TRACE VERSIONS OF YOUNG INEQUALITY AND ITS APPLICATIONS

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Abstract. In this paper, we derive a few type of trace versions of Young inequality associated with second-order cone, which can be applied to derive the Hölder inequality, Minkowski inequality. Moreover, the triangular inequality is also considered.

1. Introduction

It is well-known that the Young inequality, the Hölder inequality, and the Minkowski inequality are powerful tools in analysis and are widely applied in many fields. There exist many kinds of variants, generalizations, and refinements, which provide a variety of applications. In this paper, we explore the trace versions of Young inequality, Hölder inequality, Minkowski inequality in the setting of second-order cone (SOC for short and will be introduced in Section 2). We start with recalling these three classical inequalities [3, 11] briefly.

Suppose that $a, b \geq 0$ and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, the Young inequality is expressed by

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{1.1}$$

The Young inequality is a special case of the weighted arithmetic mean-geometric mean inequality and very useful in real analysis. In particular, it can be employed as a tool to prove the Hölder inequality:

$$\sum_{k=1}^{n} |a_k b_k| \leq \left(\sum_{k=1}^{n} |a_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |b_k|^q \right)^{\frac{1}{q}},$$

where $a_1, a_2, \cdots, a_n$, $b_1, b_2, \cdots, b_n$ are real (or complex) numbers. In light of the Hölder inequality, one can deduce the Minkowski inequality as below:

$$\left(\sum_{k=1}^{n} |a_k + b_k|^p \right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n} |a_k|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |b_k|^p \right)^{\frac{1}{p}}.$$

Key words and phrases. Second-order cone; Euclidean Jordan algebra; Young inequality; Hölder inequality; Minkowski inequality; triangular inequality.

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In 1995, Ando [1] showed the singular value version of Young inequality that

$$s_j(AB) \leq s_j\left(\frac{A^p}{p} + \frac{B^q}{q}\right) \quad \text{for all } 1 \leq j \leq n, \tag{1.2}$$

where A and B are positive definite matrices. Note that both positive semi-definite cone and second-order cone belong to symmetric cones [9]. It is natural to ask whether there is a similar version in the setting of second-order cone. First, in view of the classical Young inequality, one may conjecture that the Young inequality in the SOC setting is in form of

$$x \circ y \preceq_{\mathcal{K}^n} \frac{x^p}{p} + \frac{y^q}{q}.$$

However, this inequality does not hold in general (a counterexample is presented in Section 3). Here "$\circ$" is the Jordan product associated with second-order cone that will be introduced in Section 2. Next, according to Ando's inequality (1.2), we naively make a conjecture that the eigenvalue version of Young inequality in the SOC setting.

Conjecture 1. For any $x, y \in \mathcal{K}^n$, there holds

$$\lambda_j(x \circ y) \leq \lambda_j\left(\frac{x^p}{p} + \frac{y^q}{q}\right). \quad j = 1, 2. \tag{1.3}$$

In fact, we use some program to check this inequality and there is no counterexample. We believe it is true, but it is very complicated to prove the inequality directly due to the algebraic structure of $\frac{x^p}{p} + \frac{y^q}{q}$. Eventually, we seek another variant and establish the trace version of Young inequality. Accordingly, we further deduce the trace versions of Hölder and Minkowski inequalities.

2. Preliminary

In this section, we review the basic concepts and properties concerning Jordan algebras and symmetric cones from the book [9] which are needed in the subsequent analysis. Especially, we recall some background materials regarding second-order cone as well.

A *Euclidean Jordan algebra* is a finite dimensional inner product space $(\mathbb{V}, \langle \cdot, \cdot \rangle)$ ($\mathbb{V}$ for short) over the field of real numbers $\mathbb{R}$ equipped with a bilinear map $(x, y) \mapsto x \circ y : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$, which satisfies the following conditions:

(i) $x \circ y = y \circ x$ for all $x, y \in \mathbb{V}$;
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathbb{V}$;
(iii) $\langle x \circ y, z \rangle = \langle x, y \circ z \rangle$ for all $x, y, z \in \mathbb{V}$,

where $x^2 := x \circ x$, and $x \circ y$ is called the *Jordan product* of x and y. Moreover, if there is an (unique) element $e \in \mathbb{V}$ such that $x \circ e = x$ for all $x \in \mathbb{V}$, the element e is called the *identity element* in $\mathbb{V}$. Note that a Jordan algebra does not necessarily have an identity element. Throughout this paper, we assume that $\mathbb{V}$ is a Euclidean Jordan algebra with an identity element e.

In a Euclidean Jordan algebra $\mathbb{V}$, the set of squares $\mathcal{K} := \{x^2 : x \in \mathbb{V}\}$ is called a *symmetric cone* [9, Theorem III.2.1], which means $\mathcal{K}$ is a self-dual closed convex cone and, for any two elements $x, y \in \text{int}(\mathcal{K})$, there exists an invertible linear transformation $\Gamma : \mathbb{V} \to \mathbb{V}$ such that $\Gamma(x) = y$ and $\Gamma(\mathcal{K}) = \mathcal{K}$. An element $c \in \mathbb{V}$ is called an *idempotent* if $c^2 = c$, and it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. The idempotents c, d are said to be *orthogonal* if $c \circ d = 0$. In addition, a finite set $\{e^{(1)}, e^{(2)}, \cdots, e^{(r)}\}$ of primitive idempotents in $\mathbb{V}$ is said to be a *Jordan frame* if

$$e^{(i)} \circ e^{(j)} = 0 \quad \text{for} \quad i \neq j, \quad \text{and} \quad \sum_{i=1}^{r} e^{(i)} = e.$$

With the above, there has the spectral decomposition of an element x in $\mathbb{V}$.

