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The variants of projection extragradient algorithms for nonmonotone equilibrium problems in Hilbert spaces

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Abstract
We introduce three novel variants of the projection extragradient algorithm for solving equilibrium problems in real Hilbert spaces, without imposing any generalized monotonicity assumptions on the bifunctions involved. The proposed approach replaces the traditional shrinking projection step with a projection onto the intersection of the feasible set and a suitably chosen half-space, offering a more flexible and effective strategy. To address cases where the bifunctions do not satisfy a Lipschitz-type condition, we incorporate a general linesearch mechanism for determining the step size. When the bifunctions are of Lipschitz-type with known constants, this linesearch is rendered unnecessary; in the absence of such constants, we propose an adaptive step size rule. All three algorithms are rigorously proven to converge strongly to a solution, even without assuming the joint weak continuity of the bifunction-an assumption often required in existing literature. Numerical examples are presented to illustrate the efficiency and practical performance of the proposed methods.

Keywords Nonmonotone equilibrium problem · Armijo linesearch · Lipschitz type condition · Strong convergence

Mathematics Subject Classification 90C33 · 90C26 · 47J20 · 65K15 · 65J20

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1 Introduction

Suppose that $\mathbb{H}$ is a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. We denote the weak and strong convergence in $\mathbb{H}$ by '$\rightharpoonup$' and '$\rightarrow$' respectively.

Let Θ be an open convex subset in $\mathbb{H}$ containing a nonempty closed convex set Ω, and let $f : \Theta \times \Theta \to \mathbb{R}$ be a function satisfying $f(x, x) = 0$ for every $x \in \Theta$. Such a function is called a bifunction. We consider the equilibrium problem (shortly EP(Ω, f)), in the sense of Blum et al. [4, 22], which can be stated as follows:

$$\text{Find } x^* \in \Omega \text{ such that } f(x^*, y) \geq 0, \ \forall y \in \Omega.$$

The equilibrium problem associates with EP(Ω, f), sometimes called dual equilibrium problem or Minty equilibrium problem (MEP(Ω, f) for short) [5], is to find $u \in \Omega$ such that

$$f(y, u) \leq 0, \ \forall y \in \Omega.$$

Let us denote the solution set of EP(Ω, f) and MEP(Ω, f) by S_E and S_M respectively. It is pointed out in [22] that if $f(x, y) = \langle F(x), y - x \rangle$ in which $F : \Omega \to \mathbb{H}$ is a mapping then EP(Ω, f) and MEP(Ω, f) reduce to the variational inequality problem (VIP(Ω, F)) and the Minty variational inequality problem (MVIP(Ω, F)), respectively. Moreover, It is well-known that the equilibrium problems include many other important problems such as optimization problems, fixed point problems, saddle point problems, and Nash equilibrium problems in noncooperative games. In our best knowledge, the existence of a solution of EP(Ω, f) was first established by Fan [11] when Ω is a convex and compact set, and $f(x, \cdot)$ is quasiconvex on Ω for each $x \in \Omega$. For the case Ω is a noncompact set, the existence of a solution of EP(Ω, f) under the generalized monotonicity of bifunction *f* or without the generalized monotonicity of *f* can be founded in [2, 3, 5, 15] and the references quoted therein.

An important line of research in the study of equilibrium problems focuses on developing efficient algorithms for finding solutions to EP(Ω, f), utilizing the structure of the bifunction *f* and the feasible set Ω. In this context, a variety of approaches have been proposed in the literature, including gap function methods [18], auxiliary problem principle methods [17, 23], proximal point methods [19], and projection methods [9, 10, 13, 24, 25]. Among these, projection methods are particularly appealing due to their ability to harness tools from convex optimization to effectively compute solutions to EP(Ω, f). These methods typically rely on assumptions such as convexity in the second variable and some form of monotonicity or generalized monotonicity of the bifunction *f*. Significant progress has been made under these assumptions, as documented in various studies (see, e.g., [1, 12, 21, 30]) and comprehensive books [2, 15].

More recently, Strodiot et al. [26] (see also [6, 8, 14]) introduced the class of shrinking projection algorithms to address equilibrium problems without assuming monotonicity. These algorithms, designed for nonmonotone settings in Hilbert spaces, can be outlined as follows.

Algorithm SVN.

Initialization. Pick $x^0 = x^g \in \Omega$, choose parameters $\eta \in (0,1), \gamma > 0$.
At each iteration k $(k = 0, 1, 2, ...)$. Having x^k do the following steps:

Step 1. Compute

$$y^k = \arg\min \Big\{ f(x^k, y) + \frac{1}{2\gamma}\|y - x^k\|^2 : y \in \Omega \Big\}.$$

If $y^k = x^k$, then stop and x^k is a solution of EP(Ω, f). Otherwise, do Step 2.

Step 2. Find m_k as the smallest positive integer number *m* such that

$$\begin{cases} z^{k,m} = (1 - \eta^m)x^k + \eta^m y^k \\ f(z^{k,m}, x^k) - f(z^{k,m}, y^k) \geq \frac{1}{\gamma}\|x^k - y^k\|^2. \end{cases}$$

Set $\eta_k = \eta^{m_k}$, $z^k = z^{k,m_k}$.

Step 3. Select

$$w^k \in \partial_2 f(z^k, z^k), \;\; \Upsilon_k = \{x \in \mathbb{H} : \langle w^k, x - z^k \rangle \leq 0\}.$$

Compute

$$u^k = P_{\Delta_k}(x^k), \;\; \text{where } \Delta_k = \cap_{i=0}^k [\Omega \cap \Upsilon_i].$$

Step 4. Compute $x^{k+1} = P_{\Psi_k}(x^g)$, where $\Psi_k = \Pi_k \cap \Sigma_k \cap \Omega$,

$$\Pi_k = \{x \in \mathbb{H} : \|x - u^k\| \leq \|x - x^k\|\}, \;\; \Sigma_k = \{x \in \mathbb{H} : \langle x - x^k, x^g - x^k \rangle \leq 0\},$$

and go to Iteration *k* with *k* is replaced by $k + 1$.

They pointed out that under assumptions on the convexity and continuity of the bifunction *f*, and the non-emptiness of S_M then the sequence $\{x^k\}$ generated by Algorithm SVN converges strongly to a solution of EP(Ω, f).

One advantage of this algorithm (and algorithms in [6, 8, 14]) is that it can be used to find a solution to the equilibrium problem without requiring that *f* be neither quasimonotone nor Lipschitz-type condition, however to apply this algorithm, at each iteration, we have to solve an optimization problem in which its constraint set is the intersection of the constraint set in the previous iteration and a half space. This may lead to large computational costs, especially when dealing with a high dimensional space and a complicated constraint set. Furthermore, within this algorithm, the line-search rules are employed to determine the step length at each iteration. In some cases, where the bifunction *f* satisfies additional conditions like the Lipschitz-type condition (see assumption ($A3$) below) or exhibits a complex structure, the implementation of the linesearch rule can result in substantial computational cost. Based on these observations and recent works [7, 16, 27, 29], in this paper, we propose novel algorithms for finding a solution to nonmonotone equilibrium problems. The main contributions of this work are as follows:

- We address the nonmonotone equilibrium problem $EP(\Omega, f)$ in Hilbert spaces, where the bifunction f is of Lipschitz-type on Ω though the associated Lipschitz constants are unknown. For this setting, we propose a projection algorithm incorporating an adaptive step size rule.
- We introduce a second projection algorithm with a fixed step size for solving $EP(\Omega, f)$ when the bifunction f is Lipschitz-type on Ω with known constants L_1 and L_2. Notably, both of these algorithms avoid the use of linesearch procedures that are commonly employed in the existing literature.
- For the more general case where f is not assumed to be Lipschitz-type, we propose a projection algorithm that incorporates a linesearch strategy. In contrast to traditional approaches relying on geometric sequences $\{\eta^k\}$, our method allows for any strictly decreasing nonnegative sequence $\{\eta_k\}$ to determine the step length, offering greater flexibility.

