J_1(\mathbf{N}z(t),\mathbf{M}x(t))$.

Having in relations (11)–(13) and using the boundedness of the operators $\mathbf{N},\mathbf{M}$, the boundedness assumptions $\mathfrak{h}(P)$(b), $\mathfrak{h}(F)(d)$, and $\mathfrak{h}(Q)(b)$, for each $(x,z)\in C([0,T];P_1)\times C([0,T];P_2)$ fixed, there exists a positive constant $\mathbf{C}^*$ satisfy the following

$$\begin{aligned}
&\Xi_{\omega_1}(t,u(t),x,z,w)\\
&\quad=\langle F(t,z(t),x(t))+Q\left(t,(\mathcal{S}_1x)(t),u(t)\right),x(t)-w\rangle-\Upsilon\left((\mathcal{S}_2x)(t),x(t),w\right)\\
&\qquad+\Upsilon\left((\mathcal{S}_2x)(t),x(t),x(t)\right)-J_1^\circ\left(\mathbf{N}z(t),\mathbf{M}x(t);\mathbf{M}w-\mathbf{M}x(t)\right)\\
&\qquad-\frac{1}{2\omega_1}\|x(t)-w\|_{\mathbf{X}_1}^2\\
&\quad\le\Big(L_F\|z(t)\|_{\mathbf{X}_2}+L_Qa_{\mathcal{S}_1}T\|x\|_{L^1(0,T,\mathbf{X}_1)}+\|F(t,0_{\mathbf{X}_2},x(t))\|_{\mathbf{X}_1^*}\\
&\qquad+\left\|Q\left(t,\mathcal{S}_1(0_{\mathbf{X}_1}),u(t)\right)\right\|_{\mathbf{X}_1^*}+\varrho_\Upsilon\left(\|(\mathcal{S}_2x)(t)\|_{\mathbf{V}_1},\|x(t)\|_{\mathbf{X}_1}\right)\\
&\qquad+\left(a_{J_1}+b_{J_1}\left(\|\mathbf{M}\|\|x(t)\|_{\mathbf{X}_1}+\|\mathbf{N}\|\|z(t)\|_{\mathbf{X}_2}\right)\right)\|\mathbf{M}\|\Big)\left(\|x(t)\|_{\mathbf{X}_1}+\|w\|_{\mathbf{X}_1}\right)\\
&\quad\le\mathbf{C}^*\quad\text{for all } w\in P_1,
\end{aligned}$$

where the constant $\mathbf{C}^*$ is independent of $t\in[0,T]$ and $w\in P_1$. Hence, we have

$$0\le\Delta_{\omega_1}(t,u(t),x,z)=\sup_{w\in P_1}\Xi_{\omega_1}(t,u(t),x,z,w)\le\mathbf{C}^*,\quad\text{for all } t\in[0,T]$$

which implies that $t\mapsto\Delta_{\omega_1}(t,u(t),x,z)$ is essentially bounded. Hence, we can conclude that $t\mapsto\Delta_{\omega_1}(t,u(t),x,z)$ is a uniformly bounded function.

Thus, given any parameter $\omega_1>0$, we obtain that the function $t\mapsto\Delta_{\omega_1}(t,u(t),x,z)$ belongs to $L_+^\infty(0,T)$ for each fixed $(u,x,z)\in C([0,T];\mathbf{U})\times C([0,T];P_1)\times C([0,T];P_2)$. ■

Proposition 3.4: *Under hypotheses $\mathfrak{h}(P)$, $\mathfrak{h}(G)$, $\mathfrak{h}(\varphi)$, $\mathfrak{h}(J_2)$, $\mathfrak{h}(\mathbf{N})$ and $\mathfrak{h}(\mathbf{M})$, for any parameter $\omega_2>0$, the function Θ_{ω_2}, defined by (7), has the following properties:*

(i) *For each $t\in[0,T]$, $(z,x)\mapsto\Theta_{\omega_2}(t,z,x)$ is a lower semicontinuous function.*
(ii) *For each $(x,z)\in C([0,T];P_1)\times C([0,T];P_2)$ fixed, the function $t\mapsto\Theta_{\omega_2}(t,z,x)$ belongs to $L_+^\infty(0,T)$.*

Proof: Similar to the proof of Proposition 3.3. ■

For the study of error bounds to Problem 1.1, we need the following hypothesis, which is stronger than hypothesis $\mathfrak{h}(0)$.

$\underline{\mathfrak{h}'(0)}$: (a) There exist $\omega_1, \omega_2 > 0$ such that

$$a_F > m_{J_1}\|\mathbf{M}\|^2 + a_\Upsilon + \frac{1}{2\omega_1}, \quad a_G > m_{J_2}\|\mathbf{N}\|^2 + \frac{1}{2\omega_2};$$

(b) The following inequality holds:

$$\frac{\left(m_{J_2}\|\mathbf{N}\|\,\|\mathbf{M}\| + L_G\right)\left(m_{J_1}\|\mathbf{M}\|\,\|\mathbf{N}\| + L_F\right)}{\left(a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}\right)\left(a_G - m_{J_2}\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}\right)} < \frac{1}{2}.$$

Remark 3.2: (i) For each $\omega_1, \omega_2 > 0$, it is clear that $\mathfrak{h}'(0)$(a) implies $\mathfrak{h}(0)$(a).

(ii) If conditions $\mathfrak{h}'(0)(a, b)$ hold, then we have

$$\begin{aligned}
&\frac{\left(m_{J_2}\|\mathbf{N}\|\,\|\mathbf{M}\| + L_G\right)\left(m_{J_1}\|\mathbf{M}\|\,\|\mathbf{N}\| + L_F\right)}{\left(a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon\right)\left(a_G - m_{J_2}\|\mathbf{N}\|^2\right)} \\
&\quad < \frac{\left(m_{J_2}\|\mathbf{N}\|\,\|\mathbf{M}\| + L_G\right)\left(m_{J_1}\|\mathbf{M}\|\,\|\mathbf{N}\| + L_F\right)}{\left(a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}\right)\left(a_G - m_{J_2}\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}\right)} \\
&\quad < \frac{1}{2} < 1.
\end{aligned}$$

Thus, the assumption $\mathfrak{h}(0)$(b) is valid. Thus, $\mathfrak{h}'(0)$ implies $\mathfrak{h}(0)$.

Estimating error bounds is crucial for understanding the accuracy and reliability of the solutions to the problem. We are now ready to provide our main results on global error bounds for Problem 1.1 in the following results.

Theorem 3.5: *Let $u_0 \in \mathbf{U}$ be fixed and a triple $(u^*, x^*, z^*) \in C([0,T];\mathbf{U}) \times C([0,T];P_1) \times C([0,T];P_2)$ be the unique solution of Problem 1.1. Assume that $\mathfrak{h}(A)$, $\mathfrak{h}(P)$, $\mathfrak{h}(H)$, $\mathfrak{h}(Q)$, $\mathfrak{h}(J_1)$, $\mathfrak{h}(J_2)$, $\mathfrak{h}(\Upsilon)$, $\mathfrak{h}(\mathcal{S})$, $\mathfrak{h}(\mathbf{M})$, $\mathfrak{h}(\mathbf{N})$ and $\mathfrak{h}(\varphi)$ hold. If, in addition, parameters $\omega_1, \omega_2 > 0$ satisfy the hypothesis $\mathfrak{h}'(0)$, then for each a triple $(u, x, z) \in C([0,T];\mathbf{U}) \times C([0,T];P_1) \times C([0,T];P_2)$, there exist the functions $\Pi_i \in L^\infty_+(0,T)$ with $i = 1, 2, 3$ such that*

$$\|x(t) - x^*(t)\|_{X_1} \le \Pi_1(t), \quad \text{for all } t \in [0,T]; \tag{14}$$

$$\|z(t) - z^*(t)\|_{X_2} \le \Pi_2(t), \quad \text{for all } t \in [0,T]; \tag{15}$$

and

$$\|u(t) - u^*(t)\|_{\mathbf{U}} \le \Pi_3(t), \quad \text{for all } t \in [0,T], \tag{16}$$

where $u \in C([0,T];\mathbf{U})$ is the unique solution of the following Cauchy problem

$$\begin{cases} u'(t) = Au(t) + H(t, u(t), x(t)) & \text{for all } t \in [0,T] \\ u(0) = u_0 \end{cases} \tag{17}$$

and

$$\Pi_1(t) = \frac{\mho_{\omega_1,\omega_2}(t,u(t),x,z)}{\mathbf{C}_1} + \frac{\mathbf{C}_2}{\mathbf{C}_1^2}\int_0^t \mho_{\omega_1,\omega_2}(s,u(s),x,z)\exp\left\{\frac{\mathbf{C}_2}{\mathbf{C}_1}(t-s)\right\}\mathrm{d}s; \tag{18}$$

$$\Pi_2(t) = \sqrt{\frac{2\Theta_{\omega_2}(t,z,x)}{a_G - m_{J_2}\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}}} + \frac{L_G + m_{J_2}\|\mathbf{N}\|\,\|\mathbf{M}\|}{a_G - m_{J_2}\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}}\Pi_1(t); \tag{19}$$

$$\Pi_3(t) = a_H m_H\,\mathrm{e}^{a_H m_H T}\int_0^t \Pi_1(s)\,\mathrm{d}s, \tag{20}$$

with the function $\mho_{\omega_1,\omega_2}\colon [0,T]\times\mathbf{U}\times C([0,T];P_1)\times C([0,T];P_2)\to\mathbb{R}_+$ *given by*

$$\begin{aligned}\mho_{\omega_1,\omega_2}(t,u(t),x,z) = 2&\sqrt{\frac{\Delta_{\omega_1}(t,u(t),x,z)}{\left(a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}\right)}}\\ &+ \frac{2(L_F + m_{J_1}\|\mathbf{N}\|\,\|\mathbf{M}\|)}{\left(a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}\right)}\sqrt{\frac{2\Theta_{\omega_2}(t,z,x)}{a_G - m_{J_2}\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}}}\end{aligned} \tag{21}$$

and constants $\mathbf{C}_1, \mathbf{C}_2 > 0$ *defined by*

$$\begin{cases}\mathbf{C}_1 = 1 - \dfrac{2(L_F + m_{J_1}\|\mathbf{N}\|\,\|\mathbf{M}\|)(L_G + m_{J_2}\|\mathbf{N}\|\,\|\mathbf{M}\|)}{\left(a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}\right)\left(a_G - m_{J_2}\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}\right)};\\ \mathbf{C}_2 = \dfrac{2(L_Q a_{\mathcal{S}_1} + L_Q a_H m_H\,\mathrm{e}^{a_H m_H T} + b_\Upsilon a_{\mathcal{S}_2})}{\left(a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}\right)}.\end{cases} \tag{22}$$

Proof: Let $(u,x,z)\in C([0,T];\mathbf{U})\times C([0,T];P_1)\times C([0,T];P_2)$, $u_0\in\mathbf{U}$ be fixed and $(u^*,x^*,z^*)\in C([0,T];\mathbf{U})\times C([0,T];P_1)\times C([0,T];P_2)$ be the unique solution to Problem 1.1, where u is the unique solution of the Equation (17). Then, we obtain

$$u^*(t) = \mathcal{T}(t)u_0 + \int_0^t \mathcal{T}(t-s)H(s,u^*(s),x^*(s))\,\mathrm{d}s \quad \text{for all } t\in[0,T], \tag{23}$$

and

$$\begin{aligned}&\left\langle F(t,z^*(t),x^*(t)) + Q\left(t,\left(\mathcal{S}_1x^*\right)(t),u^*(t)\right), y - x^*(t)\right\rangle + \Upsilon\left(\left(\mathcal{S}_2x^*\right)(t),x^*(t),y\right)\\ &\quad - \Upsilon\left(\left(\mathcal{S}_2x^*\right)(t),x^*(t),x^*(t)\right) + J_1^\circ\left(\mathbf{N}z^*(t),\mathbf{M}x^*(t);\mathbf{M}y - \mathbf{M}x^*(t)\right)\geqslant 0,\\ &\langle G(t,x^*(t),z^*(t)), v - z^*(t)\rangle + J_2^\circ\left(\mathbf{M}x^*(t),\mathbf{N}z^*(t);\mathbf{N}v - \mathbf{N}z^*(t)\right)\\ &\quad + \varphi(v) - \varphi(z^*(t))\geqslant\langle f(t), v - z^*(t)\rangle,\end{aligned}$$

for all $(y,v)\in P_1\times P_2$ and $t\in[0,T]$. Inserting $y = x(t)$ and $v = z(t)$ into the above inequalities, one has

$$\begin{aligned}&\left\langle F(t,z^*(t),x^*(t)) + Q\left(t,\left(\mathcal{S}_1x^*\right)(t),u^*(t)\right), x(t) - x^*(t)\right\rangle\\ &\quad + \Upsilon\left(\left(\mathcal{S}_2x^*\right)(t),x^*(t),x(t)\right) - \Upsilon\left(\left(\mathcal{S}_2x^*\right)(t),x^*(t),x^*(t)\right)\\ &\quad + J_1^\circ\left(\mathbf{N}z^*(t),\mathbf{M}x^*(t);\mathbf{M}x(t) - \mathbf{M}x^*(t)\right)\geqslant 0\end{aligned} \tag{24}$$

and

$$\begin{aligned}&\langle G(t,x^*(t),z^*(t))-f(t),z(t)-z^*(t)\rangle+J_2^\circ\left(\mathbf{M}x^*(t),\mathbf{N}z^*(t);\mathbf{N}z(t)-\mathbf{N}z^*(t)\right)\\&\quad+\varphi(z(t))-\varphi(z^*(t))\geqslant 0.\end{aligned}\tag{25}$$

By the definition of Θ_{ω_2}, we deduce

$$\begin{aligned}&\Theta_{\omega_2}(t,z,x)\\&\quad=\sup_{v\in P_2}\Bigg(\langle G(t,x(t),z(t))-f(t),z(t)-v\rangle-J_2^\circ\left(\mathbf{M}x(t),\mathbf{N}z(t);\mathbf{N}v-\mathbf{N}z(t)\right)\\&\qquad-\varphi(v)+\varphi(z(t))-\frac{1}{2\omega_2}\|z(t)-v\|_{\mathbf{X}_2}^2\Bigg)\\&\quad\geq\langle G(t,x(t),z(t))-f(t),z(t)-z^*(t)\rangle-J_2^\circ\left(\mathbf{M}x(t),\mathbf{N}z(t);\mathbf{N}z^*(t)-\mathbf{N}z(t)\right)\\&\qquad-\varphi(z^*(t))+\varphi(z(t))-\frac{1}{2\omega_2}\|z(t)-z^*(t)\|_{\mathbf{X}_2}^2.\end{aligned}\tag{26}$$

From hypotheses $\mathfrak{h}(G)(b,c)$, we have

$$\begin{aligned}&\langle G(t,x(t),z(t)),z(t)-z^*(t)\rangle-\langle G(t,x^*(t),z^*(t)),z(t)-z^*(t)\rangle\\&\quad=\langle G(t,x(t),z(t))-G(t,x(t),z^*(t)),z(t)-z^*(t)\rangle\\&\qquad-\langle G(t,x^*(t),z^*(t))-G(t,x(t),z^*(t)),z(t)-z^*(t)\rangle\\&\quad\geq a_G\|z(t)-z^*(t)\|_{\mathbf{X}_2}^2-L_G\|x(t)-x^*(t)\|_{\mathbf{X}_1}\|z(t)-z^*(t)\|_{\mathbf{X}_2}.\end{aligned}\tag{27}$$

Using assumptions $\mathfrak{h}(\mathbf{M})$, $\mathfrak{h}(\mathbf{N})$, $\mathfrak{h}(J_2)(b)$, one has

$$\begin{aligned}&-J_2^\circ\left(\mathbf{M}x(t),\mathbf{N}z(t);\mathbf{N}z^*(t)-\mathbf{N}z(t)\right)-J_2^\circ\left(\mathbf{M}x^*(t),\mathbf{N}z^*(t);\mathbf{N}z(t)-\mathbf{N}z^*(t)\right)\\&\quad\geq-m_{J_2}\left\|\mathbf{N}z(t)-\mathbf{N}z^*(t)\right\|_{\mathbf{W}_2}\left(\left\|\mathbf{M}x(t)-\mathbf{M}x^*(t)\right\|_{\mathbf{W}_1}+\left\|\mathbf{N}z(t)-\mathbf{N}z^*(t)\right\|_{\mathbf{W}_2}\right)\\&\quad\geq-m_{J_2}\|\mathbf{N}\|\|\mathbf{M}\|\|z(t)-z^*(t)\|_{\mathbf{X}_2}\|x(t)-x^*(t)\|_{\mathbf{X}_1}-m_{J_2}\|\mathbf{N}\|^2\|z(t)-z^*(t)\|_{\mathbf{X}_2}^2.\end{aligned}\tag{28}$$