Theorem 2.1. (Spectral Decomposition Theorem) [9, Theorem III.1.2] *Let $\mathbb{V}$ be a Euclidean Jordan algebra. Then there is a number r such that, for every $x \in \mathbb{V}$, there exists a Jordan frame $\{e^{(1)}, \cdots, e^{(r)}\}$ and real numbers $\lambda_1(x), \cdots, \lambda_r(x)$ with*

$$x = \lambda_1(x)e^{(1)} + \cdots + \lambda_r(x)e^{(r)}.$$

Here, the numbers $\lambda_i(x)$ $(i = 1, \cdots, r)$ are called the spectral values of x, the expression $\lambda_1(x)e^{(1)} + \cdots + \lambda_r(x)e^{(r)}$ is called the spectral decomposition of x. Moreover, $\mathbf{tr}(x) := \sum_{i=1}^{r} \lambda_i(x)$ is called the trace of x, $\det(x) := \lambda_1(x)\lambda_2(x) \cdots \lambda_r(x)$ is called the determinant of x, and r is called the rank of $\mathbb{V}$.

The second-order cone (SOC for short) in $\mathbb{R}^n$, also called the Lorentz cone, is defined by

$$\mathcal{K}^n = \left\{ x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \,|\, \|x_2\| \leq x_1 \right\}.$$

While $n = 1$, $\mathcal{K}^n$ denotes the set of nonnegative real number $\mathbb{R}_+$. For any $x, y \in \mathbb{R}^n$, we write $x \succeq_{\mathcal{K}^n} y$ if $x - y \in \mathcal{K}^n$, and $x \succ_{\mathcal{K}^n} y$ if $x - y \in \text{int}(\mathcal{K}^n)$. The relation $\succeq_{\mathcal{K}^n}$ is a partial ordering but not a linear ordering in $\mathcal{K}^n$. For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we define their *Jordan product* as

$$x \circ y = (x^T y \ , \ y_1 x_2 + x_1 y_2).$$

Then, $(\mathbb{R}^n, \circ, \langle \cdot, \cdot \rangle)$ forms a Euclidean Jordan algebra with identity element $e = (1, 0, \ldots, 0)^T$. Notice that this Jordan product is *not associative*. However, it is power associative, i.e., $x \circ (x \circ x) = (x \circ x) \circ x$ for all $x \in \mathbb{R}^n$.

Without loss of ambiguity, we may write x^m for the product of m copies of x and $x^{m+n} = x^m \circ x^n$ for all positive integers m and n. Here, we set $x^0 = e$.

For any $x \in \mathcal{K}^n$, it is known that there exists a unique vector in $\mathcal{K}^n$ denoted by $x^{1/2}$ such that $(x^{1/2})^2 = x^{1/2} \circ x^{1/2} = x$. Indeed,

$$x^{1/2} = \left(s, \frac{x_2}{2s}\right), \quad \text{where } s = \sqrt{\frac{1}{2}\left(x_1 + \sqrt{x_1^2 - \|x_2\|^2}\right)}.$$

In the above formula, the term x_2/s is defined to be the zero vector if $s = 0$, i.e., $x = 0$. Since $x^2 \in \mathcal{K}^n$ for any $x \in \mathbb{R}^n$, there exists a unique vector $(x^2)^{1/2} \in \mathcal{K}^n$, denoted by $|x|$. It is easy to verify that $|x| \succeq_{\mathcal{K}^n} 0$ and $x^2 = |x|^2$ for any $x \in \mathbb{R}^n$. For any $x \in \mathbb{R}^n$, we define $[x]_+$ to be the nearest point projection of x onto $\mathcal{K}^n$, which is the same definition as in $\mathbb{R}^n_+$. In other words, $[x]_+$ is the optimal solution of the parametric SOCP: $[x]_+ = \arg\min\{\|x - y\| \mid y \in \mathcal{K}^n\}$. In addition, it can be verified that $[x]_+ = (x + |x|)/2$; see [9, 10].

Optimization problems involved second-order cones have been appeared in real world applications. For dealing with second-order cone programs (SOCP) and second-order cone complementarity problems (SOCCP), there needs *spectral decomposition* associated with SOC [8]. More specifically, for any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, the vector x can be decomposed as

$$x = \lambda_1 u_x^{(1)} + \lambda_2 u_x^{(2)}, \tag{2.1}$$

where λ_1, λ_2 and $u_x^{(1)}, u_x^{(2)}$ are the spectral values and the associated spectral vectors of x, respectively, given by

$$\lambda_i = x_1 + (-1)^i \|x_2\|, \tag{2.2}$$

$$u_x^{(i)} = \begin{cases} \frac{1}{2}(1, (-1)^i \frac{x_2}{\|x_2\|}) & \text{if} \quad x_2 \neq 0, \\ \frac{1}{2}(1, (-1)^i w) & \text{if} \quad x_2 = 0, \end{cases} \tag{2.3}$$

for $i = 1, 2$ with w being any vector in $\mathbb{R}^{n-1}$ satisfying $\|w\| = 1$. If $x_2 \neq 0$, the decomposition is unique. Accordingly, the determinant, the trace, and the Euclidean norm of x can all be represented in terms of λ_1 and λ_2:

$$\det(x) = \lambda_1\lambda_2, \quad \mathbf{tr}(x) = \lambda_1 + \lambda_2, \quad \|x\|^2 = \frac{1}{2}\left(\lambda_1^2 + \lambda_2^2\right).$$

From the simple calculation, we especially point out that $\mathbf{tr}(x) = 2x_1$, which we frequently use in the following paragraphs.

For any real valued function $f : \mathbb{R} \to \mathbb{R}$, the following vector-valued function associated with $\mathcal{K}^n$ $(n \geq 1)$ was considered in [6, 7]:

$$f^{\text{soc}}(x) = f(\lambda_1)u_x^{(1)} + f(\lambda_2)u_x^{(2)}, \quad \forall x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}. \tag{2.4}$$

If f is defined only on a subset of $\mathbb{R}$, then f^{soc} is defined on the corresponding subset of $\mathbb{R}^n$. The definition (2.4) is unambiguous whether $x_2 \neq 0$ or $x_2 = 0$.