Importantly, none of the proposed algorithms rely on the shrinking projection technique or the joint weak continuity of the bifunction-assumptions frequently invoked in earlier works.

The rest of paper is organized as follows. The next section provides some preliminaries that will be used in the following sections. In the third section, we present the proposed algorithms and prove their convergence. The last section is dedicated to presenting some numerical examples to illustrate the effectiveness of the proposed algorithms.

2 Main results

Let $\Omega \subset \mathbb{H}$ be a nonempty, closed and convex set. It is well known that for each $x \in \mathbb{H}$ there exists the unique nearest point in Ω, that is $P_\Omega(x) \in \Omega$, such that $\|x - P_\Omega(x)\| \leq \|y - x\|, \ \forall y \in \Omega$. The distance function to Ω is given by

$$d(x, \Omega) = \inf\{\|x - y\| : y \in \Omega\}.$$

For instance, if $\Gamma = \{y \in \mathbb{H} : \langle a, y\rangle + b \leq 0\}$ for some $a \in \mathbb{H}$ and $b \in \mathbb{R}$, then

$$d(x, \Gamma) = \begin{cases} \frac{\langle a, x\rangle + b}{\|a\|} & \text{if } \langle a, x\rangle + b > 0 \\ 0 & \text{if } \langle a, x\rangle + b \leq 0. \end{cases}$$

The following well known results on the projection operator onto a closed convex set will be used in the sequel.

Lemma 1 *Suppose that Ω is a nonempty closed convex subset in $\mathbb{H}$. Then*

(a) $z = P_\Omega(x)$ *if and only if* $\langle x - z, y - z\rangle \leq 0, \forall y \in \Omega$;
(b) $\|P_\Omega(x) - P_\Omega(y)\|^2 \leq \|x - y\|^2 - \|P_\Omega(x) - x + y - P_\Omega(y)\|^2, \ \forall x, y \in \mathbb{H}$.

Once can derive from Lemma 1 is that $P_\Omega(\cdot)$ is a nonexpansive mapping, that is $\|P_\Omega(x) - P_\Omega(y)\| \leq \|x - y\|, \ \forall x, y \in \mathbb{H}$.

Let $f : \Omega \times \Omega \to \mathbb{R}$ be a bifunction such that $f(x, \cdot)$ is convex for every $x \in \Omega$. We denote the subdifferential of $f(x, \cdot)$ at *y* by $\partial_2 f(x, y)$, that is

$$\partial_2 f(x, y) := \{w \in \mathbb{H} : f(x, z) \geq f(x, y) + \langle w, z - y\rangle, \ \forall z \in \Omega\}.$$

In particular,

$$\partial_2 f(x, x) = \{w \in \mathbb{H} : f(x, z) \geq \langle w, z - x\rangle, \ \forall z \in \Omega\}.$$

Now, let $\Omega \subset \mathbb{H}$ be a nonempty, closed and convex set, let Θ be an open convex set containing Ω and bifunction $f : \Theta \times \Theta \to \mathbb{R}$. We consider the following assumptions:

(A1) *f* is continuous on $\Omega \times \Omega$.
(A2) $f(x, \cdot)$ is convex and subdifferential on Ω for each $x \in \Omega$. In addition, if $\{x^k\}, \{y^k\} \subset \Omega$ are bounded sequences then $\{w^k\}$, with $w^k \in \partial_2 f(x^k, y^k)$, is also bounded.
(A3) *f* is Lipschitz-type on Ω with constants $L_1 > 0$ and $L_2 > 0$ (Lipschitz for short) that is

$$f(x, z) - f(x, y) - f(y, z) \leq L_1\|x - y\|^2 + L_2\|y - z\|^2, \quad \forall x, y, z \in \Omega.$$

Remark 1

- When $f(x, y) = \langle F(x), y - x\rangle$ where $F : \Omega \to \mathbb{H}$ is a continuous mapping then assumptions ($A1$) and ($A2$) are satisfied.
- To the best of our knowledge, using projection methods to find a solution of the nonmonotone equilibrium problems in literature usually assumes not only that $f(x, \cdot)$ is convex and subdifferential on Ω but also that *f* is jointly weakly continuous on $\Theta \times \Theta$. That is, if $\{x^k\}, \{y^k\} \subset \Theta$ and converge weakly to *x*, *y*, respectively, then $f(x^k, y^k)$ converges to *f*(*x*, *y*). These conditions imply that if $\{x^k\}, \{y^k\}$ are bounded sequences in Ω then $\{w^k\}$, with $w^k \in \partial_2 f(x^k, y^k)$, is also bounded (see, for instance, Proposition 4.3 in [28]). The following example shows that assumptions ($A1$), ($A2$) do not imply that *f* is jointly weakly continuous.

Example 1 Let $\mathbb{H}$ be any infinite dimension Hilbert space, $\Omega = \{x \in \mathbb{H} : \|x\| \leq 2\|\}$, and let $f(x, y) = \langle x, y - x\rangle$. Then, we have that $\partial_2 f(x, y) = \{x\}$. So *f* satisfies ($A1$) and ($A2$). However, *f* is not jointly weakly continuous on $\Omega \times \Omega$, for instance, let

$x^k = e_k$, and $y^k = 0$, where e_k is the k^{th} unit vector of $\mathbb{H}$, then we have $x^k \rightharpoonup 0$, $y^k \to 0$, but $f(x^k, y^k) = -1 \not\to 0 = f(0,0)$ as $k \to \infty$.

The following lemma is necessary for proving the convergence of our proposed algorithms.

Lemma 2 *Suppose the bifunction f satisfies the assumptions (A1) and (A2). If $\{x^k\} \subset \Omega$ is bounded, $\gamma_k \subset [\alpha, \beta]$, with $0 < \alpha \leq \beta$ and $\{y^k\}$ is a sequence such that*

$$y^k = \arg\min\{\gamma_k f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in \Omega\},$$

then $\{y^k\}$ is bounded.

Proof This can be done by the same arguments in the proof of Lemma 2.6 [8]. So we omit it. □

Now, we propose algorithms for solving EP(Ω, f), where the bifunction f is nonmonotone, and we do not use the embedding methods found in the literature. Firstly, we consider the case when f is of Lipschitz type on Ω with unknown constants L_1 and L_2.

Algorithm 1.

Initialization. Pick $x^{-1}, x^g = x^0, y^0 \in \Omega$. Choose two sequences $\{\theta_k\}, \{\xi_k\}$ satisfying $0 < \alpha \leq \theta_k < 1, 0 < \alpha \leq \xi_k < 1$ and $\theta_k + \xi_k \leq a < 1$. Choose parameter $\gamma_{-1} \in (0, \infty)$. Let $\{\delta_k\} \subset (0, \infty)$ be a nonincreasing sequence satisfying $\delta_0 < \gamma_{-1}$ and $\lim\limits_{k\to\infty} \delta_k = 0$. Set $k = 0$.