Combining (25) together with inequalities (26)–(28) yields

$$\begin{aligned}\Theta_{\omega_2}(t,z,x)&\geq\left(a_G-m_{J_2}\|\mathbf{N}\|^2-\frac{1}{2\omega_2}\right)\|z(t)-z^*(t)\|_{\mathbf{X}_2}^2\\&\quad-(L_G+m_{J_2}\|\mathbf{N}\|\|\mathbf{M}\|)\|z(t)-z^*(t)\|_{\mathbf{X}_2}\|x(t)-x^*(t)\|_{\mathbf{X}_1}\\&\quad+\langle G(t,x^*(t),z^*(t))-f(t),z(t)-z^*(t)\rangle\\&\quad+J_2^\circ\left(\mathbf{M}x^*(t),\mathbf{N}z^*(t);\mathbf{N}z(t)-\mathbf{N}z^*(t)\right)+\varphi(z(t))-\varphi(z^*(t))\\&\geq\left(a_G-m_{J_2}\|\mathbf{N}\|^2-\frac{1}{2\omega_2}\right)\|z(t)-z^*(t)\|_{\mathbf{X}_2}^2\\&\quad-(L_G+m_{J_2}\|\mathbf{N}\|\|\mathbf{M}\|)\|z(t)-z^*(t)\|_{\mathbf{X}_2}\|x(t)-x^*(t)\|_{\mathbf{X}_1}.\end{aligned}$$

The above inequality implies that

$$\left(a_G - m_{J_2}\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}\right)\|z(t) - z^*(t)\|^2_{\mathbf{X}_2}$$
$$\leq \Theta_{\omega_2}(t, z, x) + (L_G + m_{J_2}\|\mathbf{N}\|\|\mathbf{M}\|)\|z(t) - z^*(t)\|_{\mathbf{X}_2}\|x(t) - x^*(t)\|_{\mathbf{X}_1}. \tag{29}$$

Employing Young's inequality with $\varepsilon_2 = \frac{a_G - m_{J_2}\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}}{2} > 0$, we get

$$(L_G + m_{J_2}\|\mathbf{N}\|\|\mathbf{M}\|)\|z(t) - z^*(t)\|_{\mathbf{X}_2}\|x(t) - x^*(t)\|_{\mathbf{X}_1}$$
$$\leq \varepsilon_2\|z(t) - z^*(t)\|^2_{\mathbf{X}_2} + \frac{(L_G + m_{J_2}\|\mathbf{N}\|\|\mathbf{M}\|)^2}{4\varepsilon_2}\|x(t) - x^*(t)\|^2_{\mathbf{X}_1}.$$

So, the inequality (29) implies

$$\frac{a_G - m_{J_2}\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}}{2}\|z(t) - z^*(t)\|^2_{\mathbf{X}_2}$$
$$\leq \Theta_{\omega_2}(t, z, x) + \frac{(L_G + m_{J_2}\|\mathbf{N}\|\|\mathbf{M}\|)^2}{2\left(a_G - m_{J_2}\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}\right)}\|x(t) - x^*(t)\|^2_{\mathbf{X}_1}$$

for all $t \in [0, T]$. This implies that

$$\|z(t) - z^*(t)\|_{\mathbf{X}_2}$$
$$\leq \sqrt{\frac{2\Theta_{\omega_2}(t, z, x)}{a_G - m_{J_2}\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}}} + \frac{L_G + m_{J_2}\|\mathbf{N}\|\|\mathbf{M}\|}{a_G - m_{J_2}\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}}\|x(t) - x^*(t)\|_{\mathbf{X}_1} \tag{30}$$

for all $t \in [0, T]$.

On the other hand, it follows from the definition of Δ_{ω_1} that

$$\Delta_{\omega_1}(t, u(t), x, z)$$
$$= \sup_{y \in P_1}\Big(\big\langle F(t, z(t), x(t)) + Q\left(t, (\mathcal{S}_1 x)(t), u(t)\right), x(t) - y\big\rangle$$
$$- \Upsilon\left((\mathcal{S}_2 x)(t), x(t), y\right) + \Upsilon\left((\mathcal{S}_2 x)(t), x(t), x(t)\right)$$
$$- J_1^\circ\left(\mathbf{N}z(t), \mathbf{M}x(t); \mathbf{M}y - \mathbf{M}x(t)\right) - \frac{1}{2\omega_1}\|x(t) - y\|^2_{\mathbf{X}_1}\Big)$$
$$\geq \big\langle F(t, z(t), x(t)) + Q\left(t, (\mathcal{S}_1 x)(t), u(t)\right), x(t) - x^*(t)\big\rangle$$
$$- \Upsilon\left((\mathcal{S}_2 x)(t), x(t), x^*(t)\right) + \Upsilon\left((\mathcal{S}_2 x)(t), x(t), x(t)\right)$$
$$- J_1^\circ\left(\mathbf{N}z(t), \mathbf{M}x(t); \mathbf{M}x^*(t) - \mathbf{M}x(t)\right) - \frac{1}{2\omega_1}\|x(t) - x^*(t)\|^2_{\mathbf{X}_1}. \tag{31}$$

Using hypotheses $\mathfrak{h}(\Upsilon)(b)$ and $\mathfrak{h}(\mathcal{S})$ leads to

$$\begin{aligned}
&- \Upsilon\left((\mathcal{S}_2 x)(t), x(t), x^*(t)\right) + \Upsilon\left((\mathcal{S}_2 x)(t), x(t), x(t)\right) \\
&\quad - \Upsilon\left(\left(\mathcal{S}_2 x^*\right)(t), x^*(t), x(t)\right) + \Upsilon\left(\left(\mathcal{S}_2 x^*\right)(t), x^*(t), x^*(t)\right) \\
&\geq -a_\Upsilon \|x(t) - x^*(t)\|_{\mathbf{X}_1}^2 - b_\Upsilon \|x(t) - x^*(t)\|_{\mathbf{X}_1} \left\|(\mathcal{S}_2 x)(t) - (\mathcal{S}_2 x^*)(t)\right\|_{\mathbf{V}_2} \\
&\geq -a_\Upsilon \|x(t) - x^*(t)\|_{\mathbf{X}_1}^2 - b_\Upsilon a_{\mathcal{S}_2} \int_0^t \|x(s) - x^*(s)\|_{\mathbf{X}_1} \, \mathrm{d}s \|x(t) - x^*(t)\|_{\mathbf{X}_1}. \qquad (32)
\end{aligned}$$

From assumptions $\mathfrak{h}(F)(b,c)$, $\mathfrak{h}(Q)(b)$, $\mathfrak{h}(\mathcal{S})$, $\mathfrak{h}(\mathbf{M})$, $\mathfrak{h}(\mathbf{M})$ and $\mathfrak{h}(J_2)$, by similar calculations as in (26) and (27), we have

$$\begin{aligned}
&\left\langle F(t, z(t), x(t)) + Q\left(t, (\mathcal{S}_1 x)(t), u(t)\right), x(t) - x^*(t)\right\rangle \\
&\quad - J_1^\circ\left(\mathbf{N}z(t), \mathbf{M}x(t); \mathbf{M}x^*(t) - \mathbf{M}x(t)\right) \\
&\quad - \left\langle F(t, z^*(t), x^*(t)) + Q\left(t, \left(\mathcal{S}_1 x^*\right)(t), u^*(t)\right), x(t) - x^*(t)\right\rangle \\
&\quad - J_1^\circ\left(\mathbf{N}z^*(t), \mathbf{M}x^*(t); \mathbf{M}x(t) - \mathbf{M}x^*(t)\right) \\
&\geq a_F \|x(t) - x^*(t)\|_{\mathbf{X}_1}^2 - L_F \|z(t) - z^*(t)\|_{\mathbf{X}_2} \|x(t) - x^*(t)\|_{\mathbf{X}_1} \\
&\quad - L_Q \left(\left\|(\mathcal{S}_1 x)(t) - \left(\mathcal{S}_1 x^*\right)(t)\right\|_{\mathbf{V}_1} + \left\|u(t) - u^*(t)\right\|_{\mathbf{U}}\right) \|x(t) - x^*(t)\|_{\mathbf{X}_1} \\
&\quad - m_{J_1} \|\mathbf{N}\| \|\mathbf{M}\| \|x(t) - x^*(t)\|_{\mathbf{X}_1} \|z(t) - z^*(t)\|_{\mathbf{X}_2} \\
&\quad - m_{J_1} \|\mathbf{M}\|^2 \|x(t) - x^*(t)\|_{\mathbf{X}_1} \\
&\geq (a_F - m_{J_1} \|\mathbf{M}\|^2) \|x(t) - x^*(t)\|_{\mathbf{X}_1}^2 \\
&\quad - (L_F + m_{J_1} \|\mathbf{N}\| \|\mathbf{M}\|) \|x(t) - x^*(t)\|_{\mathbf{X}_1} \|z(t) - z^*(t)\|_{\mathbf{X}_2} \\
&\quad - L_Q a_{\mathcal{S}_1} \int_0^t \|x(s) - x^*(s)\|_{\mathbf{X}_1} \, \mathrm{d}s \|x(t) - x^*(t)\|_{\mathbf{X}_1} \\
&\quad - L_Q \left\|u(t) - u^*(t)\right\|_{\mathbf{U}} \|x(t) - x^*(t)\|_{\mathbf{X}_1}. \qquad (33)
\end{aligned}$$

In the point of view of (24), we combine inequalities (31)–(33) to obtain

$$\begin{aligned}
\Delta_{\omega_1}(t, u(t), x, z) \geq{}& \left(a_F - m_{J_1} \|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}\right) \|x(t) - x^*(t)\|_{\mathbf{X}_1}^2 \\
&- (L_F + m_{J_1} \|\mathbf{N}\| \|\mathbf{M}\|) \|x(t) - x^*(t)\|_{\mathbf{X}_1} \|z(t) - z^*(t)\|_{\mathbf{X}_2} \\
&- (L_Q a_{\mathcal{S}_1} + b_\Upsilon a_{\mathcal{S}_2}) \int_0^t \|x(t) - x^*(t)\|_{\mathbf{X}_1} \, \mathrm{d}s \|x(t) - x^*(t)\|_{\mathbf{X}_1} \\
&- L_Q \left\|u(t) - u^*(t)\right\|_{\mathbf{U}} \|x(t) - x^*(t)\|_{\mathbf{X}_1},
\end{aligned}$$

that is,

$$\begin{aligned}&\left(a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}\right)\|x(t) - x^*(t)\|^2_{\mathbf{X}_1}\\&\quad\leq \Delta_{\omega_1}(t,u(t),x,z) + L_Q\left\|u(t) - u^*(t)\right\|_{\mathbf{U}}\|x(t) - x^*(t)\|_{\mathbf{X}_1}\\&\qquad + (L_F + m_{J_1}\|\mathbf{N}\|\|\mathbf{M}\|)\|x(t) - x^*(t)\|_{\mathbf{X}_1}\|z(t) - z^*(t)\|_{\mathbf{X}_2}\\&\qquad + (L_Q a_{\mathcal{S}_1} + b_\Upsilon a_{\mathcal{S}_2})\int_0^t \|x(s) - x^*(s)\|_{\mathbf{X}_1}\,\mathrm{d}s\|x(t) - x^*(t)\|_{\mathbf{X}_1}.\end{aligned} \tag{34}$$

Let $\varepsilon_1 = \frac{a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}}{4} > 0$. Thanks to Young's inequality, we have

$$\begin{aligned}&L_Q\left\|u(t) - u^*(t)\right\|_{\mathbf{U}}\|x(t) - x^*(t)\|_{\mathbf{X}_1}\\&\quad\leq \varepsilon_1\|x(t) - x^*(t)\|^2_{\mathbf{X}_1} + \frac{L_Q^2}{4\varepsilon_1}\left\|u(t) - u^*(t)\right\|^2_{\mathbf{U}},\\&(L_F + m_{J_1}\|\mathbf{N}\|\|\mathbf{M}\|)\|x(t) - x^*(t)\|_{\mathbf{X}_1}\|z(t) - z^*(t)\|_{\mathbf{X}_2}\\&\quad\leq \varepsilon_1\|x(t) - x^*(t)\|^2_{\mathbf{X}_1} + \frac{(L_F + m_{J_1}\|\mathbf{N}\|\|\mathbf{M}\|)^2}{4\varepsilon_1}\|z(t) - z^*(t)\|^2_{\mathbf{X}_2}\end{aligned}$$

and

$$\begin{aligned}&(L_Q a_{\mathcal{S}_1} + b_\Upsilon a_{\mathcal{S}_2})\int_0^t \|x(t) - x^*(t)\|_{\mathbf{X}_1}\,\mathrm{d}s\|x(t) - x^*(t)\|_{\mathbf{X}_1}\\&\quad\leq \varepsilon_1\|x(t) - x^*(t)\|^2_{\mathbf{X}_1} + \frac{(L_Q a_{\mathcal{S}_1} + b_\Upsilon a_{\mathcal{S}_2})^2}{4\varepsilon_1}\left(\int_0^t \|x(s) - x^*(s)\|_{\mathbf{X}_1}\,\mathrm{d}s\right)^2.\end{aligned}$$

Substituting the above inequalities into (34), we get

$$\begin{aligned}&\frac{\left(a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}\right)}{4}\|x(t) - x^*(t)\|^2_{\mathbf{X}_1}\\&\quad\leq \Delta_{\omega_1}(t,u(t),x,z) + \frac{L_Q^2}{a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}}\left\|u(t) - u^*(t)\right\|^2_{\mathbf{U}}\\&\qquad + \frac{(L_F + m_{J_1}\|\mathbf{N}\|\|\mathbf{M}\|)^2}{a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}}\|z(t) - z^*(t)\|^2_{\mathbf{X}_2}\\&\qquad + \frac{(L_Q a_{\mathcal{S}_1} + b_\Upsilon a_{\mathcal{S}_2})^2}{a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}}\left(\int_0^t \|x(s) - x^*(s)\|_{\mathbf{X}_1}\,\mathrm{d}s\right)^2.\end{aligned}$$

for all $t \in [0, T]$. Then, we have

$$\begin{aligned}\|x(t)-x^*(t)\|_{\mathbf{X}_1} \leq 2 \sqrt{\frac{\Delta_{\omega_1}(t,u(t),x,z)}{\left(a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}\right)}} \\ + \frac{2L_Q}{\left(a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}\right)} \left\|u(t)-u^*(t)\right\|_{\mathbf{U}} \\ + \frac{2(L_F + m_{J_1}\|\mathbf{N}\|\|\mathbf{M}\|)}{\left(a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}\right)} \|z(t)-z^*(t)\|_{\mathbf{X}_2} \\ + \frac{2(L_Q a_{\mathcal{S}_1} + b_\Upsilon a_{\mathcal{S}_2})}{\left(a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}\right)} \int_0^t \|x(s)-x^*(s)\|_{\mathbf{X}_1}\,\mathrm{d}s \end{aligned} \tag{35}$$

for all $t \in [0, T]$.