The cases of $f^{\rm soc}(x) = x^{1/2}$, x^2, $\exp(x)$ are discussed in [9]. For subsequent analysis, we will frequently use the vector-valued functions corresponding to t^p $(t > 0, p > 0)$ and $|t|^p$ $(t \in \mathbb{R}, p > 0)$, respectively. In particular, they can be expressed as

$$\begin{aligned} x^p &= \lambda_1^p u_x^{(1)} + \lambda_2^p u_x^{(2)}, \quad \forall x \in \mathcal{K}^n, \\ |x|^p &= |\lambda_1|^p u_x^{(1)} + |\lambda_2|^p u_x^{(2)}, \quad \forall x \in \mathbb{R}^n. \end{aligned}$$

The spectral decomposition along with the Jordan algebra associated with SOC entail some basic properties as listed in the following text. We omit the proofs since they can be found in [6, 9, 10].

Lemma 2.2. *For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ with spectral decomposition given as in (2.1)-(2.3), there have*

(a) $|x| = (x^2)^{1/2} = |\lambda_1| u_x^{(1)} + |\lambda_2| u_x^{(2)}$;
(b) $[x]_+ = [\lambda_1]_+ u_x^{(1)} + [\lambda_2]_+ u_x^{(2)} = \frac{1}{2}(x + |x|)$;
(c) $x = [x]_+ + [x]_-$ *and* $|x| = [x]_+ + [-x]_+ = [x]_+ - [x]_-$.

Lemma 2.3. *For any $x, y \in \mathbb{R}^n$ with spectral decomposition given as in (2.1)-(2.3), the following hold.*

(a) $|x| \succeq_{\mathcal{K}^n} x$.
(b) $x \succeq_{\mathcal{K}^n} 0 \iff \langle x, y \rangle \geq 0$, $\forall y \succeq_{\mathcal{K}^n} 0$.
(c) *If* $x \succeq_{\mathcal{K}^n} y$, *then* $\lambda_i(x) \geq \lambda_i(y)$, $\forall i = 1, 2$; *and hence* $\mathbf{tr}(x) \geq \mathbf{tr}(y)$.
(d) $\mathbf{tr}(x \circ y) \leq \lambda_1(x)\lambda_1(y) + \lambda_2(x)\lambda_2(y)$.
(e) $\mathbf{tr}(\alpha x + \beta y) = \alpha \mathbf{tr}(x) + \beta \mathbf{tr}(y)$, $\forall \alpha, \beta \in \mathbb{R}$.

3. The Young Inequalities

As mentioned earlier, one may conjecture that the Young inequality in the SOC setting is in form of

$$x \circ y \preceq_{\mathcal{K}^n} \frac{x^p}{p} + \frac{y^q}{q}.$$

However, this inequality does not hold in general. For example, taking $p = 3$, $q = \frac{3}{2}$, $x = (\frac{1}{8}, \frac{1}{8}, 0)$, and $y = (\frac{1}{8}, 0, \frac{1}{8})$, we obtain $x^3 = (\frac{1}{128}, \frac{1}{128}, 0)$, $y^{\frac{3}{2}} = (\frac{1}{16}, 0, \frac{1}{16})$. Hence,

$$x \circ y = \left(\frac{1}{64}, \frac{1}{64}, \frac{1}{64} \right) \quad \text{and} \quad \frac{x^3}{3} + \frac{y^{\frac{3}{2}}}{\frac{3}{2}} = \left(\frac{17}{384}, \frac{1}{384}, \frac{16}{384} \right),$$

which says

$$\frac{x^3}{3} + \frac{y^{\frac{3}{2}}}{\frac{3}{2}} - x \circ y = \left(\frac{11}{384}, \frac{-5}{384}, \frac{10}{384} \right) \notin \mathcal{K}^n.$$

In view of this and motivated by the Ando's singular value version of Young inequality as in (1.2), we turn to derive the eigenvalue version of Young inequality in the setting of second-order cone. But, we do not succeed in achieving such type inequality. Instead, we consider the SOC trace version of the Young inequality. In fact, a SOC trace version of Young inequality is shown in [5, Theorem 4.2], which we call it Young inequality-Type I as below.

Theorem 3.1. (Young inequality-Type I) *For any $x, y \in \mathcal{K}^n$, there holds*

$$\mathbf{tr}(x \circ y) \leq \mathbf{tr}\left(\frac{x^p}{p} + \frac{y^q}{q}\right)$$

where $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We present the arguments again for completeness although it is shown in [5, Theorem 4.2]. The desired result follows by

$$\begin{aligned} \mathbf{tr}(x \circ y) &\leq \lambda_1(x)\lambda_1(y) + \lambda_2(x)\lambda_2(y) \\ &\leq \left(\frac{\lambda_1(x)^p}{p} + \frac{\lambda_1(y)^q}{q}\right) + \left(\frac{\lambda_2(x)^p}{p} + \frac{\lambda_2(y)^q}{q}\right) \\ &= \mathbf{tr}\left(\frac{x^p}{p} + \frac{y^q}{q}\right), \end{aligned}$$

where the last inequality is due to the Young inequality on real number setting. □

Remark 3.2. When $p = q = 2$, the Young inequality in Theorem 3.1 reduces to

$$2\langle x, y\rangle = \mathbf{tr}(x \circ y) \leq \mathbf{tr}\left(\frac{x^2}{2} + \frac{y^2}{2}\right) = \|x\|^2 + \|y\|^2,$$

which is equivalent to $0 \leq \|x - y\|^2$. As a matter of fact, for any $x, y \in \mathbb{R}^n$, the inequality $(x-y)^2 \succeq_{\mathcal{K}^n} 0$ always holds, which implies $2x \circ y \preceq_{\mathcal{K}^n} x^2 + y^2$. Therefore, by Lemma 2.3(c), we obtain $\mathbf{tr}(x \circ y) \leq \mathbf{tr}\left(\frac{x^2}{2} + \frac{y^2}{2}\right)$ as well.