Iteration k $(k = 0, 1, 2, ...)$. Having x^k, y^k, γ_{k-1} do the following steps:

Step 1. Select $w^k \in \partial_2 f(y^k, x^k)$, $\Upsilon_k = \{x \in \mathbb{H} : \langle w^k, x - x^k\rangle + f(y^k, x^k) \leq 0\}$. Compute $u^k = P_{\Omega_k}(x^k)$, where $\Omega_k = \Omega \cap \Upsilon_{t_k}$, $t_k \in \arg\max \{d(x^k, \Upsilon_j) : 0 \leq j \leq k\}$,
Step 2. Compute $x^{k+1} = P_{\Psi_k}(x^g)$, where $\Psi_k = \Pi_k \cap \Sigma_k \cap \Omega$,

$$\Pi_k = \{x \in \mathbb{H} : \|x - u^k\| \leq \|x - x^k\|\},\ \Sigma_k = \{x \in \mathbb{H} : \langle x - x^k, x^g - x^k\rangle \leq 0\}.$$

Step 3. If $\gamma_{k-1}(f(x^{k-1}, x^k) - f(x^{k-1}, y^k) - f(y^k, x^k)) \leq \theta_k \left\|x^{k-1} - y^k\right\|^2 + \xi_k \left\|y^k - x^k\right\|^2$ then set $\gamma_k = \gamma_{k-1}$ else set $\gamma_k = \delta_k$. Compute

$$y^{k+1} = \arg\min\left\{\gamma_k f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in \Omega\right\}.$$

If $y^{k+1} = x^k$, then stop and x^k is a solution of EP(Ω, f). Otherwise, go to Iteration k with k is replaced by $k + 1$.

Remark 2

- The inequality in Step 3 is nothing but the inequality in the definition of Lipschitz-type bifunction *f* with $L_1 = \frac{\theta_k}{\gamma_{k-1}}$, $L_2 = \frac{\xi_k}{\gamma_{k-1}}$, and $x = x^{k-1}$, $y = y^k$, and $z = x^k$. Hence, when *f* satisfies the Lipschitz-type condition on Ω with explicit constants L_1 and L_2, by choosing $\{\gamma_k\}$ such that: $\gamma_{k-1} \leq \min\{\frac{\theta_k}{L_1}, \frac{\xi_k}{L_2}\}$, for all *k* then the inequality in Step 3 is always satisfied. However, from a computational point of view, we see that this inequality may still be satisfied for some γ_{k-1} such that $\gamma_{k-1} \geq \min\left\{\frac{\theta_k}{L_1}, \frac{\xi_k}{L_2}\right\}$. Furthermore, since γ_k is proportional to the stepsize, a larger value of γ_k generally leads to faster convergence of the algorithm, as shown in Example 2.
- Since γ_k is inversely proportional to the constants L_1 and L_2, when these constants are large, γ_k becomes very small. In addition, from Step 3, we have:

$$y^{k+1} = \arg\min\Big\{f(x^k, y) + \frac{1}{2\gamma_k}\|y - x^k\|^2 : y \in \Omega\Big\}.$$

So, y^{k+1} may be close to x^k, which may lead to slow convergence of the algorithm. Therefore, in this case, Algorithm 3.1 still works well.

Example 2 Let $\mathbb{H} = \mathbb{R}$, $C = [0, 100]$ and $f(x, y) = x^2(y - x)$. Then, we have $\partial_2 f(x, y) = \{x^2\}$. It is easy to see that $S_E = S_M = \{0\}$ and *f* is Lipschitz-type on *C* with $L_1 = L_2 = 100$. So *f* satisfies Assumptions $(A1) - (A3)$.

We now compare behavior of the sequences generated by Algorithm 1 with some starting points under two different choices of γ_{k-1}, namely $\gamma_{k-1} \geq \min\left\{\frac{\theta_k}{L_1}, \frac{\theta_k}{L_2}\right\}$ and $\gamma_{k-1} \leq \min\left\{\frac{\theta_k}{L_1}, \frac{\xi_k}{L_2}\right\}$. In both cases, we choose $\theta_k = \frac{1}{3}, \xi_k = \frac{1}{4}$. For the case $\gamma_{k-1} \geq \min\left\{\frac{\theta_k}{L_1}, \frac{\xi_k}{L_2}\right\}$,weset$\gamma_{-1} = 1, \delta_k = \frac{1}{k+2}$.Forthecase$\gamma_{k-1} \leq \min\left\{\frac{\theta_k}{L_1}, \frac{\theta_k}{L_2}\right\}$, we choose $\gamma_{-1} = \frac{1}{400}, \delta_k = \frac{1}{k+402}$. The computed results are reported in Table 1.

Theorem 1 *Let f satisfy assumptions (A1) - (A3) (with unknown modulus L_1, L_2) and S_M is nonempty. Then, the sequence $\{x^k\}$ generated by Algorithm 1 converges strongly to a solution of EP(Ω, f).*

Proof We divide the proof of this theorem into five following statements.

Statement 1. $S_M \subset (\cap_{j=0}^{\infty}\Upsilon_j) \cap \Omega$ and $(\cap_{j=0}^{\infty}\Upsilon_j) \cap \Omega \subset \cap_{j=0}^{\infty}\Psi_j$. Indeed, since $S_M \neq \emptyset$, take $p \in S_M$, we have $f(y, p) \leq 0, \;\; \forall y \in \Omega$. Replacing $y = y^k \in \Omega$ into the above inequality, we get

$$f(y^k, p) \leq 0, \;\; \forall k. \tag{1}$$

Since $w^k \in \partial_2 f(y^k, x^k)$, we have

Table 1 Results on Example 2 with some starting points and γ_{-1}

Starting points	k	$\gamma_{-1}=1, \delta_k=\frac{1}{k+2}$					$\gamma_{-1}=\frac{1}{400}, \delta_k=\frac{1}{k+402}$				
		u^k	x^{k+1}	γ_k	y^{k+1}	$CPU(s)$	u^k	x^{k+1}	γ_k	y^{k+1}	$CPU(s)$
$x^{-1}=5, x^0=x^g=4, y^0=3$	0	3	3.5	0.5	0		3	3.5	0.025	3.9600	
	1	3	3.2500	0.3333	0		3	3.2500	0.0025	3.4694	
	2	3	3.1250	0.2500	0.6094		3	3.1250	0.0025	3.2236	
	3	3	3.0625	0.2000	1.1719		3	3.0625	0.0025	3.1006	
	4	0.6094	1.8359	0.1667	1.4993		3	3.0312	0.0025	3.0391	
	5	0.6094	1.2227	0.1667	1.2742		3	3.0156	0.0025	3.0083	
	2359	0.0100	0.0100	0.1667	0.0100	0.4062	0.5533	0.5535	0.0025	0.5529	
	100,000						0.0159	0.0159	0.0025	0.0159	37.8906
$x^{-1}=50,, x^0=x^g=48, y^0=46$	0	46	47	0.5	0		46	47	0.0025	42.2400	
	1	46	46.5	0.3333	0		46	46.5000	0.0025	41.4775	
	2	46	46.2500	0.2500	0		42.2400	44.3700	0.0025	41.0944	
	3	46	46.1250	0.2000	0		41.4775	42.9237	0.0025	39.4483	
	4	46	46.0625	0.1667	0		41.0944	42.0091	0.0025	38.3176	
	5	46	46.0312	0.1429	0		39.4483	40.7287	0.0025	37.5972	
	19,841	0.0100	0.0100	0.0200	0.0100	6.0625	0.0804	0.0804	0.0025	0.0804	
	100,000						0.0160	0.0160	0.0025	0.0160	90.1094

$$f(y^k, p) \geq f(y^k, x^k) + \langle w^k, p - x^k \rangle. \tag{2}$$

From (1) and (2), we get $f(y^k, x^k) + \langle w^k, p - x^k \rangle \leq 0, \quad \forall k$. This means that $p \in (\cap_{j=0}^{\infty} \Upsilon_j) \cap \Omega$. Now, to prove that $(\cap_{j=0}^{\infty} \Upsilon_j) \cap \Omega \subset \cap_{j=0}^{\infty} \Psi_j$, we will show that $(\cap_{j=0}^{\infty} \Upsilon_j) \cap \Omega \subset \Pi_k, \forall k$ and $(\cap_{j=0}^{\infty} \Upsilon_j) \cap \Omega \subset \Sigma_k, \forall k$. Firstly, take $p \in (\cap_{j=0}^{\infty} \Upsilon_j) \cap \Omega$. Then, $p \in \Omega \cap \Upsilon_{t_k} = \Omega_k$, and $P_{\Omega_k}(p) = p$. We have

$$\|u^k - p\| = \|P_{\Omega_k}(x^k) - P_{\Omega_k}(p)\| \leq \|x^k - p\| \tag{3}$$

Hence, $p \in \Pi_k, \quad \forall k$.