Moreover, by the fact that u is the unique mild solution to the problem (17), one has

$$u(t) = \mathcal{T}(t)u_0 + \int_0^t \mathcal{T}(t-s)H(s,u(s),x(s))\,\mathrm{d}s \quad \text{for all } t \in [0,T]. \tag{36}$$

Considering (23) and (36) and based on assumptions $\mathfrak{h}(A)$, $\mathfrak{h}(H)(c)$ and $\|\mathcal{T}(\cdot)\| \leq m_H$, one has

$$\begin{aligned}&\left\|u(t)-u^*(t)\right\|_{\mathbf{U}} \\ &\leq \left\|\int_0^t \mathcal{T}(t-s)H(s,u(s),x(s))\,\mathrm{d}s - \int_0^t \mathcal{T}(t-s)H(s,u^*(s),x^*(s))\,\mathrm{d}s\right\|_{\mathbf{U}} \\ &\leq \int_0^t \|\mathcal{T}(t-s)\|\|H(s,u(s),x(s)) - H(s,u^*(s),x^*(s))\|_{\mathbf{U}}\,\mathrm{d}s \\ &\leq \int_0^t a_H m_H \left\|x(s)-x^*(s)\right\|_{\mathbf{X}_1}\mathrm{d}s + \int_0^t a_H m_H \left\|u(s)-u^*(s)\right\|_{\mathbf{U}}\mathrm{d}s,\end{aligned}$$

for all $t \in [0, T]$. By applying Gronwall's inequality, we obtain the following result

$$\left\|u(t)-u^*(t)\right\|_{\mathbf{U}} \leq a_H m_H \,\mathrm{e}^{a_H m_H T} \int_0^t \left\|x(s)-x^*(s)\right\|_{\mathbf{X}_1}\mathrm{d}s, \tag{37}$$

for all $t \in [0, T]$.

Putting (30) and (37) into (35), we arrive at the estimate

$$\begin{aligned}
&\left[1-\frac{2(L_F+m_{J_1}\|\mathbf{N}\|\|\mathbf{M}\|)(L_G+m_{J_2}\|\mathbf{N}\|\|\mathbf{M}\|)}{\left(a_F-m_{J_1}\|\mathbf{M}\|^2-a_{\Upsilon}-\frac{1}{2\omega_1}\right)\left(a_G-m_{J_2}\|\mathbf{N}\|^2-\frac{1}{2\omega_2}\right)}\right]\|x(t)-x^*(t)\|_{\mathbf{X}_1}\\
&\quad\le 2\sqrt{\frac{\Delta_{\omega_1}(t,u(t),x,z)}{\left(a_F-m_{J_1}\|\mathbf{M}\|^2-a_{\Upsilon}-\frac{1}{2\omega_1}\right)}}\\
&\qquad+\frac{2(L_F+m_{J_1}\|\mathbf{N}\|\|\mathbf{M}\|)}{\left(a_F-m_{J_1}\|\mathbf{M}\|^2-a_{\Upsilon}-\frac{1}{2\omega_1}\right)}\sqrt{\frac{2\Theta_{\omega_2}(t,z,x)}{a_G-m_{J_2}\|\mathbf{N}\|^2-\frac{1}{2\omega_2}}}\\
&\qquad+\frac{2(L_Q a_{\mathcal{S}_1}+L_Q a_H m_H\,\mathrm{e}^{a_H m_H T}+b_{\Upsilon}a_{\mathcal{S}_2})}{\left(a_F-m_{J_1}\|\mathbf{M}\|^2-a_{\Upsilon}-\frac{1}{2\omega_1}\right)}\int_0^t\|x(s)-x^*(s)\|_{\mathbf{X}_1}\,\mathrm{d}s \qquad (38)
\end{aligned}$$

for all $t\in[0,T]$.

Given $\omega_i>0$ with $i=1,2$ and $u\in C([0,T];\mathbf{U})$, let $\mho_{\omega_1,\omega_2}\colon[0,T]\times\mathbf{U}\times C([0,T];P_1)\times C([0,T];P_2)\to\mathbb{R}_+$ be the function given by

$$\begin{aligned}
\mho_{\omega_1,\omega_2}(t,u(t),x,z):=&\,2\sqrt{\frac{\Delta_{\omega_1}(t,u(t),x,z)}{\left(a_F-m_{J_1}\|\mathbf{M}\|^2-a_{\Upsilon}-\frac{1}{2\omega_1}\right)}}\\
&+\frac{2(L_F+m_{J_1}\|\mathbf{N}\|\|\mathbf{M}\|)}{\left(a_F-m_{J_1}\|\mathbf{M}\|^2-a_{\Upsilon}-\frac{1}{2\omega_1}\right)}\sqrt{\frac{2\Theta_{\omega_2}(t,z,x)}{a_G-m_{J_2}\|\mathbf{N}\|^2-\frac{1}{2\omega_2}}}
\end{aligned}$$

and

$$\begin{cases}
\mathbf{C}_1:=1-\dfrac{2(L_F+m_{J_1}\|\mathbf{N}\|\|\mathbf{M}\|)(L_G+m_{J_2}\|\mathbf{N}\|\|\mathbf{M}\|)}{\left(a_F-m_{J_1}\|\mathbf{M}\|^2-a_{\Upsilon}-\frac{1}{2\omega_1}\right)\left(a_G-m_{J_2}\|\mathbf{N}\|^2-\frac{1}{2\omega_2}\right)};\\
\mathbf{C}_2:=\dfrac{2(L_Q a_{\mathcal{S}_1}+L_Q a_H m_H\,\mathrm{e}^{a_H m_H T}+b_{\Upsilon}a_{\mathcal{S}_2})}{\left(a_F-m_{J_1}\|\mathbf{M}\|^2-a_{\Upsilon}-\frac{1}{2\omega_1}\right)}.
\end{cases}$$

Moreover, by hypothesis $\mathfrak{h}'(0)$, we achieve $\mathbf{C}_1>0$ and $\mathbf{C}_2>0$. Then using the estimate (38), it follows from Gronwall's inequality that

$$\begin{aligned}
&\|x(t)-x^*(t)\|_{\mathbf{X}_1}\\
&\quad\le\frac{\mho_{\omega_1,\omega_2}(t,u(t),x,z)}{\mathbf{C}_1}+\frac{\mathbf{C}_2}{\mathbf{C}_1^2}\int_0^t\mho_{\omega_1,\omega_2}(s,u(s),x,z)\exp\left\{\frac{\mathbf{C}_2}{\mathbf{C}_1}(t-s)\right\}\mathrm{d}s
\end{aligned}$$

for all $t\in[0,T]$.

Now, we introduce the function $\Pi_1\colon[0,T]\to\mathbb{R}_+$ given by

$$\Pi_1(t):=\frac{\mho_{\omega_1,\omega_2}(t,u(t),x,z)}{\mathbf{C}_1}+\frac{\mathbf{C}_2}{\mathbf{C}_1^2}\int_0^t\mho_{\omega_1,\omega_2}(s,u(s),x,z)\exp\left\{\frac{\mathbf{C}_2}{\mathbf{C}_1}(t-s)\right\}\mathrm{d}s$$

for all $t\in[0,T]$. By Proposition 3.3(ii) and Proposition 3.4(ii), functions $t\mapsto\Delta_{\omega_1}(t,u(t),x,z)$ and $t\mapsto\Theta_{\omega_2}(t,z,x)$ belong to $L_+^\infty(0,T)$, and so the function $t\mapsto$

$\mho_{\omega_1,\omega_2}(t,u(t),x,z)$ aslo belongs to $L_+^\infty(0,T)$. Thus, we are able to admit that $\Pi_1 \in L_+^\infty(0,T)$. Then, for all $t \in [0,T]$ we can derive

$$\|x(t) - x^*(t)\|_{\mathbf{X}_1} \leq \Pi_1(t),$$

that is, the estimate (14) is valid. Using the above inequality, (30) and (37), we can find functions $\Pi_2 \in L_+^\infty(0,T)$ and $\Pi_3 \in L_+^\infty(0,T)$, which are defined by

$$\Pi_2(t) := \sqrt{\frac{2\Theta_{\omega_2}(t,z,x)}{a_G - m_{J_2}\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}}} + \frac{L_G + m_{J_2}\|\mathbf{N}\|\|\mathbf{M}\|}{a_G - m_{J_2}\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}}\Pi_1(t)$$

and

$$\Pi_3(t) := a_H m_H \,\mathrm{e}^{a_H m_H T} \int_0^t \Pi_1(s)\,\mathrm{d}s$$

for all $t \in [0,T]$, such that estimates (15) and (16) hold true. ■

Remark 3.3: (i) Error bounds $\Pi_i \in L_+^\infty(0,T)$ $(i = 1,2,3)$ provided by (28)–(30) in Theorem 3.5 illustrate the upper estimates between an arbitrary feasible triple (u,x,z) in $C([0,T];\mathbf{U}) \times C([0,T];P_1) \times C([0,T];P_2)$ and the unique solution to Problem 1.1, where u satisfies that it is the unique mild solution of the differential equation (17) corresponding to $x \in C([0,T];P_1)$. They are controlled by coupled regularized gap functions Δ_{ω_1} and Θ_{ω_2} in the form of the function $\mho_{\omega_1,\omega_2}$ defined by (21) and depended on the data of Problem 1.1.

(ii) Furthermore, if the sequence $\{(u_n,x_n,z_n)\} \subset C([0,T];\mathbf{U}) \times C([0,T];P_1) \times C([0,T];P_2)$ constructed by some methods satisfies $\mho_{\omega_1,\omega_2}(t,u_n(t),x_n,z_n) \to 0$ as $n \to \infty$, $\forall t \in [0,T]$ and u_n is the unique solution of the Cauchy problem (17) for all $n \in \mathbb{N}$, then we have $\Pi_i(t) \to 0$ $(i = 1,2,3)$. This allows us to conclude that (u_n,x_n,z_n) converges to the unique solution (u^*,x^*,z^*) of Problem 1.1 as $n \to \infty$. Additionally, we also get an advantage in obtaining error estimates between (u^*,x^*,z^*) and (u_n,x_n,z_n) for each $n \in \mathbb{N}$.

(iii) In two recent works [28] and [29], the authors successfully established gap functions and error bounds for classes of differential variational-hemivariational inequalities consisting of only ‘a single variational-hemivariational inequality’. Meanwhile, Problem 1.1 introduces a more complex system constructed by combining a differential equation with ‘coupled variational-hemivariational inequalities’ that exhibit a nested structure. Establishing error bounds for Problem 1.1 necessitates the development of advanced computational techniques. The main difficulty lies in identifying an appropriate regularized gap function to effectively evaluate the solutions of the controlled systems (2)–(3) of Problem 1.1. Up to now, there has been no contribution that deals with error bounds for this class of generalized differential variational-hemivariational inequalities based on involving gap functions. Thus, Theorem 3.5 represents a novelty result in this area, providing a meaningful contribution by presenting a new approach to deriving error bounds based on exploring the function $\mho_{\omega_1,\omega_2}$.

We now consider the following controlled history-dependent problem constructed by coupled variational-hemivariational inequalities given in the systems (2)–(3):

($\mathbf{CVHI}_{\boldsymbol{u}}$): Given $u \in \mathbf{U}$, find a pair $(x, z) \in C([0,T]; P_1) \times C([0,T]; P_2)$ such that for all $t \in [0,T]$,

$$\begin{aligned}
&\big\langle F(t, z(t), x(t)) + Q\left(t, (\mathcal{S}_1 x)(t), u\right), y - x(t)\big\rangle + \Upsilon\left((\mathcal{S}_2 x)(t), x(t), y\right) \\
&\quad - \Upsilon\left((\mathcal{S}_2 x)(t), x(t), x(t)\right) + J_1^\circ\left(\mathbf{N}z(t), \mathbf{M}x(t); \mathbf{M}y - \mathbf{M}x(t)\right) \geqslant 0, \forall y \in P_1, \\
&\langle G(t, x(t), z(t)), v - z(t)\rangle + J_2^\circ\left(\mathbf{M}x(t), \mathbf{N}z(t); \mathbf{N}v - \mathbf{N}z(t)\right) \\
&\quad + \varphi(v) - \varphi(z(t)) \geqslant \langle f(t), v - z(t)\rangle, \quad \forall\, v \in P_2.
\end{aligned}$$

We close this section with a result of error bounds for the problem (CVHI_u). This result is deduced directly from the proof of Theorem 3.5 by using the coupled regularized gap functions Δ_{ω_1} and Θ_{ω_2}.

Theorem 3.6: *Let $u \in \mathbf{U}$ be fixed and the pair $(x^*, z^*) \in C([0,T]; P_1) \times C([0,T]; P_2)$ be the unique solution of problem (CVHI_u). Assume that conditions $\mathfrak{h}(P)$, $\mathfrak{h}(Q)$, $\mathfrak{h}(J_1)$, $\mathfrak{h}(J_2)$, $\mathfrak{h}(\Upsilon)$, $\mathfrak{h}(\mathcal{S})$, $\mathfrak{h}(\mathbf{M})$, $\mathfrak{h}(\mathbf{N})$ and $\mathfrak{h}(\varphi)$ hold. In addition, parameters $\omega_1, \omega_2 > 0$ satisfy the hypothesis:*

$\underline{\mathfrak{h}''(0)}$ *: (a)* $a_F > m_{J_1}\|\mathbf{M}\|^2 + a_\Upsilon + \frac{1}{2\omega_1}$, $a_G > m_{J_2}\|\mathbf{N}\|^2 + \frac{1}{2\omega_2}$;
(b) The following inequality holds:

$$\frac{\left(m_{J_2}\|\mathbf{N}\|\,\|\mathbf{M}\| + L_G\right)\left(m_{J_1}\|\mathbf{M}\|\,\|\mathbf{N}\| + L_F\right)}{\left(a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}\right)\left(a_G - m_{J_2}\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}\right)} < \frac{1}{\sqrt{2}}.$$

Then, for each $(x, z) \in C([0,T]; P_1) \times C([0,T]; P_2)$ there exist the functions $\widetilde{\Pi}_i \in L_+^\infty(0,T)$ with i = 1, 2 satisfy the following estimations:

$$\|x(t) - x^*(t)\|_{\mathbf{X}_1} \leq \widetilde{\Pi}_1(t) \quad \text{for all } t \in [0,T]; \tag{39}$$

$$\|z(t) - z^*(t)\|_{\mathbf{X}_2} \leq \widetilde{\Pi}_2(t) \quad \text{for all } t \in [0,T], \tag{40}$$

where

$$\widetilde{\Pi}_1(t) = \frac{\widetilde{\mho}_{\omega_1,\omega_2}(t,u,x,z)}{\widetilde{\mathbf{C}}_1} + \frac{\widetilde{\mathbf{C}}_2}{\widetilde{\mathbf{C}}_1^2}\int_0^t \widetilde{\mho}_{\omega_1,\omega_2}(s,u,x,z)\exp\left\{\frac{\widetilde{\mathbf{C}}_2}{\widetilde{\mathbf{C}}_1}(t-s)\right\}\mathrm{d}s;$$

$$\widetilde{\Pi}_2(t) = \sqrt{\frac{2\Theta_{\omega_2}(t,z,x)}{a_G - m_{J_2}\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}}} + \frac{L_G + m_{J_2}\|\mathbf{N}\|\,\|\mathbf{M}\|}{a_G - m_{J_2}\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}}\widetilde{\Pi}_1(t),$$

with the function $\widetilde{\mho}_{\omega_1,\omega_2} : [0,T] \times \mathbf{U} \times C([0,T]; P_1) \times C([0,T]; P_2) \to \mathbb{R}_+$ *given by*

$$\begin{aligned}
\widetilde{\mho}_{\omega_1,\omega_2}(t,u,x,z) = &\sqrt{\frac{2\Delta_{\omega_1}(t,u,x,z)}{\left(a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}\right)}} \\
&+ \frac{2(L_F + m_{J_1}\|\mathbf{N}\|\,\|\mathbf{M}\|)}{\left(a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}\right)}\sqrt{\frac{\Theta_{\omega_2}(t,z,x)}{a_G - m_{J_2}\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}}}
\end{aligned} \tag{41}$$

and constants $\widetilde{\mathbf{C}}_1, \widetilde{\mathbf{C}}_2 > 0$ *defined by*

$$\begin{cases} \widetilde{\mathbf{C}}_1 = 1 - \dfrac{\sqrt{2}(L_F + m_{J_1}\|\mathbf{N}\|\|\mathbf{M}\|)(L_G + m_{J_2}\|\mathbf{N}\|\|\mathbf{M}\|)}{\left(a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}\right)\left(a_G - m_{J_2}\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}\right)}; \\ \widetilde{\mathbf{C}}_2 = \dfrac{\sqrt{2}(L_Q a_{\mathcal{S}_1} + b_\Upsilon a_{\mathcal{S}_2})}{\left(a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}\right)}. \end{cases} \tag{42}$$

Proof: Let $(u, x, z) \in \mathbf{U} \times C([0,T]; P_1) \times C([0,T]; P_2)$ and $(x^*, z^*) \in C([0,T]; P_1) \times C([0,T]; P_2)$ be the unique solution to problem (CVHI$_u$). Then, we obtain

$$\begin{aligned} &\left\langle F(t, z^*(t), x^*(t)) + Q\left(t, \left(\mathcal{S}_1 x^*\right)(t), u\right), y - x^*(t)\right\rangle + \Upsilon\left(\left(\mathcal{S}_2 x^*\right)(t), x^*(t), y\right) \\ &\quad - \Upsilon\left(\left(\mathcal{S}_2 x^*\right)(t), x^*(t), x^*(t)\right) + J_1^\circ\left(\mathbf{N}z^*(t), \mathbf{M}x^*(t); \mathbf{M}y - \mathbf{M}x^*(t)\right) \geqslant 0, \\ &\langle G(t, x^*(t), z^*(t)), v - z^*(t)\rangle + J_2^\circ\left(\mathbf{M}x^*(t), \mathbf{N}z^*(t); \mathbf{N}v - \mathbf{N}z^*(t)\right) \\ &\quad + \varphi(v) - \varphi(z^*(t)) \geqslant \langle f(t), v - z^*(t)\rangle, \end{aligned}$$

for all $(y, v) \in P_1 \times P_2$ and $t \in [0, T]$.