We note that the classical Young inequality can be extended to nonnegative real numbers, that is,

$$|ab| = |a| \cdot |b| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}, \quad \forall a, b \in \mathbb{R}.$$

This motivates us to consider further generalization of the SOC trace version of Young inequality as in Theorem 3.1. However, $|x| \circ |y|$ and $|x \circ y|$ are unequal in general; and no relationship between them in the partial order $\succeq_{\mathcal{K}^n}$. To see this, taking $x = (\sqrt{2}, 1, 1) \in \mathcal{K}^3$ and $y = (\sqrt{2}, 1, -1) \in \mathcal{K}^3$, yields $x \circ y = (2, 2\sqrt{2}, 0) \notin \mathcal{K}^3$. In addition, it implies

$$|x| \circ |y| = (2, 2\sqrt{2}, 0) \preceq_{\mathcal{K}^n} (2\sqrt{2}, 2, 0) = |x \circ y|.$$

On the other hand, let $x = (0, 1, 0)$, $y = (0, 1, 1)$, which give $|x| = (1, 0, 0)$, $|y| = (\sqrt{2}, 0, 0)$. However, we see that

$$|x \circ y| = (1, 0, 0) \preceq_{\mathcal{K}^n} (\sqrt{2}, 0, 0) = |x| \circ |y|.$$

From these two examples, it also indicates that there is no relationship between $\mathbf{tr}(|x| \circ |y|)$ and $\mathbf{tr}(|x \circ y|)$. In other words, there are two possible extensions of Theorem 3.1:

$$\mathbf{tr}(|x| \circ |y|) \leq \mathbf{tr}\left(\frac{|x|^p}{p} + \frac{|y|^q}{q}\right) \quad \text{or} \quad \mathbf{tr}(|x \circ y|) \leq \mathbf{tr}\left(\frac{|x|^p}{p} + \frac{|y|^q}{q}\right).$$

Fortunately, these two types of generalizations are both true, and we will prove them in Theorem 3.3 and Theorem 3.8.

Theorem 3.3. (Young inequality-Type II) *For any $x, y \in \mathbb{R}^n$, there holds*

$$\mathbf{tr}(|x| \circ |y|) \leq \mathbf{tr}\left(\frac{|x|^p}{p} + \frac{|y|^q}{q}\right)$$

where $1 < p, q < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Proof. We note that both $|x|$ and $|y|$ are in $\mathcal{K}^n$. The desired inequality follows by applying Theorem 3.1 to $|x|$ and $|y|$. $\square$

We point out that Theorem 3.3 is more general than Theorem 3.1 because it is true for all $x, y \in \mathbb{R}^n$, not necessary restricted to $x, y \in \mathcal{K}^n$. For real numbers, it is clear that $ab \leq |a| \cdot |b|$. It is natural to ask whether $\mathbf{tr}(x \circ y)$ is less than $\mathbf{tr}(|x| \circ |y|)$ or not. Before establishing the relationship, we need the following technical lemma.

Lemma 3.4. *If $x, y \in \mathcal{K}^n$, then $\mathbf{tr}(x \circ y) \geq 0$. Furthermore, if $v \preceq_{\mathcal{K}^n} u$ and $0 \preceq_{\mathcal{K}^n} w$, there holds $\mathbf{tr}(v \circ w) \leq \mathbf{tr}(u \circ w)$.*

Proof. By using the definitions of Jordan product and trace, and applying the fact that $\langle x, y \rangle \geq 0$ for any $x, y \in \mathcal{K}^n$, the desired inequalities follow. $\square$

Proposition 3.5. *For any $x, y \in \mathbb{R}^n$, there holds $\mathbf{tr}(x \circ y) \leq \mathbf{tr}\left(|x| \circ |y|\right)$.*

Proof. For any $x \in \mathbb{R}^n$, it can be expressed by $x = [x]_+ + [x]_-$, and then

$$\begin{aligned}
\mathbf{tr}(x \circ y) &= \mathbf{tr}(([x]_+ + [x]_-) \circ y) \\
&= \mathbf{tr}([x]_+ \circ y) + \mathbf{tr}((-[x]_-) \circ (-y)) \\
&\leq \mathbf{tr}([x]_+ \circ |y|) + \mathbf{tr}((-[x]_-) \circ |y|) \\
&= \mathbf{tr}(([x]_+ - [x]_-) \circ |y|) \\
&= \mathbf{tr}(|x| \circ |y|),
\end{aligned}$$

where the inequality holds by Lemma 3.4. $\square$

Remark 3.6. We elaborate the geometric view of Proposition 3.5. By the definition of trace in second-order cone, we notice

$$\mathbf{tr}(x \circ y) = 2\langle x, y\rangle = 2\|x\| \cdot \|y\| \cos\theta$$

where θ is the angle between the vectors x and y. By the definition of absolute value in second-order cone, we know the equality in Proposition 3.5 holds whenever $x, y \in \mathcal{K}^n$ or $x, y \in -\mathcal{K}^n$. Otherwise, it can be observed that the angle between $|x|$ and $|y|$ is smaller than the angle between x and y since the vector $x, |x|$ and the axis of second-order cone are in a hyperplane.

Proposition 3.7. *For any $x, y \in \mathbb{R}^n$, the following inequalities hold.*

(a) $\mathbf{tr}((x+y)^2) \le \mathbf{tr}((|x|+|y|)^2)$, *i.e.*, $\|x+y\| \le \||x|+|y|\|$.
(b) $\mathbf{tr}((x-y)^2) \ge \mathbf{tr}((|x|-|y|)^2)$, *i.e.*, $\|x-y\| \ge \||x|-|y|\|$.