Secondly, we show that $(\cap_{j=0}^{\infty} \Upsilon_j) \cap \Omega \subset \Sigma_k, \forall k$ by induction method. Indeed, if $k = 0$, we have $(\cap_{j=0}^{\infty} \Upsilon_j) \cap \Omega \subset \Sigma_0 = \mathbb{H}$. Suppose that $(\cap_{j=0}^{\infty} \Upsilon_j) \cap \Omega \subset \Sigma_k$. Take $p \in (\cap_{j=0}^{\infty} \Upsilon_j) \cap \Omega$, we have to show that $p \in \Sigma_{k+1}$. Indeed, since $x^{k+1} = P_{\Psi_k}(x^g)$, we obtain

$$\langle x - x^{k+1}, x^g - x^{k+1} \rangle \leq 0, \quad \forall x \in \Psi_k = \Pi_k \cap \Sigma_k \cap \Omega. \tag{4}$$

We have $(\cap_{j=0}^{\infty} \Upsilon_j) \cap \Omega \subset \Sigma_k$ (induction assumption) and $p \in (\cap_{j=0}^{\infty} \Upsilon_j) \cap \Omega$, so $p \in \Sigma_k$. On the other hand, $(\cap_{j=0}^{\infty} \Upsilon_j) \cap \Omega \subset \Pi_k$. Therefore, $p \in \Pi_k \cap \Sigma_k \cap \Omega = \Psi_k$. Replacing $x = p$ into (4), we have

$$\langle p - x^{k+1}, x^g - x^{k+1} \rangle \leq 0.$$

Therefore, $p \in \Sigma_{k+1}$.

Statement 2. Four sequences $\{x^k\}$, $\{y^k\}$, $\{w^k\}$, $\{u^k\}$ are bounded. Take $p \in S_M$. We have

$$\begin{aligned} \|x^{k+1} - x^k\|^2 &= \|x^{k+1} - x^g\|^2 + \|x^g - x^k\|^2 + 2\langle x^{k+1} - x^g, x^g - x^k \rangle \\ &= \|x^{k+1} - x^g\|^2 - \|x^g - x^k\|^2 + 2\langle x^{k+1} - x^k, x^g - x^k \rangle \\ &\leq \|x^{k+1} - x^g\|^2 - \|x^g - x^k\|^2, \end{aligned} \tag{5}$$

where the inequality is derived from $x^{k+1} \in \Psi_k \subset \Sigma_k$. Hence, $\{\|x^k - x^g\|\}$ is nondecreasing sequence. On the other hand, since $\Sigma_k = \{x \in \mathbb{H} : \langle x - x^k, x^g - x^k \rangle \leq 0\}$, we have $x^k = P_{\Sigma_k}(x^g)$. Furthermore, $p \in S_M \subset \cap_{j=0}^{\infty} \Psi_j = \cap_{j=0}^{\infty} (\Pi_j \cap \Sigma_j \cap \Omega)$, we have $p \in \Sigma_k$. Therefore, we obtain

$$\|x^k - x^g\| \leq \|p - x^g\|. \tag{6}$$

Hence, there exists $\lim_{k \to \infty} \|x^k - x^g\|$ and $\{x^k\}$ is bounded.

From the definition of $\{\gamma_k\}$ and the positive nonincreasing property of $\{\delta_k\}$, we deduce that $\{\gamma_k\}$ is the positive nonincreasing sequence. Now, we prove that $\{\gamma_k\} \nrightarrow 0$. By contradiction, suppose that $\gamma_k \to 0$. Then, there exists a subsequence $\{\gamma_{k_n}\} \subset \{\gamma_k\}$ such that

$$\gamma_{k_n-1}\left(f(x^{k_n-1},x^{k_n})-f(x^{k_n-1},y^{k_n})-f(y^{k_n},x^{k_n})\right)$$
$$> \theta_{k_n}\|x^{k_n-1}-y^{k_n}\|^2+\xi_{k_n}\|y^{k_n}-x^{k_n}\|^2.$$

Combining with the assumption (A3), $\theta_{k_n}, \xi_{k_n} \in [\alpha, 1)$ with $\alpha > 0$ and set $L = \max\{L_1, L_2\}$, we have

$$\begin{aligned}\alpha(\|x^{k_n-1}-y^{k_n}\|^2+\|y^{k_n}-x^{k_n}\|^2) &\le \theta_{k_n}\|x^{k_n-1}-y^{k_n}\|^2+\xi_{k_n}\|y^{k_n}-x^{k_n}\|^2\\ &< \gamma_{k_n-1}\big(f(x^{k_n-1},x^{k_n})-f(x^{k_n-1},y^{k_n})\\ &\quad -f(y^{k_n},x^{k_n})\big)\\ &\le \gamma_{k_n-1}(L_1\|x^{k_n-1}-y^{k_n}\|^2+L_2\|y^{k_n}-x^{k_n}\|^2)\\ &\le \gamma_{k_n-1}L(\|x^{k_n-1}-y^{k_n}\|^2+\|y^{k_n}-x^{k_n}\|^2).\end{aligned}$$

This means $\alpha < L\gamma_{k_n-1}$ for all $n \ge 0$. It contradicts the assumption $\gamma_k \to 0$. Hence, $\gamma_k \nrightarrow 0$. Therefore, there exists a number $k_0 > 0$ such that

$$\gamma_{k-1}\left(f(x^{k-1},x^k)-f(x^{k-1},y^k)-f(y^k,x^k)\right) \le \theta_k\|x^{k-1}-y^k\|^2+\xi_k\|y^k-x^k\|^2, \quad (7)$$

and $\gamma_k = \gamma_{k-1}$ for all $k \ge k_0$.

If $k \ge k_0$ then $\gamma_k = \gamma_{k-1} = ... = \gamma_{k_0}$ and

$$y^{k+1} = \arg\min\{\gamma_{k_0} f(x^k, y) + \frac{1}{2}\|y-x^k\|^2 : y \in \Omega\}.$$

By Lemma 2, we get the boundedness of the sequence $\{y^k\}_{k\ge k_0}$. If $k < k_0$, set $M_1 = \max\{\|y^0\|, \|y^1\|, ..., \|y^{k_0-1}\|\}$ we always have $\|y^k\| \le M_1$ for all $k < k_0$. Inconclusion, $\{y^k\}$ is bounded. Consider $\{w^k\}$ with $w^k \in \partial_2 f(y^k, x^k)$, by Assumption $(A2)$, we obtain the boundedness of $\{w^k\}$. By (3), we also have the boundedness of $\{u^k\}$.