First, we observe that inequality (30) holds in the initial part of the proof of Theorem 3.5. Moreover, by an argument similar to that used in deriving (35), and noting that the terms involving $\|u(t) - u^*(t)\|_{\mathbf{U}}$ vanish, we obtain

$$\begin{aligned} \|x(t) - x^*(t)\|_{\mathbf{X}_1} \leq\ & \sqrt{\frac{2\Delta_{\omega_1}(t, u(t), x, z)}{\left(a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}\right)}} \\ & + \frac{\sqrt{2}(L_F + m_{J_1}\|\mathbf{N}\|\|\mathbf{M}\|)}{\left(a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}\right)}\|z(t) - z^*(t)\|_{\mathbf{X}_2} \\ & + \frac{\sqrt{2}(L_Q a_{\mathcal{S}_1} + b_\Upsilon a_{\mathcal{S}_2})}{\left(a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}\right)} \int_0^t \|x(s) - x^*(s)\|_{\mathbf{X}_1}\,\mathrm{d}s \end{aligned} \tag{43}$$

for all $t \in [0, T]$.

Substituting (30) into (43), we obtain the following inequality:

$$\begin{aligned} &\left[1 - \frac{\sqrt{2}(L_F + m_{J_1}\|\mathbf{N}\|\|\mathbf{M}\|)(L_G + m_{J_2}\|\mathbf{N}\|\|\mathbf{M}\|)}{\left(a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}\right)\left(a_G - m_{J_2}\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}\right)}\right]\|x(t) - x^*(t)\|_{\mathbf{X}_1} \\ &\leq \sqrt{\frac{2\Delta_{\omega_1}(t, u(t), x, z)}{\left(a_F - m_{J_1}\|\mathbf{M}\|^2 - a_\Upsilon - \frac{1}{2\omega_1}\right)}} \end{aligned}$$

$$+\frac{2(L_F+m_{J_1}\|\mathbf{N}\|\|\mathbf{M}\|)}{\left(a_F-m_{J_1}\|\mathbf{M}\|^2-a_\Upsilon-\frac{1}{2\omega_1}\right)}\sqrt{\frac{\Theta_{\omega_2}(t,z,x)}{a_G-m_{J_2}\|\mathbf{N}\|^2-\frac{1}{2\omega_2}}}$$
$$+\frac{\sqrt{2}(L_Q a_{\mathcal{S}_1}+b_\Upsilon a_{\mathcal{S}_2})}{\left(a_F-m_{J_1}\|\mathbf{M}\|^2-a_\Upsilon-\frac{1}{2\omega_1}\right)}\int_0^t \|x(s)-x^*(s)\|_{\mathbf{X}_1}\,\mathrm{d}s \tag{44}$$

for all $t \in [0,T]$.

By defining the function $\widetilde{\mho}_{\omega_1,\omega_2}$ in (41), together with the constants $\widetilde{\mathbf{C}}_1$ and $\widetilde{\mathbf{C}}_2$ in (42), and employing an argument similar to the final part of the proof of Theorem 3.5 for (44), we obtain the estimates (39) and (40). ■

Remark 3.4: The function $\widetilde{\mho}_{\omega_1,\omega_2}$ given in (41) is also established by coupled regularized gap functions Δ_{ω_1} and Θ_{ω_2}. By Proposition 3.1 and Proposition 3.2, we can show that $\widetilde{\mho}_{\omega_1,\omega_2}$ is a gap function for the variational control system (CVHI$_u$), namely, the function $\widetilde{\mho}_{\omega_1,\omega_2}$ satisfies two properties: for all $t \in [0,T]$,

(a) $\widetilde{\mho}_{\omega_1,\omega_2}(t,u,x,z) \geq 0$ for all $(x,z) \in C([0,T];P_1) \times C([0,T];P_2)$;
(b) $(x^*,z^*) \in C([0,T];P_1) \times C([0,T];P_2)$ satisfies $\widetilde{\mho}_{\omega_1,\omega_2}(t,u,x^*,z^*) = 0$ if and only if (x^*,z^*) is a solution for problem (CVHI$_u$).

Lastly, we point out some facts and possible future directions. The result of Theorem 3.6 offers a novel method for analysing error bounds in systems of coupled variational-hemivariational inequalities with a nested structure. This new approach allows us to use these computational techniques to explore error bounds in similar systems, such as those discussed in the works of Bai-Costea-Zeng [30], Costea [31], and Migórski-Ogorzały-Dudek [32], which are based on regularized gap functions.

Yamashita-Fukushima [18] originally introduced the regularized gap function of the Moreau—Yosida type constructed by the regularized gap function of the Fukushima formulation for variational inequalities. They demonstrated that this gap function in the form of the Moreau—Yosida regularization possesses advantageous theoretical properties that are not found in other gap functions (see [18, Section 2]). Consequently, we may develop coupled regularized gap functions of the Moreau—Yosida type based on the coupled regularized gap functions Δ_{ω_1} and Θ_{ω_2}. This enables us to establish new error bounds to Problem 1.1 and the variational control system (CVHI$_u$) based on the new gap function constructed from these coupled regularized gap functions of the Moreau—Yosida type.

4. Application to quasistatic contact problems of two elastic bodies

This section applies the error bound results established in Section 3 to the classes of quasistatic contact problems involving two elastic bodies, which are formulated as nonlinear variational problems described in Problem 1.1 and the problem (CVHI$_u$).

Given open bounded sets $\mathbb{B}_1, \mathbb{B}_2 \subset \mathbb{R}^k$ with $k = 2, 3$, for each $p = 1, 2$ let $\Sigma^p = \partial\mathbb{B}_p$ be a Lipschitz boundary of $\mathbb{B}_p$ which consisted of four disjoint measurable parts Σ_1^p, Σ_2^p, Σ_3^p and Σ_4 with meas$(\Sigma_1^p) > 0$ Figure 2. Denote by $\mathbb{S}^k$ the space of second-order symmetric

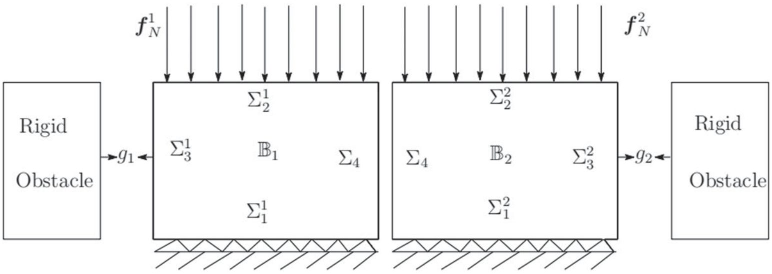

Figure 2. Quasistatic contact problem of two elastic bodies (Hao-Wang-Han [1]).

$k \times k$ matrices. We now revisit the following class of contact models introduced by Hao-Wang-Han [1].

Problem 4.1: Find displacement fields $\boldsymbol{w}^1 : \mathbb{B}_1 \times [0,T] \to \mathbb{R}^k, \boldsymbol{w}^2 : \mathbb{B}_2 \times [0,T] \to \mathbb{R}^k$, stress fields $\boldsymbol{\sigma}^1 : \mathbb{B}_1 \times [0,T] \to \mathbb{S}^k, \boldsymbol{\sigma}^2 : \mathbb{B}_2 \times [0,T] \to \mathbb{S}^k$, and $\boldsymbol{u} : \Sigma_3^1 \times [0,T] \to \mathbb{R}$ such that,

$$\mathrm{Div}\boldsymbol{\sigma}^p(t) + \boldsymbol{f}_0^p(t) = \boldsymbol{0} \quad \text{in } \mathbb{B}_p \times (0,T), \tag{45}$$

$$\boldsymbol{\sigma}^1(t) = \mathcal{A}^1\left(t, \boldsymbol{\varepsilon}\left(\boldsymbol{w}^1(t)\right)\right) + \int_0^t \mathcal{L}(t-s)\boldsymbol{\varepsilon}\left(\boldsymbol{w}^1(s)\right) \mathrm{d}s \quad \text{in } \mathbb{B}_1 \times (0,T), \tag{46}$$

$$\boldsymbol{\sigma}^2(t) \in \mathcal{A}^2\left(t, \boldsymbol{\varepsilon}\left(\boldsymbol{w}^2(t)\right)\right) + \partial^c \mathcal{G}\left(\boldsymbol{\varepsilon}\left(\boldsymbol{w}^2(t)\right)\right) \quad \text{in } \mathbb{B}_2 \times (0,T), \tag{47}$$

$$\boldsymbol{w}^p(t) = \boldsymbol{0} \quad \text{on } \Sigma_1^p \times (0,T), \tag{48}$$

$$\boldsymbol{\sigma}^p(t)\boldsymbol{\nu} = \boldsymbol{f}_N^p(t) \quad \text{on } \Sigma_2^p \times (0,T), \tag{49}$$

$$\begin{cases} w_\nu^1(t) \leqslant g_1, \sigma_\nu^1(t) + \boldsymbol{u}(t) \leqslant 0 \\ \left(\sigma_\nu^1(t) + \boldsymbol{u}(t)\right)\left(w_\nu^1 - g_1\right) = 0, \\ -\sigma_\nu^1(t) \in \partial j_\nu^1\left(w_\nu^1(t)\right), \\ \boldsymbol{u}'(t) = \mathcal{Q}\left(t, \boldsymbol{u}(t), w_\nu^1(t)\right), \quad \text{on } \Sigma_3^1 \times (0,T), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \\ \left\|\boldsymbol{\sigma}_\tau^1(t)\right\| \leqslant \mathcal{G}_b\left(w_\nu^1(t)\right), \\ \boldsymbol{\sigma}_\tau^1(t) = -\mathcal{G}_b\left(w_\nu^1(t)\right)\frac{\boldsymbol{w}_\tau^1(t)}{\|\boldsymbol{w}_\tau^1(t)\|} \quad \text{if } \left\|\boldsymbol{w}_\tau^1(t)\right\| \neq 0 \quad \text{on } \Sigma_3^1 \times (0,T), \end{cases} \tag{50}$$

$$\begin{cases} w_\nu^2(t) \leqslant g_2, \sigma_\nu^2 \leqslant 0 \\ \left(w_\nu^2(t) - g_2\right)\sigma_\nu^2(t) = 0, \\ -\sigma_\nu^2(t) \in \partial j_\nu^2\left(w_\nu^2(t)\right) \\ \boldsymbol{\sigma}_\tau^2 = \boldsymbol{0}, \quad \text{on } \Sigma_3^2 \times (0,T) \end{cases} \tag{51}$$

$$\begin{cases} -\sigma_\nu^1(t) \in \partial J_\nu^1\left(w_\nu^2(t), w_\nu^1(t)\right), \\ -\sigma_\nu^2(t) \in \partial J_\nu^2\left(w_\nu^1(t), w_\nu^2(t)\right), \\ \boldsymbol{\sigma}_\tau^1 = \boldsymbol{\sigma}_\tau^2 = \boldsymbol{0}, \quad \text{on } \Sigma_4 \times (0,T). \end{cases} \tag{52}$$

Here, $\boldsymbol{0}$ stands the zero element of $\mathbb{R}^k$ and $\mathbb{S}^k$. The inner products and the Euclidean norms on $\mathbb{R}^k$ and $\mathbb{S}^k$ are defined by

$$\boldsymbol{w} \cdot \boldsymbol{v} = w_i v_i, \quad \|\boldsymbol{w}\| = (\boldsymbol{w} \cdot \boldsymbol{w})^{\frac{1}{2}}, \quad \text{for all } \boldsymbol{w} = (w_i), \boldsymbol{v} = (v_i) \in \mathbb{R}^k;$$

$$\boldsymbol{\sigma}:\boldsymbol{\omega}=\sigma_{ij}\omega_{ij},\quad \|\boldsymbol{\sigma}\|=(\boldsymbol{\sigma}:\boldsymbol{\sigma})^{\frac{1}{2}},\quad \text{for all } \boldsymbol{\sigma}=(\sigma_{ij})\in\mathbb{S}^k,\boldsymbol{\omega}=(\omega_{ij})\in\mathbb{S}^k.$$

For each $p=1,2$, $w_\nu^p:=\boldsymbol{w}^p\cdot\boldsymbol{\nu}^p$ and $\sigma_\nu^p:=(\boldsymbol{\sigma}^p\boldsymbol{\nu}^p)\cdot\boldsymbol{\nu}^p$ stand for the normal components of $\boldsymbol{w}^p$ and $\boldsymbol{\sigma}^p$, respectively on the boundary $\partial\mathbb{B}_p$ and $\boldsymbol{w}_\tau^p:=\boldsymbol{w}^p-w_\nu^p\boldsymbol{\nu}^p$ and $\boldsymbol{\sigma}_\tau^p:=\boldsymbol{\sigma}^p\boldsymbol{\nu}^p-\sigma_\nu^p\boldsymbol{\nu}^p$ stand for the tangential components of $\boldsymbol{w}^p$ and $\boldsymbol{\sigma}^p$, respectively on the boundary $\partial\mathbb{B}_p$, where $\boldsymbol{\nu}^p=(\nu_i^p)$ is the unit outward normal vector on the boundary $\partial\mathbb{B}_p$. Moreover, the linearized strain tensor defined by

$$\boldsymbol{\varepsilon}\left(\boldsymbol{w}^p\right)=\left(\varepsilon_{ij}\left(\boldsymbol{w}^p\right)\right),\varepsilon_{ij}\left(\boldsymbol{w}^p\right)=\frac{1}{2}\left(w_{i,j}^p+w_{j,i}^p\right)\quad\text{in }\mathbb{B}_p,$$

where $w_{i,j}^p=\frac{\partial w_i^p}{\partial x_j}$.

Next, conditions in Problem 4.1 will be described together with the assumptions on the data.

Equation (45) is the equilibrium equation in which $\operatorname{Div}\boldsymbol{\sigma}^p=(\sigma_{ij,j}^p)=(\frac{\partial\sigma_{ij}^p}{\partial x_j})$ and the density of the body forces $\boldsymbol{f}_0^1,\boldsymbol{f}_0^2$ satisfy the following conditions.