Proof. (a) From Proposition 3.5, we have

$$\begin{aligned}\mathbf{tr}\left((x+y)^2\right) &= \mathbf{tr}\left(x^2 + 2x\circ y + y^2\right)\\ &\le \mathbf{tr}\left(|x|^2 + 2|x|\circ|y| + y^2\right) = \mathbf{tr}\left((|x|+|y|)^2\right).\end{aligned}$$

It is equivalent to $\|x+y\|^2 \le \||x|+|y|\|^2$, which implies $\|x+y\| \le \||x|+|y|\|$.
(b) The proof is similar to part(a). □

In contrast to Proposition 3.5, applying Lemma 2.3, it is obvious that $\mathbf{tr}(x\circ y) \le \mathbf{tr}\left(|x\circ y|\right)$ because $x \circ y \preceq_{\mathcal{K}^n} |x\circ y|$. In view of this, we try to achieve another extension as below.

Theorem 3.8. (Young inequality-Type III) *For any $x, y \in \mathbb{R}^n$, there holds*

$$\mathbf{tr}(|x\circ y|) \le \mathbf{tr}\left(\frac{|x|^p}{p} + \frac{|y|^q}{q}\right)$$

where $1 < p, q < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Proof. For analysis needs, we write $x = (x_1, x_2) \in \mathbb{R}\times\mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R}\times\mathbb{R}^{n-1}$. Note that if $x\circ y \in \mathcal{K}^n \cup (-\mathcal{K}^n)$, the desired inequality holds immediately by Theorem 3.3 and Proposition 3.5. Thus, it suffices to show the inequality holds for $x \circ y \notin \mathcal{K}^n \cup (-\mathcal{K}^n)$. In fact, we only need to show the inequality for the case of $x_1 \ge 0$ and $y_1 \ge 0$. The other cases can be derived by suitable changing variable like

$$|x\circ y| = |-(x\circ y)| = |(-x)\circ y| = |x\circ(-y)| = |(-x)\circ(-y)|.$$

To proceed, we first claim the following inequality

$$2\|x_1y_2 + y_1x_2\| \le |\lambda_1(x)\lambda_1(y)| + |\lambda_2(x)\lambda_2(y)|, \tag{3.1}$$

which is also equivalent to $4\|x_1y_2+y_1x_2\|^2 \le (|\lambda_1(x)\lambda_1(y)| + |\lambda_2(x)\lambda_2(y)|)^2$. Indeed, we observe that

$$4\|x_1y_2 + y_1x_2\|^2 = 4\left(x_1^2\|y_2\|^2 + y_1^2\|x_2\|^2 + 2x_1y_1x_2^Ty_2\right).$$

On the other hand,

$$\begin{aligned}
& (|\lambda_1(x)\lambda_1(y)| + |\lambda_2(x)\lambda_2(y)|)^2 \\
= \; & [\lambda_1(x)\lambda_1(y)]^2 + [\lambda_2(x)\lambda_2(y)]^2 + 2\,|\lambda_1(x)\lambda_1(y)\lambda_2(x)\lambda_2(y)| \\
= \; & 2(x_1y_1 + \|x_2\|\|y_2\|)^2 + 2(x_1\|y_2\| + y_1\|x_2\|)^2 \\
& + 2\left|\left(x_1^2 - \|x_2\|^2\right)\left(y_1^2 - \|y_2\|^2\right)\right| \\
= \; & 2\left(x_1^2y_1^2 + \|x_2\|^2\|y_2\|^2 + x_1^2\|y_2\|^2 + y_1^2\|x_2\|^2\right) + 8x_1y_1\|x_2\|\|y_2\| \\
& + 2\left|\left(x_1^2 - \|x_2\|^2\right)\left(y_1^2 - \|y_2\|^2\right)\right|.
\end{aligned}$$

Therefore, we conclude that (3.1) is satisfied by checking

$$\begin{aligned}
& (|\lambda_1(x)\lambda_1(y)| + |\lambda_2(x)\lambda_2(y)|)^2 - 4\|x_1y_2 + y_1x_2\|^2 \\
= \; & 2\left(x_1^2y_1^2 + \|x_2\|^2\|y_2\|^2 + x_1^2\|y_2\|^2 + y_1^2\|x_2\|^2\right) \\
& +8x_1y_1\|x_2\|\|y_2\| + \; 2\left|\left(x_1^2 - \|x_2\|^2\right)\left(y_1^2 - \|y_2\|^2\right)\right| \\
& -4\left(x_1^2\|y_2\|^2 + y_1^2\|x_2\|^2 + 2x_1y_1x_2^Ty_2\right) \\
= \; & 2\left(x_1^2y_1^2 + \|x_2\|^2\|y_2\|^2 - x_1^2\|y_2\|^2 - y_1^2\|x_2\|^2\right) \\
& +8x_1y_1\left(\|x_2\|\|y_2\| - x_2^Ty_2\right) + 2\left|\left(x_1^2 - \|x_2\|^2\right)\left(y_1^2 - \|y_2\|^2\right)\right| \\
= \; & 2\left(x_1^2 - \|x_2\|^2\right)\left(y_1^2 - \|y_2\|^2\right) + 2\left|\left(x_1^2 - \|x_2\|^2\right)\left(y_1^2 - \|y_2\|^2\right)\right| \\
& +8x_1y_1(\|x_2\|\|y_2\| - x_2^Ty_2) \\
\geq \; & 0,
\end{aligned}$$

where the last inequality is due to the Cauchy-Schwarz inequality.