Statement 3. $\lim_{k\to\infty} d(x^k, \Upsilon_k) = \lim_{k\to\infty} d(x^k, \Upsilon_{t_k}) = \lim_{k\to\infty} f(y^k, x^k) = \lim_{k\to\infty}\|x^{k+1}-x^k\| = \lim_{k\to\infty}\|u^k-x^k\| = 0$ From (5) and the fact that there exists $\lim_{k\to\infty}\|x^k - x^g\|$, we deduce $\lim_{k\to\infty}\|x^{k+1}-x^k\| = 0$. On the other hand, $x^{k+1} \in \Psi_k \subset \Pi_k$, i.e., $\|x^{k+1}-u^k\| \le \|x^{k+1}-x^k\|$. So,

$$\begin{aligned}\|u^k-x^k\| &\le \|u^k-x^{k+1}\|+\|x^{k+1}-x^k\|\\ &\le 2\|x^{k+1}-x^k\|.\end{aligned}$$

Combining with $\lim_{k\to\infty}\|x^{k+1}-x^k\| = 0$, we obtain $\lim_{k\to\infty} d(x^k, \Omega_k) = \lim_{k\to\infty}\|u^k-x^k\| = 0$. By definition of t_k, $\Omega_k = \Omega \cap \Upsilon_{t_k}$, and $\|w^k\| \le M_2$, we have

$$0 \le \frac{|f(y^k,x^k)|}{M_2} \le \frac{|f(y^k,x^k)|}{\|w^k\|} = d(x^k,\Upsilon_k) \le d(x^k,\Upsilon_{t_k}) \le d(x^k,\Omega_k).$$

Therefore, $\lim_{k\to\infty} d(x^k, \Upsilon_k) = \lim_{k\to\infty} d(x^k, \Upsilon_{t_k}) = \lim_{k\to\infty} f(y^k, x^k) = 0$.

Statement 4. $\{x^k\}, \{u^k\}$ converges strongly to $x^* = P_{\cap_{j=0}^{\infty}\Psi_j}(x^g)$. Replacing $p = x^*$ into (6), we have

$$\|x^k - x^g\| \leq \|P_{\cap_{j=0}^{\infty}\Psi_j}(x^g) - x^g\|. \tag{8}$$

Now, we show that any weak accumulation of $\{x^k\}$ belongs to $\cap_{j=0}^{\infty}\Psi_j$. Indeed, let $\bar{x}$ be a weak accumulation of $\{x^k\}$. Then, there exists $\{x^{k_l}\} \subset \{x^k\}$ and $x^{k_l} \rightharpoonup \bar{x}$ as $l \to \infty$. Now, we have to prove that $\bar{x} \in \cap_{j=0}^{\infty}\Psi_j$. From definition of t_k, we have

$$0 \leq d(x^k, \Upsilon_i) \leq d(x^k, \Upsilon_{t_k}), \quad \text{for each} \quad i = 0, 1, ..., k.$$

Combining with $\lim_{k\to\infty} d(x^k, \Upsilon_{t_k}) = 0$, we have $\lim_{k\to\infty} d(x^k, \Upsilon_i) = 0$ for any fixed $i \geq 0$. So $\lim_{l\to\infty} d(x^{k_l}, \Upsilon_i) = 0$. Set $t^{k_l} := P_{\Upsilon_i}(x^{k_l})$. It is clear that $\lim_{l\to\infty} \|x^{k_l} - t^{k_l}\| = 0$. We have for all $y \in \mathbb{H}$

$$\begin{aligned}\langle t^{k_l} - \bar{x}, y\rangle &= \langle t^{k_l} - x^{k_l}, y\rangle + \langle x^{k_l} - \bar{x}, y\rangle \\ &\leq \|t^{k_l} - x^{k_l}\|.\|y\| + \langle x^{k_l} - \bar{x}, y\rangle.\end{aligned}$$

Combining with $\lim_{l\to\infty} \|x^{k_l} - t^{k_l}\| = 0.$ and $x^{k_l} \rightharpoonup \bar{x}$ as $l \to \infty$, we have $t^{k_l} \rightharpoonup \bar{x}$ as $l \to \infty$. Furthermore, $\{t^{k_l}\} \subset \Upsilon_i$ and Υ_i is weakly closed (because Υ_i is closed convex in $\mathbb{H}$). Hence, $\bar{x} \in \Upsilon_i, \forall i$. Similiary, $\bar{x} \in \Omega$. Therefore, $\bar{x} \in (\cap_{j=0}^{\infty}\Upsilon_j) \cap \Omega \subset \cap_{j=0}^{\infty}\Psi_j$. Next, we show that $x^k \rightharpoonup P_{\cap_{j=0}^{\infty}\Psi_j}(x^g)$. Indeed, let v be any weak accumulation of $\{x^k\}$. Then, there exists $\{x^{k_l}\} \subset \{x^k\}$ and $\{x^{k_l}\} \rightharpoonup v$ as $l \to \infty$. By (8), we have

$$\|x^{k_l} - x^g\| \leq \|P_{\cap_{j=0}^{\infty}\Psi_j}(x^g) - x^g\|.$$

By the weak lower-semicontinuity of the norm and $x^{k_l} \rightharpoonup v$ as $l \to \infty$, we get

$$\liminf_{l\to\infty} \|x^{k_l} - x^g\| \geq \|v - x^g\|.$$

From two above inequalities, we obtain

$$\|v - x^g\| \leq \|P_{\cap_{j=0}^{\infty}\Psi_j}(x^g) - x^g\|.$$

Since $v \in \cap_{j=0}^{\infty}\Psi_j$ and by the property of projection, we have

$$\|P_{\cap_{j=0}^{\infty}\Psi_j}(x^g) - x^g\| \leq \|v - x^g\|.$$

It implies that

$$\|v - x^g\| = \|P_{\cap_{j=0}^{\infty}\Psi_j}(x^g) - x^g\|.$$

On the other hand, by the property of projection, we have

$$\|v - P_{\cap_{j=0}^{\infty}\Psi_j}(x^g)\|^2 \leq \|v - x^g\|^2 - \|x^g - P_{\cap_{j=0}^{\infty}\Psi_j}(x^g)\|^2.$$

Therefore, $v = P_{\cap_{j=0}^{\infty}\Psi_j}(x^g)$ for any weak accumulation v of $\{x^k\}$. Hence, $x^k \rightharpoonup P_{\cap_{j=0}^{\infty}\Psi_j}(x^g)$. Now, we have

$$\begin{aligned}
\|x^k - P_{\cap_{j=0}^{\infty}\Psi_j}(x^g)\|^2 &= \|x^k - x^g + x^g - P_{\cap_{j=0}^{\infty}\Psi_j}(x^g)\|^2 \\
&= \|x^k - x^g\|^2 + \|x^g - P_{\cap_{j=0}^{\infty}\Psi_j}(x^g)\|^2 \\
&\quad + 2\langle x^k - x^g, x^g - P_{\cap_{j=0}^{\infty}\Psi_j}(x^g)\rangle \\
&\leq 2(\|x^g - P_{\cap_{j=0}^{\infty}\Psi_j}(x^g)\|^2 + \langle x^k - x^g, x^g - P_{\cap_{j=0}^{\infty}\Psi_j}(x^g)\rangle),
\end{aligned}$$

where the inequality is derived from (8). We have $\|x^g - P_{\cap_{j=0}^{\infty}\Psi_j}(x^g)\|^2 + \langle x^k - x^g, x^g - P_{\cap_{j=0}^{\infty}\Psi_j}(x^g)\rangle \to 0$ as $k \to \infty$ because $x^k \rightharpoonup P_{\cap_{j=0}^{\infty}\Psi_j}(x^g)$. Therefore, we get $x^k \to P_{\cap_{j=0}^{\infty}\Psi_j}(x^g)$. Because $\lim_{k\to\infty} \|u^k - x^k\| = 0$, we get that $u^k \to P_{\cap_{j=0}^{\infty}\Psi_j}(x^g)$.