$\underline{\mathfrak{h}(\boldsymbol{f}_0^p)}:\boldsymbol{f}_0^p\in C([0,T];L^2(\mathbb{B}_p;\mathbb{R}^k)),p=1,2.$

Equation (46) represents the elastic constitutive laws with long memory. The assumptions on the elastic operator $\mathcal{A}^1$ are as follows:

$\underline{\mathfrak{h}(\mathcal{A}^1)}:\mathcal{A}^1:\mathbb{B}_1\times(0,T)\times\mathbb{S}^k\to\mathbb{S}^k$ is such that

(a) $\mathcal{A}^1(\cdot,t,\boldsymbol{\varepsilon})$ is measurable on $\mathbb{B}_1$, for all $(t,\boldsymbol{\varepsilon})\in(0,T)\times\mathbb{S}^k$;
(b) $\mathcal{A}^1(\boldsymbol{x},\cdot,\cdot)$ is a continuous function on $(0,T)\times\mathbb{S}^k$ for a.e. $\boldsymbol{x}\in\mathbb{B}_1$;
(c) There exists $m_{\mathcal{A}^1}>0$ satisfy $(\mathcal{A}^1(\boldsymbol{x},t,\boldsymbol{\varepsilon}_1)-\mathcal{A}^1(\boldsymbol{x},t,\boldsymbol{\varepsilon}_2):\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)\geqslant m_{\mathcal{A}^1}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|_{\mathbb{S}^k}^2$ for a.e. $\boldsymbol{x}\in\mathbb{B}_1,t\in(0,T),\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^k$;
(d) There exist $c_{\mathcal{A}^1},b_{\mathcal{A}^1}>0$ such that the inequality $\|\mathcal{A}^1(\boldsymbol{x},t,\boldsymbol{\varepsilon})\|_{\mathbb{S}^k}\leqslant c_{\mathcal{A}^1}(\boldsymbol{x},t)+b_{\mathcal{A}^1}\|\boldsymbol{\varepsilon}\|_{\mathbb{S}^k}$ holds for all $\boldsymbol{\varepsilon}\in\mathbb{S}^k$ and for a.e. $(\boldsymbol{x},t)\in\mathbb{B}_1\times(0,T)$.

The linear relaxation tensor $\mathcal{L}$ satisfies the following conditions.

$\underline{\mathfrak{h}(\mathcal{L})}:\mathcal{L}:\mathbb{B}_1\times(0,T)\times\mathbb{S}^k\to\mathbb{S}^k$ is such that

(a) $\mathcal{L}(\boldsymbol{x},t,\xi)=\mathcal{E}(\boldsymbol{x},t)\xi$ for all $\xi\in\mathbb{S}^k$, a.e. $(\boldsymbol{x},t)\in\mathbb{B}_1\times(0,T)$;
(b) $\mathcal{E}(\boldsymbol{x},t)=(e_{ijkl}(\boldsymbol{x},t))$ with $e_{ijkl}=e_{jkkl}=e_{lkij}\in L^2([0,T];L^\infty(\mathbb{B}_1))$.

Inclusion (47) describes the elastic constitutive laws, where $\mathcal{A}^2$ stands for the elasticity operator and satisfies the following conditions.

$\underline{\mathfrak{h}(\mathcal{A}^2)}:\mathcal{A}_2:\mathbb{B}_2\times(0,T)\times\mathbb{S}^k\to\mathbb{S}^k$ is such that

(a) $\mathcal{A}^2(\cdot,t,\boldsymbol{\varepsilon})$ is measurable on $\mathbb{B}_2$, for all $(t,\boldsymbol{\varepsilon})\in(0,T)\times\mathbb{S}^k$;
(b) $\mathcal{A}^2(\boldsymbol{x},\cdot,\cdot)$ is a continuous function on $(0,T)\times\mathbb{S}^k$ for a.e. $\boldsymbol{x}\in\mathbb{B}_2$;
(c) There exists $m_{\mathcal{A}^2}>0$ such that $(\mathcal{A}^2(\boldsymbol{x},t,\boldsymbol{\varepsilon}_1)-\mathcal{A}^2(\boldsymbol{x},t,\boldsymbol{\varepsilon}_2):\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)\geqslant m_{\mathcal{A}^2}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|_{\mathbb{S}^k}^2$ for a.e. $\boldsymbol{x}\in\mathbb{B}_2,t\in(0,T),\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^k$;
(d) There exist $c_{\mathcal{A}^2},b_{\mathcal{A}^2}>0$ such that $\|\mathcal{A}^2(\boldsymbol{x},t,\boldsymbol{\varepsilon})\|_{\mathbb{S}^k}\leqslant c_{\mathcal{A}^2}(\boldsymbol{x},t)+b_{\mathcal{A}^2}\|\boldsymbol{\varepsilon}\|_{\mathbb{S}^k}$ for all $\boldsymbol{\varepsilon}\in\mathbb{S}^k$ and for a.e. $(\boldsymbol{x},t)\in\mathbb{B}_2\times(0,T)$.

and $\partial^c\mathcal{G}$ denotes the convex subdifferential operator of the function $\mathcal{G}$ with
$\underline{\mathfrak{h}(\mathcal{G})}$: $\mathcal{G} : \mathbb{B}_2 \times \mathbb{S}^k \to \mathbb{R}$ satisfies the following conditions.

(a) $\mathcal{G}(\cdot, \boldsymbol{\varepsilon})$ is measurable on $\mathbb{B}_2$ for all $\boldsymbol{\varepsilon} \in \mathbb{S}^k$;
(b) $\mathcal{G}(\cdot, \boldsymbol{\zeta}(\cdot)) \in L^1(\mathbb{B}_2)$ with $\boldsymbol{\zeta} \in L^2(\mathbb{B}_2 \times [0, T]; \mathbb{S}^k)$;
(c) $\mathcal{G}(\boldsymbol{x}, \cdot)$ is a convex and lower semicontinuous function for a.e. $\boldsymbol{x} \in \mathbb{B}_2$.

Equations (48) and (49) represents the clamped boundary condition on Σ_1^p and the surface traction boundary on Σ_2^p, respectively. The density $\boldsymbol{f}_N^p$ satisfies
$\underline{\mathfrak{h}(\boldsymbol{f}_N^p)}$: $\boldsymbol{f}_N^p \in C([0, T]; L^2(\Sigma_2^p; \mathbb{R}^k)), p = 1, 2.$

The unilateral boundary condition is described by Equation (50), where $g_1 > 0$ stands for the thickness of the elastic layer, the Clarke's subdifferential of a locally Lipschitz function j_ν^1 represents the frictional contact between foundation and elastic body and the normal compliance function j_ν^1
$\underline{\mathfrak{h}(j_\nu^1)}$: $j_\nu^1 : \Sigma_3^1 \times \mathbb{R} \to \mathbb{R}$ is such that

(a) $j_\nu^1(\cdot, r)$ is measurable on Σ_3^1 for all $r \in \mathbb{R}$, and $j_\nu^1(\cdot, \boldsymbol{e}(\cdot)) \in L^1(\Sigma_3^1)$ with $\boldsymbol{e} \in L^2(\Sigma_3^1)$;
(b) $j_\nu^1(\boldsymbol{x}, \cdot)$ is a locally Lipschitz function on $\mathbb{R}$ for a.e. $\boldsymbol{x} \in \Sigma_3^1$;
(c) There exists $c_{j_\nu^1} > 0$ such that $|\partial j_\nu^1(\boldsymbol{x}, \zeta)| \leqslant c_{j_\nu^1}$ for a.e. $\boldsymbol{x} \in \Sigma_3^1$ and $\zeta \in \mathbb{R}$;
(d) There exists $m_{j_\nu^1} > 0$ such that $j_\nu^{\circ 1}(\boldsymbol{x}, \zeta_1; \zeta_2 - \zeta_1) + j_\nu^{\circ 1}(\boldsymbol{x}, \zeta_2; \zeta_1 - \zeta_2) \leqslant m_{j_\nu^1} |\zeta_1 - \zeta_2|^2$ for a.e. $\boldsymbol{x} \in \Sigma_3^1$ and all $\zeta_1, \zeta_2 \in \mathbb{R}$.

Moreover, the equation

$$\begin{cases} \boldsymbol{u}'(t) = \mathcal{Q}\left(t, \boldsymbol{u}(t), w_\nu^1(t)\right), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \end{cases} \tag{53}$$

describes the memory behaviour during the contact process, where $\mathcal{Q}$ satisfies the following assumption.
$\underline{\mathfrak{h}(\mathcal{Q})}$: $\mathcal{Q} : (0, T) \times \Sigma_3^1 \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is such that

(a) $\mathcal{Q}(\cdot, \cdot, \zeta, \beta)$ is measurable on $(0, T) \times \Sigma_3^1$ for all $\beta \leqslant 0, \zeta \in \mathbb{R}_+$
(b) $|\mathcal{Q}(t, \boldsymbol{x}, \zeta_1, \beta_1) - \mathcal{Q}(t, \boldsymbol{x}, \zeta_2, \beta_2)| \leqslant L_{\mathcal{Q}}(|\zeta_1 - \zeta_2| + |\beta_1 - \beta_2|)$ for a.e. $(t, \boldsymbol{x}) \in (0, T) \times \Sigma_3^1$ and all $(\zeta_i, \beta_i) \in \mathbb{R}_+ \times \mathbb{R}$ with $(i = 1, 2)$ and $L_{\mathcal{Q}} > 0$
(c) $\mathcal{Q}(t, \boldsymbol{x}, \zeta, \beta) = 0$ for all $\beta \leqslant 0, \zeta \in \mathbb{R}_+$ and a.e. $(t, \boldsymbol{x}) \in (0, T) \times \Sigma_3^1$.

The condition

$$\begin{cases} \left\|\boldsymbol{\sigma}_\tau^1(t)\right\| \leqslant \mathcal{G}_b\left(w_\nu^1(t)\right) \\ \boldsymbol{\sigma}_\tau^1(t) = -\mathcal{G}_b\left(w_\nu^1(t)\right) \frac{\boldsymbol{w}_\tau^1(t)}{\|\boldsymbol{w}_\tau^1(t)\|} \quad \text{if } \left\|\boldsymbol{w}_\tau^1(t)\right\| \neq 0 \end{cases}$$

stands Coulomb's law of dry friction, where the friction bound $\mathcal{G}_b$ satisfies
$\underline{\mathfrak{h}(\mathcal{G}_b)}$: $\mathcal{G}_b : \Sigma_3^1 \times \mathbb{R} \to \mathbb{R}$ is such that

(a) $\mathcal{G}_b(\cdot, \beta)$ is measurable for all $\beta \in \mathbb{R}$;

(b) there exists $L_{\mathcal{G}_b} > 0$ such that $|\mathcal{G}_b(\boldsymbol{x},\beta_1) - \mathcal{G}_b(\boldsymbol{x},\beta_2)| \leqslant L_{\mathcal{G}_b}|\beta_1 - \beta_2|$ for a.e. $\boldsymbol{x} \in \Sigma_3^1, \beta_i \in \mathbb{R}, i = 1,2$;
(c) $\mathcal{G}_b(\boldsymbol{x},\beta) \geqslant 0$, for $\beta \in \mathbb{R}$ and a.e. $\boldsymbol{x} \in \Sigma_3^1$.

Equation (51) represents the frictionless contact on boundary Σ_3^2, where the thickness of the elastic layer is given by g_2 and $\sigma_\nu^2(t) \in \partial j_\nu^2(w_\nu^2(t))$ describes the contact with j_ν^2 satisfying
$\underline{\mathfrak{h}(j_\nu^2)} : j_\nu^2 : \Sigma_3^2 \times \mathbb{R} \to \mathbb{R}$ is such that

(a) $j_\nu^2(\cdot,\zeta)$ is measurable on Σ_3^2 for all $\zeta \in \mathbb{R}$, and $j_\nu^2(\cdot, e(\cdot)) \in L^1(\Sigma_3^2)$ with $e \in L^2(\Sigma_3^2)$;
(b) $j_\nu^2(\boldsymbol{x},\cdot)$ is a locally Lipschitz function on $\mathbb{R}$ for a.e. $\boldsymbol{x} \in \Sigma_3^2$;
(c) There exists $c_{j_\nu^2} > 0$ such that $|\partial j_\nu^2(\boldsymbol{x},\zeta)| \leqslant c_{j_\nu^2}$ for a.e. $\boldsymbol{x} \in \Sigma_3^2$ and $\zeta \in \mathbb{R}$;
(d) There exists $m_{j_\nu^2} > 0$ such that $j_\nu^{\circ 2}(\boldsymbol{x},\zeta_1;\zeta_2 - \zeta_1) + j_\nu^{\circ 2}(\boldsymbol{x},\zeta_2;\zeta_1 - \zeta_2) \leqslant m_{j_\nu^2}|\zeta_1 - \zeta_2|^2$ for a.e. $\boldsymbol{x} \in \Sigma_3^2$ and all $\zeta_1, \zeta_2 \in \mathbb{R}$.

Finally, the contact conditions caused by the contact between two elastic bodies $\mathbb{B}_1$ and $\mathbb{B}_2$ are described by Equation (52), where the contact is given by the multivalued normal compliance conditions of the forms $-\sigma_\nu^1(t) \in \partial J_\nu^1(w_\nu^2(t), w_\nu^1(t))$ and $-\sigma_\nu^2(t) \in \partial J_\nu^2(w_\nu^1(t), w_\nu^2(t))$ with J_ν^p satisfying
$\underline{\mathfrak{h}(J_\nu^p)} : J_\nu^p : \Sigma_4 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $(p = 1,2)$ is such that

(a) $J_\nu^p(\cdot,\zeta,\epsilon)$ is measurable on Σ_4 for all $\zeta,\epsilon \in \mathbb{R}$, and $J_\nu^p(\cdot,0,0) \in L^1(\Sigma_4)$;
(b) $J_\nu^p(\boldsymbol{x},\cdot,\epsilon)$ is a continuous function on $\mathbb{R}$ for a.e $\boldsymbol{x} \in \Sigma_4$ and all $\epsilon \in \mathbb{R}$;
(c) $J_\nu^p(\boldsymbol{x},\zeta,\cdot)$ is a locally Lipschitz function on $\mathbb{R}$ for a.e $\boldsymbol{x} \in \Sigma_4$ and all $\zeta \in \mathbb{R}$;
(d) There exists $c_{J_\nu^p} > 0$ such that $|\partial J_\nu^p(\boldsymbol{x},\zeta,\epsilon)| \leqslant c_{J_\nu^p}$ for a.e $\boldsymbol{x} \in \Sigma_4$ and all $\zeta,\epsilon \in \mathbb{R}$;
(e) There exists $m_{J_\nu^p} > 0$ such that

$$J_\nu^{0k}(\boldsymbol{x},\zeta_1,\epsilon_1;\epsilon_2 - \epsilon_1) + J_\nu^{0k}(\boldsymbol{x},\zeta_2,\epsilon_2;\epsilon_1 - \epsilon_2)$$
$$\leqslant m_{J_\nu^p}\left(|\zeta_1 - \zeta_2| + |\epsilon_1 - \epsilon_2|\right)|\epsilon_1 - \epsilon_2|$$

for a.e. $\boldsymbol{x} \in \Sigma_4$ and all $\zeta_1, \zeta_2, \epsilon_1, \epsilon_2 \in \mathbb{R}$.

A two-dimensional example of the model of two elastic bodies considered in Problem 4.1 can be found in [1, Remark 5.1]. For each $p = 1,2$, let $\mathbb{H}_p$ and $\mathbb{V}_p$ be Hilbert spaces defined by

$$\mathbb{H}_p = L^2\left(\mathbb{B}_p;\mathbb{S}^k\right),$$
$$\mathbb{V}_p = \left\{\boldsymbol{v} \in H^1\left(\mathbb{B}_p;\mathbb{R}^k\right) \mid \boldsymbol{v} = \boldsymbol{0} \quad \text{on } \Sigma_1^p\right\}$$

with the inner product

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathbb{H}_p} = \int_{\mathbb{B}_p} \sigma_{ij}(\boldsymbol{x})\tau_{ij}(\boldsymbol{x})\mathrm{d}x, \quad \text{for all } \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathbb{H}_p,$$
$$(\boldsymbol{w},\boldsymbol{v})_{\mathbb{V}_p} = (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathbb{H}_p}, \quad \text{for all } \boldsymbol{w},\boldsymbol{v} \in \mathbb{V}_p$$

and the norms $\|\cdot\|_{\mathbb{H}_p}$, and $\|\cdot\|_{\mathbb{V}_p}$, respectively. Further, the set of admissible velocity fields $\mathbb{K}_p$ are given by

$$\mathbb{K}_p = \left\{ \boldsymbol{v} \in \mathbb{V}_p \mid v_\nu \leqslant g_p \text{ on } \Sigma_3^p \right\}, \quad p = 1, 2$$

Since $\mathbb{V}_p$ $(p = 1, 2)$ are Hilbert spaces, there exist the trace operators $\mathbf{M} : \mathbb{V}_1 \to L^2(\Sigma^1; \mathbb{R}^k)$ and $\mathbf{N} : \mathbb{V}_2 \to L^2(\Sigma^2; \mathbb{R}^k)$ with the norms $\|\mathbf{M}\|$ and $\|\mathbf{N}\|$, respectively such that

$$\|\mathbf{M}\boldsymbol{\theta}\|_{L^2(\Sigma^1;\mathbb{R}^k)} \leq \|\mathbf{M}\| \|\boldsymbol{\theta}\|_{\mathbb{V}_1} \quad \text{and} \quad \|\mathbf{N}\boldsymbol{z}\|_{L^2(\Sigma^2;\mathbb{R}^k)} \leq \|\mathbf{N}\| \|\boldsymbol{z}\|_{\mathbb{V}_2},$$

for all $\boldsymbol{\theta} \in \mathbb{V}_1$ and $\boldsymbol{z} \in \mathbb{V}_2$. In addition, we need some conditions of constants on Problem 4.1 as follows:

Hypothesis $\mathfrak{h}(1)$:

(a) $m_{\mathcal{A}^1} > \left(m_{j_\nu^1} + m_{J_\nu^1}\right) \|\mathbf{M}\|^2 + L_{\mathcal{G}_b} \|\mathbf{M}\|, m_{\mathcal{A}^2} > \left(m_{j_\nu^2} + m_{J_\nu^2}\right) \|\mathbf{N}\|^2$.