Suppose that $x \circ y \notin \mathcal{K}^n \cup (-\mathcal{K}^n)$. From the simple calculation, we have

$$|x \circ y| = \left(\|x_1y_2 + y_1x_2\|,\ \frac{x_1y_1 + x_2^Ty_2}{\|x_1y_2 + y_1x_2\|}(x_1y_2 + y_1x_2)\right),$$

which says $\mathbf{tr}(|x \circ y|) = 2\|x_1y_2 + y_1x_2\|$. Using inequality (3.1), we obtain

$$\begin{aligned}
\mathbf{tr}(|x \circ y|) & \leq |\lambda_1(x)\lambda_1(y)| + |\lambda_2(x)\lambda_2(y)| \\
& \leq \left(\frac{|\lambda_1(x)|^p}{p} + \frac{|\lambda_1(y)|^q}{q}\right) + \left(\frac{|\lambda_2(x)|^p}{p} + \frac{|\lambda_2(y)|^q}{q}\right) \\
& = \mathbf{tr}\left(\frac{|x|^p}{p} + \frac{|y|^q}{q}\right),
\end{aligned}$$

where the last inequality holds by the classical Young inequality on real number setting. □

Following the argument of SOC trace versions of Young inequality, we now turn to derive the trace versions of Young inequality in Euclidean Jordan algebra. We recall the crucial inequality which is established in [2, Theorem 23].

Theorem 3.9. [2, Theorem 23] *Let $\mathbb{V}$ be a simple Euclidean Jordan algebra with rank r. For any $x, y \in \mathbb{V}$, there holds*

$$\mathbf{tr}(x \circ y) \leq \sum_{i=1}^{r} \lambda_i(x)\lambda_i(y),$$

where $\lambda_i(x)$ and $\lambda_i(y)$ are the spectral values of x and y with decreasing order, respectively.

Now, we are able to establish some trace versions of Young inequality in Euclidean Jordan algebra.

Theorem 3.10. (EJA Young inequality-Type I) *Let $\mathbb{V}$ be a simple Euclidean Jordan algebra with rank r. For any $x, y \in \mathcal{K}$, there holds*

$$\mathbf{tr}(x \circ y) \leq \mathbf{tr}\left(\frac{x^p}{p} + \frac{y^q}{q}\right)$$

where $1 < p, q < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Proof. Following the same argument of Theorem 3.1, we obtain

$$\begin{aligned} \mathbf{tr}(x \circ y) &\leq \sum_{i=1}^{r} \lambda_i(x)\lambda_i(y) \\ &\leq \sum_{i=1}^{r} \left(\frac{\lambda_i(x)^p}{p} + \frac{\lambda_i(y)^q}{q}\right) = \mathbf{tr}\left(\frac{x^p}{p} + \frac{y^q}{q}\right), \end{aligned}$$

by Theorem 3.9 and the Young inequality on real number setting. □

Theorem 3.11. (EJA Young inequality-Type II) *Let $\mathbb{V}$ be a simple Euclidean Jordan algebra with rank r. For any $x, y \in \mathbb{V}$, there holds*

$$\mathbf{tr}(|x| \circ |y|) \leq \mathbf{tr}\left(\frac{|x|^p}{p} + \frac{|y|^q}{q}\right)$$

where $1 < p, q < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and $|x| = |\lambda_1(x)|e^{(1)} + \cdots + |\lambda_r(x)|e^{(r)}$.

Proof. We note that $|x|, |y| \in \mathcal{K}$, then the desired result follows by Theorem 3.10. □

4. Applications

In real analysis, Young inequality as in (1.1) is the main tool to derive the Hölder inequality, and then the Minkowski inequality can be derived by Hölder inequality as well. As a matter of fact, Tao et al. [13] establish a trace p-norm in Euclidean Jordan algebra, that is, the authors directly show the trace version of Minkowski inequality, see [13, Theorem 4.1]. As an application of trace versions of Young inequality, we use the approach which follows the same idea as in real analysis to derive the trace versions of Hölder inequality. Furthermore, the SOC trace version of Minkowski inequality is also deduced.

Theorem 4.1. (Hölder inequality-Type I) *For any $x, y \in \mathbb{R}^n$, there holds*

$$\mathbf{tr}(|x| \circ |y|) \leq [\mathbf{tr}(|x|^p)]^{\frac{1}{p}} \cdot [\mathbf{tr}(|y|^q)]^{\frac{1}{q}}$$

where $1 < p, q < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Proof. Let $\alpha = [\mathbf{tr}(|x|^p)]^{\frac{1}{p}}$, $\beta = [\mathbf{tr}(|y|^q)]^{\frac{1}{q}}$. Since $\alpha = 0$ (or $\beta = 0$) implies $x = 0$ ($y = 0$, respectively), the inequality will hold automatically. Suppose $\alpha \cdot \beta \neq 0$, by Theorem 3.3, we have

$$\begin{aligned}
\mathbf{tr}\left(\frac{|x|}{\alpha} \circ \frac{|y|}{\beta}\right) &\leq \mathbf{tr}\left(\frac{|\frac{|x|}{\alpha}|^p}{p} + \frac{|\frac{|y|}{\beta}|^q}{q}\right) \\
&= \frac{1}{p}\mathbf{tr}\left(\frac{|x|^p}{\alpha^p}\right) + \frac{1}{q}\mathbf{tr}\left(\frac{|y|^q}{\beta^q}\right) = \frac{1}{p} + \frac{1}{q} = 1.
\end{aligned}$$

Therefore, we conclude that

$$\mathbf{tr}(|x| \circ |y|) \leq \alpha \cdot \beta = [\mathbf{tr}(|x|^p)]^{\frac{1}{p}} \cdot [\mathbf{tr}(|y|^q)]^{\frac{1}{q}}$$

since $\alpha, \beta > 0$. □

Theorem 4.2. (Hölder inequality-Type II) *For any $x, y \in \mathbb{R}^n$, there holds*

$$\mathbf{tr}(|x \circ y|) \leq [\mathbf{tr}(|x|^p)]^{\frac{1}{p}} \cdot [\mathbf{tr}(|y|^q)]^{\frac{1}{q}}$$

where $1 < p, q < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Proof. The proof is similar to Theorem 4.1 by using Theorem 3.8. □

Remark 4.3. When $p = q = 2$, both inequalities in Theorem 4.1 and Theorem 4.2 deduce

$$2\langle x, y\rangle = \mathbf{tr}(x \circ y) \leq \left[\mathbf{tr}(|x|^2)\right]^{\frac{1}{2}} \cdot \left[\mathbf{tr}(|y|^2)\right]^{\frac{1}{2}} = 2\|x\| \cdot \|y\|,$$

which is equivalent to the Cauchy-Schwarz inequality in $\mathbb{R}^n$.