Statement 5. x^* is a solution of EP(Ω, f). Now, from Step 3 of Algorithm 1, we have

$$y^{k+1} = \arg\min\{\gamma_k f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in \Omega\}.$$

Using the optimality condition, we have

$$0 \in \gamma_k \partial_2 f(x^k, y^{k+1}) + (y^{k+1} - x^k) + N_\Omega(y^{k+1}).$$

Hence, there exists $v^k \in \partial_2 f(x^k, y^{k+1})$ such that

$$-\gamma_k v^k - (y^{k+1} - x^k) \in N_\Omega(y^{k+1}).$$

This means

$$\langle \gamma_k v^k + y^{k+1} - x^k, z - y^{k+1}\rangle \geq 0, \ \forall z \in \Omega,$$

or

$$\langle x^k - y^{k+1}, z - y^{k+1}\rangle \leq \gamma_k \langle v^k, z - y^{k+1}\rangle, \ \forall z \in \Omega. \tag{9}$$

Since $v^k \in \partial_2 f(x^k, y^{k+1})$, we have

$$f(x^k, z) - f(x^k, y^{k+1}) \geq \langle v^k, z - y^{k+1}\rangle, \ \forall z \in \Omega. \tag{10}$$

From (9) and (10), we derive

$$\langle x^k - y^{k+1}, z - y^{k+1}\rangle \le \gamma_k \left(f(x^k, z) - f(x^k, y^{k+1})\right), \ \forall z \in \Omega. \tag{11}$$

Since (A1) and $x^k \to x^*$, we get

$$\lim_{k\to\infty} f(x^k, x^{k+1}) = 0. \tag{12}$$

For all $k \ge k_0$, we have

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &= \|x^{k+1} - x^k + x^k - x^*\|^2 \\
&= \|x^k - x^*\|^2 + \|x^{k+1} - x^k\|^2 + 2\langle x^{k+1} - x^k, x^k - x^*\rangle \\
&= \|x^k - x^*\|^2 + \|x^{k+1} - x^k\|^2 + 2\langle x^{k+1} - x^k, x^k - x^{k+1} \\
&\quad + x^{k+1} - x^*\rangle \\
&= \|x^k - x^*\|^2 - \|x^{k+1} - x^k\|^2 + 2\langle x^{k+1} - x^k, x^{k+1} - x^*\rangle.
\end{aligned} \tag{13}$$

On the other hand, from (11), replacing $z = x^k$, we have

$$\|y^{k+1} - x^k\|^2 \le -\gamma_k f(x^k, y^{k+1}).$$

Hence,

$$-2\gamma_k f(x^k, y^{k+1}) - 2\|y^{k+1} - x^k\|^2 \ge 0. \tag{14}$$

By adding (13) and (14) side by side, for all $k \ge k_0$, we get

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &\le \|x^k - x^*\|^2 - 2\|y^{k+1} - x^k\|^2 - \|x^{k+1} - x^k\|^2 \\
&\quad - 2\gamma_k f(x^k, y^{k+1}) \\
&\quad + 2\langle x^{k+1} - x^k, x^{k+1} - x^*\rangle \\
&\le \|x^k - x^*\|^2 - 2\|y^{k+1} - x^k\|^2 - \|x^{k+1} - x^k\|^2 \\
&\quad + 2\gamma_k (f(y^{k+1}, x^{k+1}) - f(x^k, x^{k+1})) \\
&\quad + 2\theta_{k+1}\|x^k - y^{k+1}\|^2 + 2\xi_{k+1}\|y^{k+1} - x^{k+1}\|^2 \\
&\quad + 2\langle x^{k+1} - x^k, x^{k+1} - x^*\rangle \\
&= \|x^k - x^*\|^2 - (2 - 2\theta_{k+1})\|y^{k+1} - x^k\|^2 - \|x^{k+1} \\
&\quad - x^k\|^2 + 2\xi_{k+1}\|y^{k+1} - x^{k+1}\|^2 \\
&\quad + 2\gamma_{k_0}[f(y^{k+1}, x^{k+1}) - f(x^k, x^{k+1})] \\
&\quad + 2\langle x^{k+1} - x^k, x^{k+1} - x^*\rangle,
\end{aligned} \tag{15}$$

where the second inequality is derived from (7) by setting $k := k - 1$, and the equality is derived from $\gamma_k = \gamma_{k-1}$ for all $k \ge k_0$. Moreover,

$$\|x^{k+1}-y^{k+1}\|^2 = \|y^{k+1}-x^k\|^2 + \|x^{k+1}-x^k\|^2 + 2\left\langle x^{k+1}-x^k, x^k-y^{k+1}\right\rangle.$$

Replacing the above inequality into (15), we obtain

$$\begin{aligned}\|x^{k+1}-x^*\|^2 \le \|x^k-x^*\|^2 &- 2(1-\theta_{k+1}-\xi_{k+1})\|y^{k+1}-x^k\|^2\\ &-(1-2\xi_{k+1})\|x^{k+1}-x^k\|^2\\ &+2\gamma_{k_0}[f(y^{k+1},x^{k+1})-f(x^k,x^{k+1})]\\ &+4\xi_{k+1}\langle x^{k+1}-x^k, x^k-y^{k+1}\rangle\\ &+2\langle x^{k+1}-x^k, x^{k+1}-x^*\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}0 \le 2(1-\theta_{k+1}-\xi_{k+1})\|y^{k+1}-x^k\|^2 \le \|x^k-x^*\|^2 &- \|x^{k+1}-x^*\|^2\\ &-(1-2\xi_{k+1})\|x^{k+1}-x^k\|^2\\ &+2\gamma_{k_0}[f(y^{k+1},x^{k+1})-f(x^k,x^{k+1})]\\ &+4\xi_{k+1}\langle x^{k+1}-x^k, x^k-y^{k+1}\rangle\\ &+2\langle x^{k+1}-x^k, x^{k+1}-x^*\rangle.\end{aligned}$$

Letting $k\to\infty$, using $\theta_{k+1},\xi_{k+1}\in[\alpha,1),(\alpha>0),\theta_{k+1}+\xi_{k+1}\le a<1$, $\lim_{k\to\infty}\|x^k-x^*\| = \lim_{k\to\infty}\|x^{k+1}-x^k\| = \lim_{k\to\infty} f(y^k,x^k)=0$, the boundedness of $\{x^k\}$ and $\{y^k\}$ and (12) we obtain

$$\lim_{k\to\infty}\|x^k-y^{k+1}\| = 0. \tag{16}$$

Combining with $x^k\to x^*$, we get $y^{k+1}\to x^*$. From (11), we have for all $z\in\Omega$

$$\begin{aligned}\gamma_{k_0} f(x^k,z) &\ge \langle x^k-y^{k+1}, z-y^{k+1}\rangle + \gamma_{k_0} f(x^k,y^{k+1})\\ &\ge -\|x^k-y^{k+1}\|.\|z-y^{k+1}\| - \gamma_{k_0}\|t^k\|\|y^{k+1}-x^k\|.\end{aligned}\tag{17}$$

where $t^k\in\partial_2 f(x^k,x^k)$. Using $x^k\to x^*$ and Assumption $(A2)$, we have the boundedness of $\{t^k\}$. Letting $k\to\infty$ in (17) and using (16), Assumption $(A2)$, and the boundedness of $\{t^k\},\{y^k\}$, $x^k\to x^*, y^{k+1}\to x^*$, we get

$$\gamma_{k_0} f(x^*,z)\ge 0,\ \forall z\in\Omega,$$

or $f(x^*,z)\ge 0$, $\forall z\in\Omega$. Therefore, x^* is a solution of EP(Ω,f). $\square$

When the equilibrium problem reduces to the variational inequality, that is, $f(x,y)=\langle F(x), y-x\rangle$ then Algorithm 1 becomes the following algorithm, which, to the best of our knowledge, is also a new method for solving the nonmonotone variation inequality in Hilbert space.