(b) The following inequality holds:

$$\frac{\left(m_{j_\nu^1} + m_{J_\nu^1}\right)\left(m_{j_\nu^2} + m_{J_\nu^2}\right) \|\mathbf{M}\|^2 \|\mathbf{N}\|^2}{\left(m_{\mathcal{A}^1} - \left(m_{j_\nu^1} + m_{J_\nu^1}\right) \|\mathbf{M}\|^2 - L_{\mathcal{G}_b} \|\mathbf{M}\|\right)\left(m_{\mathcal{A}^2} - \left(m_{j_\nu^2} + m_{J_\nu^2}\right) \|\mathbf{N}\|^2\right)} < 1.$$

Setting $\boldsymbol{\theta} = \boldsymbol{w}^1$, $\boldsymbol{z} = \boldsymbol{w}^2$, $\boldsymbol{\gamma} = \boldsymbol{v}^1$ and $\boldsymbol{v} = \boldsymbol{v}^2$, the below coupling variational system are derived from Problem 4.1 by using the Green formula.

Problem 4.2 (see [1, Problem 6]): Find displacement fields $\boldsymbol{\theta} : [0, T] \to \mathbb{V}_1$, $\boldsymbol{z} : [0, T] \to \mathbb{V}_2$ and $\boldsymbol{u} : (0, T) \to L^2(\Sigma_3^1)$ such that for all $t \in [0, T]$,

$$\begin{cases} \boldsymbol{u}'(t) = \mathcal{Q}\left(t, \boldsymbol{u}(t), \theta_\nu(t)\right) \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \end{cases} \tag{54}$$

$$\begin{aligned} &\left\langle \mathcal{A}^1\left(t, \boldsymbol{\varepsilon}\left(\boldsymbol{\theta}(t)\right)\right), \boldsymbol{\varepsilon}\left(\boldsymbol{\gamma}(t)\right) - \boldsymbol{\varepsilon}\left(\boldsymbol{\theta}(t)\right)\right\rangle_{\mathbb{H}_1} \\ &\quad + \left\langle \int_0^t \mathcal{L}(t-s)\boldsymbol{\varepsilon}\left(\boldsymbol{\theta}(s)\right) \mathrm{d}s, \boldsymbol{\varepsilon}\left(\boldsymbol{\gamma}(t)\right) - \boldsymbol{\varepsilon}\left(\boldsymbol{\theta}(t)\right)\right\rangle_{\mathbb{H}_1} \\ &\quad + \int_{\Sigma_3^1} \boldsymbol{u}(t)\left(\gamma_\nu - \theta_\nu\right) \mathrm{d}\Sigma + \int_{\Sigma_3^1} j_\nu^{\circ 1}\left(\theta_\nu(t); \gamma_\nu - \theta_\nu(t)\right) \mathrm{d}\Sigma \\ &\quad + \int_{\Sigma_3^1} \mathcal{G}_b\left(\theta_\nu(t)\right)\left(\left\|\boldsymbol{\gamma}_\tau\right\| - \left\|\boldsymbol{\theta}_\tau\right\|\right) \mathrm{d}\Sigma + \int_{\Sigma_4} j_\nu^{\circ 1}\left(z_\nu(t), \theta_\nu(t); \gamma_\nu - \theta_\nu(t)\right) \mathrm{d}\Sigma \\ &\geqslant \left\langle \boldsymbol{f}_N^1(t), \boldsymbol{\gamma} - \boldsymbol{\theta}(t)\right\rangle_{L^2(\Sigma_2^1;\mathbb{R}^k)} + \left\langle \boldsymbol{f}_0^1(t), \boldsymbol{\gamma} - \boldsymbol{\theta}(t)\right\rangle_{\mathbb{H}_1} \quad \text{for all } \boldsymbol{\gamma} \in \mathbb{K}_1, \end{aligned} \tag{55}$$

$$\begin{aligned} &\left\langle \mathcal{A}^2\left(t, \boldsymbol{\varepsilon}\left(\boldsymbol{z}(t)\right)\right), \boldsymbol{\varepsilon}\left(\boldsymbol{v}\right) - \boldsymbol{\varepsilon}\left(\boldsymbol{z}(t)\right)\right\rangle_{\mathbb{H}_2} + \int_{\mathbb{B}_2} \mathcal{G}\left(\boldsymbol{\varepsilon}\left(\boldsymbol{v}\right)\right) \mathrm{d}x - \int_{\mathbb{B}_2} \mathcal{G}\left(\boldsymbol{\varepsilon}\left(\boldsymbol{z}(t)\right)\right) \mathrm{d}x \\ &\quad + \int_{\Sigma_3^2} j_\nu^{\circ 2}\left(z_\nu(t); v_\nu - z_\nu(t)\right) \mathrm{d}\Sigma + \int_{\Sigma_4} j_\nu^{\circ 2}\left(\theta_\nu(t), z_\nu(t); v_\nu - z_\nu(t)\right) \mathrm{d}\Sigma \\ &\geqslant \left\langle \boldsymbol{f}_N^2(t), \boldsymbol{v} - \boldsymbol{z}(t)\right\rangle_{\mathcal{L}^2(\Sigma_2^2;\mathbb{R}^k)} + \left\langle \boldsymbol{f}_0^2(t), \boldsymbol{v} - \boldsymbol{z}(t)\right\rangle_{\mathbb{H}_2} \quad \text{for all } \boldsymbol{v} \in \mathbb{K}_2. \end{aligned} \tag{56}$$

Let us consider $\mathbf{U} = L^2(\Sigma_3^1), \mathbf{X}_1 = \mathbb{V}_1, \mathbf{X}_2 = \mathbb{V}_2, P_1 = \mathbb{K}_1, P_2 = \mathbb{K}_2, \mathcal{W}_1 = L^2(\Sigma_4; \mathbb{R}^k)$, $\mathcal{W}_2 = L^2(\Sigma_3^1 \cup \Sigma_4; \mathbb{R}^k)$, $\mathcal{W}_3 = L^2(\Sigma_3^1; \mathbb{R}^k)$, $\mathcal{W}_4 = L^2(\Sigma_3^2 \cup \Sigma_4; \mathbb{R}^k)$, $\mathbf{V}_1 = \mathbb{V}_1^*$. Then, we introduce the functions $H : (0, T) \times \mathbf{X}_1 \times \mathbf{U} \to \mathbf{U}$, $\mathcal{S}_1 : L^2(0, T; \mathbf{X}_1) \to L^2(0, T; \mathbf{V}_1)$, $F : (0, T) \times \mathbf{X}_1 \to \mathbf{X}_1^*$, $Q : (0, T) \times \mathbf{V}_1 \times \mathbf{U} \to \mathbf{X}$, $J_1 : \mathbf{W}_2 \times \mathbf{W}_1 \to \mathbb{R}$, $\Upsilon : \mathbf{X}_1 \times \mathbf{X}_1 \to \mathbb{R}$, $J_2 : \mathbf{W}_1 \times \mathbf{W}_2 \to \mathbb{R}$, $G : (0, T) \times \mathbf{X}_2 \to \mathbf{X}_2^*$, $\varphi : \mathbf{X}_2 \to \mathbb{R}$, and $\boldsymbol{f} : (0, T) \to \mathbf{X}_2^*$ defined by

$$\begin{aligned} & H(t, \boldsymbol{u}, \boldsymbol{\theta})(\boldsymbol{x}) = \mathcal{Q}\left(\boldsymbol{x}, t, \boldsymbol{u}(\boldsymbol{x}, t), \theta_\nu(\boldsymbol{x}, t)\right) \\ & \qquad \text{for all } (\boldsymbol{x}, t, \boldsymbol{u}, \boldsymbol{\theta}) \in \Sigma_3^1 \times [0, T] \times \mathbf{U} \times \mathbf{X}_1, \end{aligned} \tag{57}$$

$$(\mathcal{S}_1\boldsymbol{\theta})(t) = \int_0^l \mathcal{L}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{\theta}(s))\, \mathrm{d}s \quad \text{for all } t \in (0, T), \boldsymbol{\theta} \in L^2\left([0, T]; \mathbf{X}_1\right), \tag{58}$$

$$\langle F(t, \boldsymbol{\theta}), \boldsymbol{\gamma}\rangle_{\mathbf{X}_1^* \times \mathbf{X}_1} = \left\langle \mathcal{A}^1(t, \boldsymbol{\varepsilon}(\boldsymbol{\theta}(t))), \boldsymbol{\varepsilon}(\boldsymbol{\gamma}(t))\right\rangle_{\mathbb{H}_1} \quad \text{for all } t \in [0, T], \boldsymbol{\theta}, \boldsymbol{\gamma} \in \mathbf{X}_1, \tag{59}$$

$$\begin{aligned} \langle Q(t, \zeta, \boldsymbol{u}), \boldsymbol{\gamma}\rangle_{\mathbf{X}_1^* \times \mathbf{X}_1} &= \langle \zeta(t), \boldsymbol{\gamma}\rangle_{\mathbb{H}_1} + \int_{\Sigma_3^1} \boldsymbol{u}(t)\gamma_\nu \mathrm{d}\Sigma - \left\langle \boldsymbol{f}_N^1(t), \boldsymbol{\gamma}\right\rangle_{\mathcal{L}^2(\Sigma_2^1; \mathbb{R}^k)} \\ & \quad - \left\langle \boldsymbol{f}_0^1(t), \boldsymbol{\gamma}\right\rangle_{\mathbb{H}_1} \quad \text{for all } \boldsymbol{\gamma} \in \mathbf{X}_1, \boldsymbol{u} \in \mathbf{U}, \zeta \in \mathbf{V}_1, \end{aligned} \tag{60}$$

$$J_1(\mu, \beta) = \int_{\Sigma_3^1} j_\nu^1(\beta(t))\mathrm{d}\Sigma + \int_{\Sigma_4} J_\nu^1(\mu(t), \beta(t))\mathrm{d}\Sigma \quad \text{for all } (\mu, \beta) \in \mathcal{W}_1 \times \mathcal{W}_2, \tag{61}$$

$$\Upsilon(\boldsymbol{\theta}, \boldsymbol{\gamma}) = \int_{\Sigma_3^1} \mathcal{G}_b\left(\theta_\nu(t)\right)\left(\|\boldsymbol{\gamma}_\tau\|\right) \mathrm{d}\Sigma \quad \text{for all } \boldsymbol{\theta}, \boldsymbol{\gamma} \in \mathbf{X}_1, \tag{62}$$

$$\langle G(t, \boldsymbol{z}), \boldsymbol{v}\rangle_{\mathbf{X}_2^* \times \mathbf{X}_2} = \left\langle \mathcal{A}^2(t, \boldsymbol{\varepsilon}(\boldsymbol{z}(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}(t))\right\rangle_{\mathbb{H}_2} \quad \text{for all } t \in [0, T], \boldsymbol{z}, \boldsymbol{v} \in \mathbf{X}_2, \tag{63}$$

$$J_2(\rho, \varrho) = \int_{\Sigma_4} J_\nu^2(\rho(t), \varrho(t))\mathrm{d}\Sigma + \int_{\Sigma_3^2} j_\nu^2(\varrho(t))\mathrm{d}\Sigma \quad \text{for all } (\rho, \varrho) \in \mathcal{W}_1 \times \mathcal{W}_4, \tag{64}$$

$$\langle \boldsymbol{f}(t), \boldsymbol{v}\rangle_{\mathbf{X}_2^* \times \mathbf{X}_2} = \left\langle \boldsymbol{f}_N^2(t), \boldsymbol{v}\right\rangle_{L^2(\Sigma_2^2, \mathbb{R}^k)} + \left\langle \boldsymbol{f}_0^2(t), \boldsymbol{v}\right\rangle_{\mathbb{H}_2} \quad \text{for all } \boldsymbol{v} \in \mathbf{X}_2, \tag{65}$$

$$\varphi(\boldsymbol{z}) = \int_{\mathbb{B}_2} \mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{z}(t)))\mathrm{d}x \text{ for all } \boldsymbol{z} \in \mathbf{X}_2. \tag{66}$$

Taking in account of relations (57)–(66), we observe that Problem 4.2 can be rewritten in the following problem:

Problem 4.3: Find $\boldsymbol{\theta} \in C([0, T]; \mathbf{X}_1), \boldsymbol{z} \in C([0, T]; \mathbf{X}_2)$ and $\boldsymbol{u} \in C([0, T]; \mathbf{U})$ such that for all $t \in (0, T), \boldsymbol{\theta}(t) \in P_1, \boldsymbol{z}(t) \in P_2$, and

$$\begin{aligned} & \boldsymbol{u}'(t) = H(t, \boldsymbol{u}(t), \boldsymbol{\theta}(t)), \boldsymbol{u}(0) = \boldsymbol{u}_0, \\ & \langle F(t, \boldsymbol{\theta}(t)) + Q\left(t, (\mathcal{S}_1\boldsymbol{\theta})(t), \boldsymbol{u}(t)\right), \boldsymbol{\gamma} - \boldsymbol{\theta}(t)\rangle + \Upsilon\left(\boldsymbol{\theta}(t), \boldsymbol{\gamma}\right) \\ & \quad - \Upsilon\left(\boldsymbol{\theta}(t), \boldsymbol{\theta}(t)\right) + J_1^\circ\left(\mathbf{N}\boldsymbol{z}(t), \mathbf{M}\boldsymbol{\theta}(t); \mathbf{M}\boldsymbol{\gamma} - \mathbf{M}\boldsymbol{\theta}(t)\right) \geqslant 0, \quad \forall\, \boldsymbol{\gamma} \in P_1, \\ & \langle G(t, \boldsymbol{z}(t)), \boldsymbol{v} - \boldsymbol{z}(t)\rangle + J_2^\circ\left(\mathbf{M}\boldsymbol{\theta}(t), \mathbf{N}\boldsymbol{z}(t); \mathbf{N}\boldsymbol{v} - \mathbf{N}\boldsymbol{z}(t)\right) \\ & \quad + \varphi(\boldsymbol{v}) - \varphi(\boldsymbol{z}(t)) \geqslant \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{z}(t)\rangle, \quad \forall\, \boldsymbol{v} \in P_2. \end{aligned}$$

We see that Problem 4.3 is a special form of Problem 1.1. Then, under the assumptions on the data of Problem 4.1, it is easy to verify that all conditions of Theorem 2.7 are satisfied with

$$\begin{cases} a_H = L_{\mathcal{Q}}\|\mathbf{M}\|, a_F = m_{\mathcal{A}^1}, L_F = 0, L_Q = 0, a_{\mathcal{S}_1} = a_{\mathcal{S}_1}, a_{\mathcal{S}_2} = 0, \\ a_\Upsilon = L_{\mathcal{G}_b}\|\mathbf{M}\|, b_\Upsilon = 0, m_{J_1} = m_{j_\nu^1} + m_{J_\nu^1}, m_{J_2} = m_{j_\nu^2} + m_{J_\nu^2}, \\ a_G = m_{\mathcal{A}^2}, L_G = 0. \end{cases} \tag{67}$$

Therefore, together with Theorem 2.7, we achieve the existence and uniqueness results to Problem 4.2.