Theorem 4.4. (EJA Hölder inequality) *Let $\mathbb{V}$ be a simple Euclidean Jordan algebra with rank r. For any $x, y \in \mathbb{V}$, there holds*

$$\mathbf{tr}(|x| \circ |y|) \leq [\mathbf{tr}(|x|^p)]^{\frac{1}{p}} \cdot [\mathbf{tr}(|y|^q)]^{\frac{1}{q}}$$

where $1 < p, q < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Proof. Following the same argument of Theorem 4.1 the desired inequality can be derived by Theorem 3.11. □

Next, we derive the SOC trace version of Minkowski inequality by using the SOC trace version of Hölder inequality.

Theorem 4.5. (Minkowski inequality) *For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, and $p > 1$, there holds*

$$[\mathbf{tr}(|x+y|^p)]^{\frac{1}{p}} \leq [\mathbf{tr}(|x|^p)]^{\frac{1}{p}} + [\mathbf{tr}(|y|^p)]^{\frac{1}{p}}.$$

Proof. We partition the proof into three parts. Let $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.
(i) For $x + y \in \mathcal{K}^n$, we have $|x + y| = x + y$, then we have

$$\begin{aligned}
& \mathbf{tr}(|x+y|^p) \\
= \;& \mathbf{tr}(|x+y| \circ |x+y|^{p-1}) = \mathbf{tr}((x+y) \circ |x+y|^{p-1}) \\
= \;& \mathbf{tr}(x \circ |x+y|^{p-1}) + \mathbf{tr}(y \circ |x+y|^{p-1}) \\
\leq \;& [\mathbf{tr}(|x|^p)]^{\frac{1}{p}} \cdot \left[\mathbf{tr}(|x+y|^{(p-1)q})\right]^{\frac{1}{q}} + [\mathbf{tr}(|y|^p)]^{\frac{1}{p}} \cdot \left[\mathbf{tr}(|x+y|^{(p-1)q})\right]^{\frac{1}{q}} \\
= \;& \left([\mathbf{tr}(|x|^p)]^{\frac{1}{p}} + [\mathbf{tr}(|y|^p)]^{\frac{1}{p}}\right) \cdot [\mathbf{tr}(|x+y|^p)]^{\frac{1}{q}},
\end{aligned}$$

which implies $[\mathbf{tr}(|x+y|^p)]^{\frac{1}{p}} \leq [\mathbf{tr}(|x|^p)]^{\frac{1}{p}} + [\mathbf{tr}(|y|^p)]^{\frac{1}{p}}$.

(ii) For $x + y \in -\mathcal{K}^n$, we have $|x + y| = -x - y$, then we have

$$\begin{aligned}
& \mathbf{tr}(|x+y|^p) \\
= \;& \mathbf{tr}((-x) \circ |x+y|^{p-1}) + \mathbf{tr}((-y) \circ |x+y|^{p-1}) \\
\leq \;& [\mathbf{tr}(|x|^p)]^{\frac{1}{p}} \cdot \left[\mathbf{tr}(|x+y|^{(p-1)q})\right]^{\frac{1}{q}} + [\mathbf{tr}(|y|^p)]^{\frac{1}{p}} \cdot \left[\mathbf{tr}(|x+y|^{(p-1)q})\right]^{\frac{1}{q}} \\
= \;& \left([\mathbf{tr}(|x|^p)]^{\frac{1}{p}} + [\mathbf{tr}(|y|^p)]^{\frac{1}{p}}\right) \cdot [\mathbf{tr}(|x+y|^p)]^{\frac{1}{q}},
\end{aligned}$$

which also implies $[\mathbf{tr}(|x+y|^p)]^{\frac{1}{p}} \leq [\mathbf{tr}(|x|^p)]^{\frac{1}{p}} + [\mathbf{tr}(|y|^p)]^{\frac{1}{p}}$.

(iii) For $x + y \notin \mathcal{K}^n \cup (-\mathcal{K}^n)$, we note that $\lambda_1(x+y) < 0$ and $\lambda_2(x+y) > 0$, which say,

$$\begin{aligned}
|\lambda_1(x+y)| &= \|x_2 + y_2\| - x_1 - y_1 \leq \|x_2\| + \|y_2\| - x_1 - y_1, \\
|\lambda_2(x+y)| &= x_1 + y_1 + \|x_2 + y_2\| \leq x_1 + y_1 + \|x_2\| + \|y_2\|.
\end{aligned}$$

This yields

$$\begin{aligned}
[\mathbf{tr}(|x+y|^p)]^{\frac{1}{p}} &= [|\lambda_1(x+y)|^p + |\lambda_2(x+y)|^p]^{\frac{1}{p}} \\
&\le [(\|x_2\| + \|y_2\| - x_1 - y_1)^p + (\|x_2\| + \|y_2\| + x_1 + y_1)^p]^{\frac{1}{p}} \\
&= [(-\lambda_1(x) - \lambda_1(y))^p + (\lambda_2(x) + \lambda_2(y))^p]^{\frac{1}{p}} \\
&= [|\lambda_1(x) + \lambda_1(y)|^p + |\lambda_2(x) + \lambda_2(y)|^p]^{\frac{1}{p}} \\
&\le [|\lambda_1(x)|^p + |\lambda_2(x)|^p]^{\frac{1}{p}} + [|\lambda_1(y)|^p + |\lambda_2(y)|^p]^{\frac{1}{p}} \\
&= [\mathbf{tr}(|x|^p)]^{\frac{1}{p}} + [\mathbf{tr}(|y|^p)]^{\frac{1}{p}},
\end{aligned}$$

where the last inequality holds by the classical Minkowski inequality on real number setting. □

Remark 4.6. We elaborate more about Theorem 4.5. It can define a norm $|||\cdot|||_p$ on $\mathbb{R}^n$ by

$$|||x|||_p := [\mathbf{tr}(|x|^p)]^{\frac{1}{p}},$$

and hence it induces a distance $d(x,y) = |||x-y|||_p$ on $\mathbb{R}^n$. In particular, this norm will deduce the Euclidean-norm when $p=2$, and the inequality reduces to the triangular inequality. In addition, this norm is similar to Schatten p-norm, which arise when applying the p-norm to the vector of singular values of a matrix. For more details, please refer to [4].