Algorithm 2.

Initialization. Pick $x^{-1}, x^0 = x^g, y^0 \in \Omega$ and $\gamma_{-1} \in (0, \infty)$. Choose two sequences $\{\theta_k\}, \{\xi_k\}$ satisfying $0 < \alpha \leq \theta_k < 1, 0 < \alpha \leq \xi_k < 1$ and $\theta_k + \xi_k \leq a < 1$. Choose parameter $\gamma_{-1} \in (0, \infty)$. Let $\{\delta_k\} \subset (0, \infty)$ be a non-increasing sequence satisfying $\delta_0 < \gamma_{-1}$ and $\lim\limits_{k\to\infty} \delta_k = 0$. Set $k = 0$.

Iteration k. (k = 0, 1, 2,...) Having x^k, y^k, γ_{k-1} do the following steps:

Step 1. Set $\Gamma_k = \{x \in \mathbb{H} : \langle F(y^k), x - y^k\rangle \leq 0\}$, and compute

$$u^k = P_{C_k}(x^k), \quad \text{where } \Omega_k = \Omega \cap \Gamma_{t_k}, t_k \in \arg\max\{d(x^k, \Gamma_j) : 0 \leq j \leq k\}.$$

Step 2. Compute $x^{k+1} = P_{\Psi_k}(x^g)$, where

$$\Pi_k = \{x \in \mathbb{H} : \|x - u^k\| \leq \|x - x^k\|\}, \Sigma_k = \{x \in \mathbb{H} : \langle x - x^k, x^g - x^k\rangle \leq 0\}.$$

$$\Psi_k = \Pi_k \cap \Sigma_k \cap \Omega.$$

Step 3. If $\gamma_{k-1}\langle F(x^{k-1}) - F(y^k), x^k - y^k\rangle \leq \theta_k\|x^{k-1} - y^k\|^2 + \xi_k \|y^k - x^k\|^2$ then set $\gamma_k = \gamma_{k-1}$ else set $\gamma_k = \delta_k$. Compute

$$y^{k+1} = P_\Omega(x^k - \gamma_k F(x^k)).$$

If $y^{k+1} = x^k$, then stop, x^k is a solution of VIP(*C*, *F*). Otherwise, go to Iteration *k* with *k* is replaced by $k + 1$.
The following corollary is immediate from Theorem 1.

Corollary 1 *Let F be Lipschitz continuous on Ω (with the Lipschitz constant is unknown). Suppose that the solution set of MVIP(Ω, F) is nonempty. Then the sequence $\{x^k\}$ generated by Algorithm 2 converges strongly to a solution of VIP(Ω, F).*

One advantage of Algorithm 1 is that we can use it to find a solution to the non-monotone equilibrium problem even when the Lipschitz constants of the bifunction may be unknown. The following algorithm shows that if these constants are explicit, then we do not need to check the condition in Step 3.

Algorithm 3.

Initialization. Pick $x^0 = x^g \in \Omega$, choose parameters $\{\gamma_k\} \subset [\alpha, \beta]$ such that $0 < \beta < \frac{1-\mu}{L_2}, 0 < \lambda \leq \sqrt{\frac{1-\mu-\beta L_2}{\beta L_1}}$ or $0 < \beta < \frac{\mu}{L_2}, 0 < \lambda \leq \sqrt{\frac{\mu-\beta L_2}{\beta L_1}}$.
Iteration k $(k = 0, 1, 2, ...)$. Having x^k do the following steps:

Step 1. Compute

$$y^k = \arg\min\left\{\gamma_k f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in \Omega\right\}.$$

If $y^k = x^k$ then stop, x^k is a solution of EP(Ω, f). Otherwise, do Step 2.

Step 2. Set $z^k = (1-\lambda)x^k + \lambda y^k$, $w^k \in \partial_2 f(z^k, x^k)$, and

$$T_k = \{x \in \mathbb{H} : \langle w^k, x - x^k\rangle + f(z^k, x^k) \leq 0\}.$$

Compute $u^k = P_{\Omega_k}(x^k)$, where

$$\Omega_k = \Omega \cap T_{t_k},\ \ t_k \in \arg\max\{d(x^k, T_j) : 0 \leq j \leq k\}.$$

Step 3. Compute $x^{k+1} = P_{\Psi_k}(x^g)$, where $\Psi_k = \Pi_k \cap \Sigma_k \cap \Omega$.

$$\Pi_k = \{x \in \mathbb{H} : \|x - u^k\| \leq \|x - x^k\|\},\ \ \Sigma_k = \{x \in \mathbb{H} : \langle x - x^k, x^g - x^k\rangle \leq 0\}.$$

Go to Step 1 with k is replaced by $k+1$.
The following theorem gives us the convergence of Algorithm 3.

Theorem 2 *Suppose that f satisfies assumptions $(A1) - (A3)$ (with the explicit constants L_1, L_2), and $S_M \neq \emptyset$. Then $\{x^k\}$ and $\{u^k\}$ generated by Algorithm 3 converges strongly to a solution of EP(Ω, f).*

Proof The proof of convergence can be done by the similar way as in Theorem 1, so we omit it. For instance, instead of using $y = y^k \in \Omega$ in (1), we use $y = z^k \in \Omega$ and Υ_k is replaced by the corresponding T_k.

Now, we will show that x^* is a solution of EP(Ω, f). Indeed, from Step 1 of Algorithm 3, we have

$$y^k = \arg\min\{\gamma_k f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in \Omega\}$$

Using the optimality condition, we obtain

$$0 \in \gamma_k \partial_2 f(x^k, y^k) + (y^k - x^k) + N_\Omega(y^k).$$

Therefore, there exists $w \in \partial_2 f(x^k, y^k)$ such that $-\gamma_k w - (y^k - x^k) \in N_\Omega(y^k)$. This means that

$$\langle -\gamma_k w - (y^k - x^k), y - y^k\rangle \leq 0.$$

By definition of subdifferentiable $w \in \partial_2 f(x^k, y^k)$, we get

$$f(x^k, y) - f(x^k, y^k) \geq \langle w, y - y^k \rangle, \quad \forall y \in \Omega.$$

From two above inequalities, we obtain

$$\gamma_k [f(x^k, y) - f(x^k, y^k)] \geq \langle y^k - x^k, y^k - y \rangle, \quad \forall y \in \Omega. \tag{18}$$

Replacing $y = x^k$ into the above inequality, we get

$$f(x^k, y^k) \leq -\frac{1}{\gamma_k} \|y^k - x^k\|^2 \leq -\frac{1}{\beta} \|y^k - x^k\|^2, \tag{19}$$

where the second inequality is derived from $0 < \alpha \leq \gamma_k \leq \beta$.

Now, we consider two following cases.