Theorem 4.1 (see [1, Theorem 5.2]): *Under hypotheses* $\mathfrak{h}(\boldsymbol{f}_0^p)$, $\mathfrak{h}(\boldsymbol{f}_N^p)$, $\mathfrak{h}(\mathcal{A}^p)$, $\mathfrak{h}(\mathcal{L})$, $\mathfrak{h}(j_\nu^p)$, $\mathfrak{h}(J_\nu^p)$, $\mathfrak{h}(\mathcal{G})$ *and* $\mathfrak{h}(\mathcal{G}_b)$ *and* $\mathfrak{h}(1)$ *for all* $p = 1, 2$, *Problem 4.2 has a unique solution.*

We point out that according to the similar arguments in the proof of [36, Theorem 8], we are able to examine the hypotheses $\mathfrak{h}(J_1)(d)$, $\mathfrak{h}(J_2)(d)$ and $\mathfrak{h}(\Upsilon)(b, c)$. To treat error bounds for Problem 4.2 by applying the results established in Section 3, we first consider the coupled regularized gap functions of the system given by (55) and (56) under the setting of the Equations (58)–(66). Let $\omega_i > 0$ with $i = 1, 2$ and $\boldsymbol{u} \in C([0, T]; \mathbf{U})$ be fixed. We now introduce two functions $\widehat{\Delta}_{\omega_1} \colon [0, T] \times \mathbf{U} \times C([0, T]; \mathbb{K}_1) \times C([0, T]; \mathbb{K}_2) \to \mathbb{R}$ and $\widehat{\Theta}_{\omega_2} \colon [0, T] \times C([0, T]; \mathbb{K}_2) \times C([0, T]; \mathbb{K}_1) \to \mathbb{R}$ given by

$$\begin{aligned} &\widehat{\Delta}_{\omega_1}(t, \boldsymbol{u}(t), \boldsymbol{\theta}, \boldsymbol{z}) \\ &\quad = \sup_{\boldsymbol{\gamma} \in \mathbb{K}_1} \Bigg(\langle F(t, \boldsymbol{\theta}(t)) + Q\left(t, \left(\mathcal{S}_1\boldsymbol{\theta}\right)(t), \boldsymbol{u}(t)\right), \boldsymbol{\theta}(t) - \boldsymbol{\gamma} \rangle - \Upsilon\left(\boldsymbol{\theta}(t), \boldsymbol{\gamma}\right) \\ &\qquad + \Upsilon\left(\boldsymbol{\theta}(t), \boldsymbol{\theta}(t)\right) - J_1^\circ\left(\mathbf{N}\boldsymbol{z}(t), \mathbf{M}\boldsymbol{\theta}(t); \mathbf{M}\boldsymbol{\gamma} - \mathbf{M}\boldsymbol{\theta}(t)\right) - \frac{1}{2\omega_1}\|\boldsymbol{\theta}(t) - \boldsymbol{\gamma}\|_{\mathbf{X}_1}^2 \Bigg) \end{aligned} \tag{68}$$

and

$$\begin{aligned} &\widehat{\Theta}_{\omega_2}(t, \boldsymbol{z}, \boldsymbol{\theta}) \\ &\quad = \sup_{\boldsymbol{v} \in \mathbb{K}_2} \Bigg(\langle G(t, \boldsymbol{z}(t)) - \boldsymbol{f}(t), \boldsymbol{z}(t) - \boldsymbol{v} \rangle - J_2^\circ\left(\mathbf{M}\boldsymbol{\theta}(t), \mathbf{N}\boldsymbol{z}(t); \mathbf{N}\boldsymbol{v} - \mathbf{N}\boldsymbol{z}(t)\right) \\ &\qquad - \varphi(\boldsymbol{v}) + \varphi(\boldsymbol{z}(t)) - \frac{1}{2\omega_2}\|\boldsymbol{z}(t) - \boldsymbol{v}\|_{\mathbf{X}_2}^2 \Bigg), \end{aligned} \tag{69}$$

for all $(\boldsymbol{\theta}, \boldsymbol{z}) \in C([0, T]; \mathbb{K}_1) \times C([0, T]; \mathbb{K}_2)$ and $t \in [0, T]$.

From Propositions 3.1–3.4, the below consequence follows.

Proposition 4.2: *Suppose that conditions* $\mathfrak{h}(\boldsymbol{f}_0^1)$, $\mathfrak{h}(\boldsymbol{f}_N^1)$, $\mathfrak{h}(\mathcal{A}^1)$, $\mathfrak{h}(\mathcal{L})$, $\mathfrak{h}(j_\nu^1)$, $\mathfrak{h}(J_\nu^1)$ *and* $\mathfrak{h}(\mathcal{G}_b)$ *hold. Then, for each* $\omega_1 > 0$, *the function* $\widehat{\Delta}_{\omega_1} \colon [0, T] \times \mathbf{U} \times C([0, T]; \mathbb{K}_1) \times C([0, T]; \mathbb{K}_2) \to \mathbb{R}$ *defined by* (68) *satisfies the following properties:*

(i) *Given* $(\boldsymbol{u}, \boldsymbol{z}) \in C([0, T]; \mathbf{U}) \times C([0, T]; \mathbb{K}_2)$, $\widehat{\Delta}_{\omega_1}$ *is the regularized gap function of the variational-hemivariational inequality corresponding to* $(\boldsymbol{u}, \boldsymbol{z})$ *formulated by the system* (55).
(ii) *For each* $t \in [0, T]$, $(\boldsymbol{\theta}, \boldsymbol{z}) \mapsto \Delta_{\omega_1}(t, \boldsymbol{u}(t), \boldsymbol{\theta}, \boldsymbol{z})$ *is a lower semicontinuous function.*
(iii) *For each* $(\boldsymbol{\theta}, \boldsymbol{z}) \in C([0, T]; \mathbb{K}_1) \times C([0, T]; \mathbb{K}_2)$ *fixed, the regularized gap function* $t \mapsto \Delta_{\omega_1}(t, \boldsymbol{u}(t), \boldsymbol{\theta}, \boldsymbol{z})$ *belongs to* $L_+^\infty(0, T)$.

Proposition 4.3: *Suppose that conditions* $\mathfrak{h}(\boldsymbol{f}_0^2)$, $\mathfrak{h}(\boldsymbol{f}_N^2)$, $\mathfrak{h}(\mathcal{A}^2)$, $\mathfrak{h}(j_\nu^2)$, $\mathfrak{h}(J_\nu^2)$ *and* $\mathfrak{h}(\mathcal{G})$ *hold. Then, for each* $\omega_2 > 0$, *the function* $\widehat{\Theta}_{\omega_2} : [0,T] \times C([0,T];\mathbb{K}_2) \times C([0,T];\mathbb{K}_1) \to \mathbb{R}$ *defined by* (69) *satisfies the following properties:*

(i) *Given* $\boldsymbol{\theta} \in C([0,T];\mathbb{K}_1)$, $\widehat{\Theta}_{\omega_2}$ *is the regularized gap function of the variational-hemivariational inequality corresponding to* $\boldsymbol{\theta}$ *formulated by the system* (56).
(ii) *For each* $t \in [0,T]$, $(\boldsymbol{z},\boldsymbol{\theta}) \mapsto \widehat{\Theta}_{\omega_2}(t,\boldsymbol{z},\boldsymbol{\theta})$ *is a lower semicontinuous function.*
(iii) *For each* $(\boldsymbol{\theta},\boldsymbol{z}) \in C([0,T];\mathbb{K}_1) \times C([0,T];\mathbb{K}_2)$ *fixed, the function* $t \mapsto \widehat{\Theta}_{\omega_2}(t,\boldsymbol{z},\boldsymbol{\theta})$ *belongs to* $L_+^\infty(0,T)$.

Using coupled regularized gap functions $\widehat{\Delta}_{\omega_1}$ and $\widehat{\Theta}_{\omega_2}$, we build up the error bounds for Problem 4.2 deduced from Theorem 3.5 with constants given by (67).

Theorem 4.4: *Let* $\boldsymbol{u}_0 \in \mathbf{U}$ *be fixed and a triple* $(\boldsymbol{u}^*,\boldsymbol{\theta}^*,\boldsymbol{z}^*) \in C([0,T];\mathbf{U}) \times C([0,T];\mathbb{K}_1) \times C([0,T];\mathbb{K}_2)$ *be the unique solution of Problem* 4.2. *Assume that* $\mathfrak{h}(\boldsymbol{f}_0^p)$, $\mathfrak{h}(\boldsymbol{f}_N^p)$, $\mathfrak{h}(\mathcal{A}^p)$, $\mathfrak{h}(\mathcal{L})$, $\mathfrak{h}(j_\nu^p)$, $\mathfrak{h}(J_\nu^p)$, $\mathfrak{h}(\mathcal{G})$ *and* $\mathfrak{h}(\mathcal{G}_b)$ *are satisfied for all* $p = 1, 2$. *In addition, the following conditions hold (denoted by* $\underline{\mathfrak{h}'(1)}$*):*

(a) $m_{\mathcal{A}^1} > (m_{j_\nu^1} + m_{J_\nu^1})\|\mathbf{M}\|^2 + L_{\mathcal{G}_b}\|\mathbf{M}\| + \frac{1}{2\omega_1}$, $\ m_{\mathcal{A}^2} > (m_{j_\nu^2} + m_{J_\nu^2})\|\mathbf{N}\|^2 + \frac{1}{2\omega_2}$;
(b) *the parameters* $\omega_1, \omega_2 > 0$ *satisfy*

$$\frac{\left(m_{j_\nu^1} + m_{J_\nu^1}\right)\left(m_{j_\nu^2} + m_{J_\nu^2}\right)\|\mathbf{M}\|^2\,\|\mathbf{N}\|^2}{\left(m_{\mathcal{A}^1} - \left(m_{j_\nu^1} + m_{J_\nu^1}\right)\|\mathbf{M}\|^2 - L_{\mathcal{G}_b}\|\mathbf{M}\| - \frac{1}{2\omega_1}\right)\left(m_{\mathcal{A}^2} - \left(m_{j_\nu^2} + m_{J_\nu^2}\right)\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}\right)}$$
$$< \frac{1}{2}.$$

Then, for each $(\boldsymbol{u},\boldsymbol{\theta},\boldsymbol{z}) \in C([0,T];L^2(\Sigma_3^1)) \times C([0,T];\mathbb{K}_1) \times C([0,T];\mathbb{K}_2)$ *there exist the functions* $\widehat{\Pi}_i \in L_+^\infty(0,T)$ *with* $i = 1, 2, 3$ *such that*

$$\|\boldsymbol{\theta}(t) - \boldsymbol{\theta}^*(t)\|_{\mathbb{V}_1} \le \widehat{\Pi}_1(t), \quad \text{for all } t \in [0,T];$$
$$\|\boldsymbol{z}(t) - \boldsymbol{z}^*(t)\|_{\mathbb{V}_2} \le \widehat{\Pi}_2(t), \quad \text{for all } t \in [0,T];$$

and

$$\|\boldsymbol{u}(t) - \boldsymbol{u}^*(t)\|_{L^2(\Sigma_3^1)} \le \widehat{\Pi}_3(t), \quad \text{for all } t \in [0,T],$$

where $\boldsymbol{u} \in C([0,T];L^2(\Sigma_3^1))$ *is the unique mild solution of the following equation*

$$\begin{cases} \boldsymbol{u}_t'(\boldsymbol{x},t) = H(\boldsymbol{x},t,\boldsymbol{u}(\boldsymbol{x},t),\boldsymbol{\theta}(\boldsymbol{x},t)) & \text{in } \Sigma_3^1 \times (0,T), \\ \boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}) & \text{in } \Sigma_3^1 \end{cases}$$

and

$$\widehat{\Pi}_1(t) = \frac{\widehat{\mho}_{\omega_1,\omega_2}(t,\boldsymbol{u}(t),\boldsymbol{\theta},\boldsymbol{z})}{\widehat{\mathbf{C}}_1} + \frac{\widehat{\mathbf{C}}_2}{\widehat{\mathbf{C}}_1^2}\int_0^t \widehat{\mho}_{\omega_1,\omega_2}(s,\boldsymbol{u}(s),\boldsymbol{\theta},\boldsymbol{z}) \exp\left\{\frac{\widehat{\mathbf{C}}_2}{\widehat{\mathbf{C}}_1}(t-s)\right\} \mathrm{d}s;$$

$$\widehat{\Pi}_2(t) = \sqrt{\frac{2\widehat{\Theta}_{\omega_2}(t,\boldsymbol{z},\boldsymbol{\theta})}{m_{\mathcal{A}^2} - \left(m_{j^2_\nu} + m_{J^2_\nu}\right)\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}}} + \frac{\left(m_{j^2_\nu} + m_{J^2_\nu}\right)\|\mathbf{N}\|\,\|\mathbf{M}\|}{m_{\mathcal{A}^2} - \left(m_{j^2_\nu} + m_{J^2_\nu}\right)\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}}\widehat{\Pi}_1(t);$$

$$\widehat{\Pi}_3(t) = L_{\mathcal{Q}}\|\mathbf{M}\|\,\mathrm{e}^{L_{\mathcal{Q}}T}\int_0^t \widehat{\Pi}_1(s)\,\mathrm{d}s,$$

with the function $\widehat{\mho}_{\omega_1,\omega_2} : [0,T] \times \mathbf{U} \times C([0,T];\mathbb{K}_1) \times C([0,T];\mathbb{K}_2) \to \mathbb{R}_+$ *given by*

$$\begin{aligned}
&\widehat{\mho}_{\omega_1,\omega_2}(t,\boldsymbol{u}(t),\boldsymbol{\theta},\boldsymbol{z})\\
&\quad = 2\sqrt{\frac{\widehat{\Delta}_{\omega_1}(t,\boldsymbol{u}(t),\boldsymbol{\theta},\boldsymbol{z})}{\left(m_{\mathcal{A}^1} - \left(m_{j^1_\nu} + m_{J^1_\nu}\right)\|\mathbf{M}\|^2 - L_{\mathcal{G}_b}\|\mathbf{M}\| - \frac{1}{2\omega_1}\right)}}\\
&\qquad + \frac{2\left(m_{j^1_\nu} + m_{J^1_\nu}\right)\|\mathbf{N}\|\,\|\mathbf{M}\|}{\left(m_{\mathcal{A}^1} - \left(m_{j^1_\nu} + m_{J^1_\nu}\right)\|\mathbf{M}\|^2 - L_{\mathcal{G}_b}\|\mathbf{M}\| - \frac{1}{2\omega_1}\right)}\\
&\qquad \times \sqrt{\frac{2\widehat{\Theta}_{\omega_2}(t,\boldsymbol{z},\boldsymbol{\theta})}{m_{\mathcal{A}^2} - \left(m_{j^2_\nu} + m_{J^2_\nu}\right)\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}}}
\end{aligned}$$

and constants $\widehat{\mathbf{C}}_1, \widehat{\mathbf{C}}_2 > 0$ *defined by*

$$\begin{cases}
\widehat{\mathbf{C}}_1 = 1 - \dfrac{2\left(m_{j^1_\nu} + m_{J^1_\nu}\right)\left(m_{j^2_\nu} + m_{J^2_\nu}\right)\|\mathbf{M}\|^2\,\|\mathbf{N}\|^2}{\left(m_{\mathcal{A}^1} - \left(m_{j^1_\nu} + m_{J^1_\nu}\right)\|\mathbf{M}\|^2 - L_{\mathcal{G}_b}\|\mathbf{M}\| - \frac{1}{2\omega_1}\right)\left(m_{\mathcal{A}^2} - \left(m_{j^2_\nu} + m_{J^2_\nu}\right)\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}\right)};\\
\widehat{\mathbf{C}}_2 = \dfrac{2(a_{\mathcal{S}_1} + L_{\mathcal{Q}}\|\mathbf{M}\|\,\mathrm{e}^{L_{\mathcal{Q}}T})}{\left(m_{\mathcal{A}^1} - \left(m_{j^1_\nu} + m_{J^1_\nu}\right)\|\mathbf{M}\|^2 - L_{\mathcal{G}_b}\|\mathbf{M}\| - \frac{1}{2\omega_1}\right)}.
\end{cases}$$

Let us now present a specific category of quasistatic contact problems involving two elastic bodies, where there is no memory effect during the contact process, as described by (53) on $\Sigma_4 \times (0,T)$. This is a particular case of Problem 4.1.