According to the arguments in Theorem 4.5, if we wish to establish the SOC trace version of Minkowski inequality in the same framework, the crucial key is verifying the SOC triangular inequality

$$|x+y| \preceq_{\mathcal{K}^n} |x| + |y|.$$

Unfortunately, this inequality does not hold in general. To see this, checking $x = (\sqrt{2}, 1, -1)$ and $y = (-\sqrt{2}, -1, 0)$ will lead to a counterexample. More specifically, $x \in \mathcal{K}^n, y \in -\mathcal{K}^n$, and $x + y = (0,0,-1) \notin \mathcal{K}^n \cup (-\mathcal{K}^n)$, which say $|x+y| = (1,0,0)$ and $|x| + |y| = x + (-y) = (2\sqrt{2}, 2, -1)$. Hence,

$$|x| + |y| - |x+y| = (2\sqrt{2}-1, 2, -1) \notin \mathcal{K}^n \cup (-\mathcal{K}^n).$$

Moreover, we have

$$\begin{aligned}
\lambda_1(|x+y|) = 1 &> 2\sqrt{2} - \sqrt{5} = \lambda_1(|x| + |y|), \\
\lambda_2(|x+y|) = 1 &< 2\sqrt{2} + \sqrt{5} = \lambda_2(|x| + |y|).
\end{aligned}$$

Nonetheless, we build another SOC trace version of triangular inequality as below. In fact, the trace version of triangular inequality holds for any Euclidean Jordan algebra (see [12, Proposition 4.3] and [13, Corollary 3.1]). In the setting of second-order cone, we can prove the inequality by discussing in three cases directly.

Theorem 4.7. (Triangular inequality) *For any* $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ *and* $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$*, there holds*

$$\mathbf{tr}(|x+y|) \leq \mathbf{tr}(|x|) + \mathbf{tr}(|y|).$$

Proof. In order to complete the proof, we discuss three cases.
(i) If $x + y \in \mathcal{K}^n$, then $|x+y| = x + y \preceq_{\mathcal{K}^n} |x| + |y|$, and hence

$$\mathbf{tr}(|x+y|) \leq \mathbf{tr}(|x|) + \mathbf{tr}(|y|)$$

by Lemma 2.3(c).
(ii) If $x + y \in -\mathcal{K}^n$, then $|x+y| = -x - y \preceq_{\mathcal{K}^n} |x| + |y|$, and hence

$$\mathbf{tr}(|x+y|) \leq \mathbf{tr}(|x|) + \mathbf{tr}(|y|).$$

(iii) Suppose $x + y \notin \mathcal{K}^n \cup (-\mathcal{K}^n)$, we have

$$|x+y| = \left(\|x_2 + y_2\|, \frac{x_1 + y_1}{\|x_2 + y_2\|}(x_2 + y_2) \right)$$

from simple calculation, and then

$$\mathbf{tr}(|x+y|) = 2\|x_2 + y_2\|.$$

If one of x, y is in $\mathcal{K}^n$, say $x \in \mathcal{K}^n$, we have two subcases: $y \in -\mathcal{K}^n$ and $y \notin \mathcal{K}^n \cup (-\mathcal{K}^n)$. For $y \in -\mathcal{K}^n$, we have $|y| = -y$ and $-y_1 \geq \|y_2\|$, and hence

$$\mathbf{tr}(|x| + |y|) = 2(x_1 - y_1) \geq 2(\|x_2\| + \|y_2\|) \geq 2\|x_2 + y_2\| = \mathbf{tr}(|x+y|).$$

For $y \notin \mathcal{K}^n \cup (-\mathcal{K}^n)$, we have $|y| = \left(\|y_2\|, \frac{y_1}{\|y_2\|} y_2 \right)$, and hence

$$\mathbf{tr}(|x| + |y|) = 2(x_1 + \|y_2\|) \geq 2(\|x_2\| + \|y_2\|) \geq 2\|x_2 + y_2\| = \mathbf{tr}(|x+y|).$$

If one of x, y is in $-\mathcal{K}^n$, then the argument is similar. To complete the proof, it remains to show the inequality holds for $x, y \notin \mathcal{K}^n \cup (-\mathcal{K}^n)$. Indeed, in this case, we have

$$\mathbf{tr}(|x| + |y|) = 2(\|x_2\| + \|y_2\|) \geq 2\|x_2 + y_2\| = \mathbf{tr}(|x+y|).$$

Hence, we complete the proof. $\square$

5. Conclusion

In this paper, we have established the trace versions of Young inequality, and then Hölder inequality, Minkowski inequality, and triangular inequality can be deduced. There are still some directions which deserve to be further investigated. We outline them as below.

- Does the eigenvalue version of inequality (1.3) parallel to Ando's inequality (1.2) hold?

- Lastly, as remarked in Remark 4.6, it deduces a norm on $\mathbb{R}^n$ for some certain values of p. With this norm, we suspect that some algorithms based on proximal distance, for example, proximal point algorithm and proximal-like algorithm can be applied accordingly to solve second-order cone programming.

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