Problem 4.4: Given $\boldsymbol{u} \in L^2(\Sigma^1_3)$, find displacement fields $\boldsymbol{w}^1 : \mathbb{B}_1 \times [0,T] \to \mathbb{R}^k, \boldsymbol{w}^2 : \mathbb{B}_2 \times [0,T] \to \mathbb{R}^k$ and stress fields $\boldsymbol{\sigma}^1 : \mathbb{B}_1 \times [0,T] \to \mathbb{S}^k, \boldsymbol{\sigma}^2 : \mathbb{B}_2 \times [0,T] \to \mathbb{S}^k$ such that

$$\begin{aligned}
&\operatorname{Div}\boldsymbol{\sigma}^p(t) + \boldsymbol{f}^p_0(t) = \mathbf{0} \quad \text{in } \mathbb{B}_p \times (0,T),\\
&\boldsymbol{\sigma}^1(t) = \mathcal{A}^1\left(t, \boldsymbol{\varepsilon}\left(\boldsymbol{w}^1(t)\right)\right) + \int_0^t \mathcal{L}(t-s)\boldsymbol{\varepsilon}\left(\boldsymbol{w}^1(s)\right)\mathrm{d}s \quad \text{in } \mathbb{B}_1 \times (0,T),
\end{aligned}$$

$$
\begin{aligned}
&\boldsymbol{\sigma}^2(t) \in \mathcal{A}^2\left(t, \boldsymbol{\varepsilon}\left(\boldsymbol{w}^2(t)\right)\right) + \partial^c \mathcal{G}\left(\boldsymbol{\varepsilon}\left(\boldsymbol{w}^2(t)\right)\right) \quad \text{in } \mathbb{B}_2 \times (0,T),\\
&\boldsymbol{w}^p(t) = \mathbf{0} \quad \text{on } \Sigma_1^p \times (0,T),\\
&\boldsymbol{\sigma}^p(t)\boldsymbol{\nu} = \boldsymbol{f}_N^p(t) \quad \text{on } \Sigma_2^p \times (0,T),\\
&\begin{cases}
w_\nu^1(t) \leqslant g_1, \sigma_\nu^1(t) + \boldsymbol{u} \leqslant 0\\
\left(\sigma_\nu^1(t) + \boldsymbol{u}\right)\left(w_\nu^1 - g_1\right) = 0,\\
-\sigma_\nu^1(t) \in \partial j_\nu^1\left(w_\nu^1(t)\right),\\
\left\|\boldsymbol{\sigma}_\tau^1(t)\right\| \leqslant \mathcal{G}_b\left(w_\nu^1(t)\right),\\
\boldsymbol{\sigma}_\tau^1(t) = -\mathcal{G}_b\left(w_\nu^1(t)\right)\dfrac{\boldsymbol{w}_\tau^1(t)}{\left\|\boldsymbol{w}_\tau^1(t)\right\|} \quad \text{if } \left\|\boldsymbol{w}_\tau^1(t)\right\| \neq 0 \quad \text{on } \Sigma_3^1 \times (0,T),
\end{cases}\\
&\begin{cases}
w_\nu^2(t) \leqslant g_2, \sigma_\nu^2 \leqslant 0\\
\left(w_\nu^2(t) - g_2\right)\sigma_\nu^2(t) = 0,\\
-\sigma_\nu^2(t) \in \partial j_\nu^2\left(w_\nu^2(t)\right)\\
\boldsymbol{\sigma}_\tau^2 = \mathbf{0}, \quad \text{on } \Sigma_3^2 \times (0,T)
\end{cases}\\
&\begin{cases}
-\sigma_\nu^1(t) \in \partial J_\nu^1\left(w_\nu^2(t), w_\nu^1(t)\right),\\
-\sigma_\nu^2(t) \in \partial J_\nu^2\left(w_\nu^1(t), w_\nu^2(t)\right),\\
\boldsymbol{\sigma}_\tau^1 = \boldsymbol{\sigma}_\tau^2 = \mathbf{0}, \quad \text{on } \Sigma_4 \times (0,T).
\end{cases}
\end{aligned}
$$

Let $\boldsymbol{\theta} = \boldsymbol{w}^1$, $\boldsymbol{z} = \boldsymbol{w}^2$, $\boldsymbol{\gamma} = \boldsymbol{v}^1$ and $\boldsymbol{v} = \boldsymbol{v}^2$. Similar to the variational formula of Problem 4.1 in Problem 4.2, we also obtain the coupling variational system of Problem 4.4.

Problem 4.5: Given $\boldsymbol{u} \in L^2(\Sigma_3^1)$, find displacement fields $\boldsymbol{\theta} : [0,T] \to \mathbb{V}_1$ and $\boldsymbol{z} : [0,T] \to \mathbb{V}_2$ such that for all $t \in [0,T]$,

$$
\begin{aligned}
&\left\langle \mathcal{A}^1\left(t, \boldsymbol{\varepsilon}\left(\boldsymbol{\theta}(t)\right)\right), \boldsymbol{\varepsilon}\left(\boldsymbol{\gamma}(t)\right) - \boldsymbol{\varepsilon}\left(\boldsymbol{\theta}(t)\right)\right\rangle_{\mathbb{H}_1} + \left\langle \int_0^t \mathcal{L}(t-s)\boldsymbol{\varepsilon}\left(\boldsymbol{\theta}(s)\right)\mathrm{d}s, \boldsymbol{\varepsilon}\left(\boldsymbol{\gamma}(t)\right) - \boldsymbol{\varepsilon}\left(\boldsymbol{\theta}(t)\right)\right\rangle_{\mathbb{H}_1}\\
&\quad + \int_{\Sigma_3^1} \boldsymbol{u}\left(\gamma_\nu - \theta_\nu\right)\mathrm{d}\Sigma + \int_{\Sigma_3^1} j_\nu^{\circ 1}\left(\theta_\nu(t); \gamma_\nu - \theta_\nu(t)\right)\mathrm{d}\Sigma\\
&\quad + \int_{\Sigma_3^1} \mathcal{G}_b\left(\theta_\nu(t)\right)\left(\left\|\boldsymbol{\gamma}_\tau\right\| - \left\|\boldsymbol{\theta}_\tau\right\|\right)\mathrm{d}\Sigma + \int_{\Sigma_4} j_\nu^{\circ 1}\left(z_\nu(t), \theta_\nu(t); \gamma_\nu - \theta_\nu(t)\right)\mathrm{d}\Sigma\\
&\quad \geqslant \left\langle \boldsymbol{f}_N^1(t), \boldsymbol{\gamma} - \boldsymbol{\theta}(t)\right\rangle_{L^2(\Sigma_2^1;\mathbb{R}^k)} + \left\langle \boldsymbol{f}_0^1(t), \boldsymbol{\gamma} - \boldsymbol{\theta}(t)\right\rangle_{\mathbb{H}_1} \quad \text{for all } \boldsymbol{\gamma} \in \mathbb{K}_1,\\
&\left\langle \mathcal{A}^2\left(t, \boldsymbol{\varepsilon}\left(\boldsymbol{z}(t)\right)\right), \boldsymbol{\varepsilon}\left(\boldsymbol{v}\right) - \boldsymbol{\varepsilon}\left(\boldsymbol{z}(t)\right)\right\rangle_{\mathbb{H}_2} + \int_{\mathbb{B}_2} \mathcal{G}\left(\boldsymbol{\varepsilon}\left(\boldsymbol{v}\right)\right)\mathrm{d}x - \int_{\mathbb{B}_2} \mathcal{G}\left(\boldsymbol{\varepsilon}\left(\boldsymbol{z}(t)\right)\right)\mathrm{d}x\\
&\quad + \int_{\Sigma_3^2} j_\nu^{\circ 2}\left(z_\nu(t); v_\nu - z_\nu(t)\right)\mathrm{d}\Sigma + \int_{\Sigma_4} j_\nu^{\circ 2}\left(\theta_\nu(t), z_\nu(t); v_\nu - z_\nu(t)\right)\mathrm{d}\Sigma\\
&\quad \geqslant \left\langle \boldsymbol{f}_N^2(t), \boldsymbol{v} - \boldsymbol{z}(t)\right\rangle_{\mathcal{L}^2(\Sigma_2^2;\mathbb{R}^k)} + \left\langle \boldsymbol{f}_0^2(t), \boldsymbol{v} - \boldsymbol{z}(t)\right\rangle_{\mathbb{H}_2} \quad \text{for all } \boldsymbol{v} \in \mathbb{K}_2.
\end{aligned}
$$

Lastly, we establish the error bound results for Problem 4.5 from using Theorem 3.6 and Theorem 4.4 via coupled regularized gap functions $\widehat{\Delta}_{\omega_1}$ and $\widehat{\Theta}_{\omega_2}$ defined by (68) and (69), respectively.

Theorem 4.5: *Let* $\boldsymbol{u} \in L^2(\Sigma_3^1)$ *be fixed and a pair* $(\boldsymbol{\theta}^*, \boldsymbol{z}^*) \in C([0,T];\mathbb{K}_1) \times C([0,T];\mathbb{K}_2)$ *be the unique solution of Problem 4.5. Assume that* $\mathfrak{h}(\boldsymbol{f}_0^p)$, $\mathfrak{h}(\boldsymbol{f}_N^p)$, $\mathfrak{h}(\mathcal{A}^p)$, $\mathfrak{h}(\mathcal{L})$, $\mathfrak{h}(j_\nu^p)$, $\mathfrak{h}(J_\nu^p)$ *and* $\mathfrak{h}(\mathcal{G}_b)$ *are satisfied for all* $p = 1, 2$. *In addition, the following conditions (denoted by* $\underline{\mathfrak{h}''(1)}$*) hold:*

(a) $m_{\mathcal{A}^1} > \left(m_{j_\nu^1} + m_{J_\nu^1}\right)\|\mathbf{M}\|^2 + L_{\mathcal{G}_b}\|\mathbf{M}\| + \frac{1}{2\omega_1}$, $m_{\mathcal{A}^2} > \left(m_{j_\nu^2} + m_{J_\nu^2}\right)\|\mathbf{N}\|^2 + \frac{1}{2\omega_2}$;

(b) *the parameters* $\omega_1, \omega_2 > 0$ *satisfy*

$$\frac{\left(m_{j_\nu^1} + m_{J_\nu^1}\right)\left(m_{j_\nu^2} + m_{J_\nu^2}\right)\|\mathbf{M}\|^2\|\mathbf{N}\|^2}{\left(m_{\mathcal{A}^1} - \left(m_{j_\nu^1} + m_{J_\nu^1}\right)\|\mathbf{M}\|^2 - L_{\mathcal{G}_b}\|\mathbf{M}\| - \frac{1}{2\omega_1}\right)\left(m_{\mathcal{A}^2} - \left(m_{j_\nu^2} + m_{J_\nu^2}\right)\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}\right)} < \frac{1}{\sqrt{2}}.$$

Then, for each $(\boldsymbol{\theta}, \boldsymbol{z}) \in C([0,T];\mathbb{K}_1) \times C([0,T];\mathbb{K}_2)$ *there exist the functions* $\overline{\Pi}_i \in L_+^\infty(0,T)$ *with* $i = 1, 2$ *such that*

$$\|\boldsymbol{\theta}(t) - \boldsymbol{\theta}^*(t)\|_{\mathbb{V}_1} \le \overline{\Pi}_1(t), \quad \text{for all } t \in [0,T]$$

and

$$\|\boldsymbol{z}(t) - \boldsymbol{z}^*(t)\|_{\mathbb{V}_2} \le \overline{\Pi}_2(t), \quad \text{for all } t \in [0,T],$$

where

$$\overline{\Pi}_1(t) = \frac{\overline{\mho}_{\omega_1,\omega_2}(t,\boldsymbol{u},\boldsymbol{\theta},\boldsymbol{z})}{\overline{\mathbf{C}}_1} + \frac{\overline{\mathbf{C}}_2}{\overline{\mathbf{C}}_1^2}\int_0^t \overline{\mho}_{\omega_1,\omega_2}(s,\boldsymbol{u},\boldsymbol{\theta},\boldsymbol{z})\exp\left\{\frac{\overline{\mathbf{C}}_2}{\overline{\mathbf{C}}_1}(t-s)\right\}\mathrm{d}s;$$

$$\overline{\Pi}_2(t) = \sqrt{\frac{2\widehat{\Theta}_{\omega_2}(t,\boldsymbol{z},\boldsymbol{\theta})}{m_{\mathcal{A}^2} - \left(m_{j_\nu^2} + m_{J_\nu^2}\right)\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}}} + \frac{\left(m_{j_\nu^2} + m_{J_\nu^2}\right)\|\mathbf{N}\|\|\mathbf{M}\|}{m_{\mathcal{A}^2} - \left(m_{j_\nu^2} + m_{J_\nu^2}\right)\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}}\overline{\Pi}_1(t).$$

with the function $\overline{\mho}_{\omega_1,\omega_2} : [0,T] \times L^2(\Sigma_3^1) \times C([0,T];\mathbb{K}_1) \times C([0,T];\mathbb{K}_2) \to \mathbb{R}_+$ *given by*

$$\begin{aligned}
&\overline{\mho}_{\omega_1,\omega_2}(t,\boldsymbol{u},\boldsymbol{\theta},\boldsymbol{z}) \\
&\quad = \sqrt{\frac{2\widehat{\Delta}_{\omega_1}(t,\boldsymbol{u},\boldsymbol{\theta},\boldsymbol{z})}{\left(m_{\mathcal{A}^1} - \left(m_{j_\nu^1} + m_{J_\nu^1}\right)\|\mathbf{M}\|^2 - L_{\mathcal{G}_b}\|\mathbf{M}\| - \frac{1}{2\omega_1}\right)}} \\
&\qquad + \frac{2\left(m_{j_\nu^1} + m_{J_\nu^1}\right)\|\mathbf{N}\|\|\mathbf{M}\|}{\left(m_{\mathcal{A}^1} - \left(m_{j_\nu^1} + m_{J_\nu^1}\right)\|\mathbf{M}\|^2 - L_{\mathcal{G}_b}\|\mathbf{M}\| - \frac{1}{2\omega_1}\right)} \\
&\qquad \times \sqrt{\frac{\widehat{\Theta}_{\omega_2}(t,\boldsymbol{z},\boldsymbol{\theta})}{m_{\mathcal{A}^2} - \left(m_{j_\nu^2} + m_{J_\nu^2}\right)\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}}}
\end{aligned}$$

and constants $\overline{\mathbf{C}}_1, \overline{\mathbf{C}}_2 > 0$ *defined by*

$$\begin{cases} \overline{\mathbf{C}}_1 = 1 - \dfrac{\sqrt{2}\left(m_{j_\nu^1} + m_{J_\nu^1}\right)\left(m_{j_\nu^2} + m_{J_\nu^2}\right)\|\mathbf{M}\|^2\,\|\mathbf{N}\|^2}{\left(m_{\mathcal{A}^1} - \left(m_{j_\nu^1} + m_{J_\nu^1}\right)\|\mathbf{M}\|^2 - L_{\mathcal{G}_b}\|\mathbf{M}\| - \frac{1}{2\omega_1}\right)\left(m_{\mathcal{A}^2} - \left(m_{j_\nu^2} + m_{J_\nu^2}\right)\|\mathbf{N}\|^2 - \frac{1}{2\omega_2}\right)}; \\ \overline{\mathbf{C}}_2 = \dfrac{\sqrt{2}(a_{\mathcal{S}_1} + L_{\mathcal{Q}}\|\mathbf{M}\|\,\mathrm{e}^{L_{\mathcal{Q}}T})}{\left(m_{\mathcal{A}^1} - \left(m_{j_\nu^1} + m_{J_\nu^1}\right)\|\mathbf{M}\|^2 - L_{\mathcal{G}_b}\|\mathbf{M}\| - \frac{1}{2\omega_1}\right)}. \end{cases}$$

Acknowledgments

The authors would like to thank the anonymous referees for their careful reading of the manuscript and their insightful comments, which have helped improve the quality of the paper.

Data availability statement

This manuscript has no associated data.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

The first author's work is supported by the Ministry of Education and Training of Vietnam [grant number B2024-CTT-03]. The second author's work is supported by the National Science and Technology Council, Taiwan [grant numbers NSTC 114-2115-M-003-00-MY2; NSTC 114-2124-M-003-004] and the NTNU Higher Education Sprout Project – Center for Optimal Intelligent Data Analytics and Prediction.

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