Case 1. Two parameters β, λ satisfy $0 < \beta < \frac{1-\mu}{L_2}, 0 < \lambda \leq \sqrt{\frac{1-\mu-\beta L_2}{\beta L_1}}$. Because f is Lipschitz-type continuous on Ω with constants $L_1 > 0$ and $L_2 > 0$, so we have

$$\begin{aligned} f(z^k, y^k) - f(z^k, x^k) - f(x^k, y^k) &\leq L_1 \|z^k - x^k\|^2 + L_2 \|x^k - y^k\|^2 \\ &= (L_1 \lambda^2 + L_2) \|x^k - y^k\|^2 \\ &\leq (L_1 \frac{1 - \mu - \beta L_2}{\beta L_1} + L_2) \|x^k - y^k\|^2 \\ &= \frac{1 - \mu}{\beta} \|x^k - y^k\|^2. \end{aligned} \tag{20}$$

Adding (19) to (20), we obtain

$$f(z^k, y^k) - f(z^k, x^k) \leq -\frac{\mu}{\beta} \|x^k - y^k\|^2. \tag{21}$$

Using Assumption (*A*2), we have

$$0 = f(z^k, z^k) = f(z^k, (1 - \lambda) x^k + \lambda y^k) \leq (1 - \lambda) f(z^k, x^k) + \lambda f(z^k, y^k).$$

Combining with (21), we get

$$f(z^k, x^k) \geq \lambda (f(z^k, x^k) - f(z^k, y^k)) \geq \frac{\lambda \mu}{\beta} \|x^k - y^k\|^2 > 0. \tag{22}$$

Case 2. Two parameters β, λ satisfy $0 < \beta < \frac{\mu}{L_2}, 0 < \lambda \leq \sqrt{\frac{\mu-\beta L_2}{\beta L_1}}$. Because f is Lipschitz-type continuous on Ω with constants $L_1 > 0$ and $L_2 > 0$, $0 < \lambda \leq \sqrt{\frac{\mu-\beta L_2}{\beta L_1}}$, so we have

$$\begin{aligned} f(z^k, x^k) + f(x^k, y^k) - f(z^k, y^k) &\geq -L_1\|z^k - x^k\|^2 - L_2\|x^k - y^k\|^2 \\ &= -(L_1\lambda^2 + L_2)\|x^k - y^k\|^2 \\ &\geq -(L_1\frac{\mu - \beta L_2}{\beta L_1} + L_2)\|x^k - y^k\|^2 \\ &= -\frac{\mu}{\beta}\|x^k - y^k\|^2. \end{aligned} \tag{23}$$

Adding (19) to (23), we obtain

$$f(z^k, x^k) - f(z^k, y^k) \geq \frac{1-\mu}{\beta}\|x^k - y^k\|^2.$$

Using the same argument as Case 1, we have

$$f(z^k, x^k) \geq \lambda(f(z^k, x^k) - f(z^k, y^k)) \geq \frac{\lambda(1-\mu)}{\beta}\|x^k - y^k\|^2 > 0. \tag{24}$$

Therefore, in both cases, from (22) or (24), and $\lim_{k\to\infty} f(z^k, x^k) = 0$, we get that

$$\lim_{k\to\infty} \|x^k - y^k\| = 0.$$

Remember that $x^k \to x^*$, it follows from the above equality that $y^k \to x^*$. From (18), we get

$$\gamma_k[f(x^k, y) - f(x^k, y^k)] \geq -\|y^k - x^k\|.\|y^k - y\|.$$

Letting $k \to \infty$, using (A1), $0 < \alpha \leq \gamma_k \leq \beta$, $x^k \to x^*$, $y^k \to x^*$, we get

$$f(x^*, y) \geq 0, \quad \forall y \in \Omega.$$

Therefore, x^* is a solution of EP(Ω, f). $\square$

One advantage of Algorithms 1 and 3 is that they can be used to find a solution to the nonmonotone equilibrium problem $EP(\Omega, f)$ when f is of Lipschitz type. However, in some cases, the bifunction is not of Lipschitz type, so we may not be able to use them to solve $EP(\Omega, f)$ directly. The following algorithm may be applied to find a solution for such equilibrium problems.

Algorithm 4.

Initialization. Pick $x^0 = x^g \in \Omega$, choose parameters $\{\gamma_k\} \subset [\alpha, \beta]$ with $0 < \alpha \leq \beta$, $\mu \in (0, 1)$. Take a sequence $\{\eta_i\} \subset (0, 1)$ strictly decreasing to zero.
Iteration k $(k = 0, 1, 2, ...)$. Having x^k do the following steps:

Step 1. Compute

$$y^k = \arg\min\left\{\gamma_k f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in \Omega\right\}.$$

If $y^k = x^k$ then stop and x^k is a solution of EP(Ω, f). Otherwise, do Step 2.

Step 2. Find m_k be the smallest nonnegative integer *m* such that

$$\begin{cases} z^{k,m} = (1-\eta_m)x^k + \eta_m y^k \\ f(z^{k,m}, x^k) - f(z^{k,m}, y^k) \geq \frac{\mu}{\gamma_k}\|y^k - x^k\|^2. \end{cases} \tag{25}$$

Set $\theta_k = \eta_{m_k}, z^k = z^{k,m_k}$.

Step 3. Select $w^k \in \partial_2 f(z^k, x^k)$, set $T_k = \{x \in \mathbb{H} : \langle w^k, x - x^k\rangle + f(z^k, x^k) \leq 0\}$. Compute $u^k = P_{\Omega_k}(x^k)$, where

$$\Omega_k = \Omega \cap T_{t_k}, t_k \in \arg\max\{d(x^k, T_j) : 0 \leq j \leq k\}.$$

Step 4. Compute $x^{k+1} = P_{\Psi_k}(x^g)$, where $\Psi_k = \Omega_k \cap \Sigma_k \cap \Omega$,

$$\Pi_k = \{x \in \mathbb{H} : \|x - u^k\| \leq \|x - x^k\|\}, \ \Sigma_k = \{x \in \mathbb{H} : \langle x - x^k, x^g - x^k\rangle \leq 0\}.$$

Go to Step 1 with *k* is replaced by $k+1$.
We need the following lemma to prove the convergence of Algorithm 4.

Lemma 3 *At iteration k, if $y^k \neq x^k$, then we have:*

(i) *The linesearch is well-defined;*
(ii) $f(z^k, x^k) > 0$.

The proof of this lemma is quite similar to the proof of Lemma 4.5 in [25], so we omit it.

We need the following assumption:

(A1') $f(\cdot,\cdot)$ is strongly weakly continuous on $\Omega \times \Omega$ that is if $\{x^k\}, \{y^k\}$ are sequences in Ω such that $x^k \to x$ and $y^k \rightharpoonup y$ then we have $f(x^k, y^k) \to f(x, y)$..

Remark 3 Since $x^k \to x$ we also have $x^k \rightharpoonup x$, so the joint weak continuity of *f* implies that *f* is also strongly weakly continuous on $\Omega \times \Omega$. However, the inverse may not be true. Indeed, for instance $f(x, y) = \langle F(x), y - x\rangle$, where $F : \Omega \to \mathbb{H}$ is a continuous mapping. Then we can verify that *f* is strongly weakly continuous on $\Omega \times \Omega$. Recall Example 1, we have that *f* is not necessarily jointly weakly continuous.

Theorem 3 *Suppose that f satisfies assumptions $(A1'), (A2)$, and $S_M \neq \emptyset$. Then $\{x^k\}$ and $\{u^k\}$ generated by Algorithm 4 converge strongly to a solution of EP(Ω, f).*

Proof We divide the proof into five statements as the followings.

Statement 1. $S_M \subset (\cap_{j=0}^{\infty} T_j) \cap \Omega$ and $(\cap_{j=0}^{\infty} T_j) \cap \Omega \subset \cap_{j=0}^{\infty} \Psi_j$.
Statement 2. Five sequences $\{x^k\}, \{y^k\}, \{z^k\}, \{w^k\}, \{u^k\}$ are bounded.
Statement 3. $\lim_{k\to\infty} d(x^k, T_k) = \lim_{k\to\infty} d(x^k, T_{t_k}) = \lim_{k\to\infty} f(z^k, x^k) = \lim_{k\to\infty} \|x^{k+1} - x^k\| = \lim_{k\to\infty} \|u^k - x^k\| = 0$
Statement 4. $\{x^k\}, \{u^k\}$ converges strongly to $x^* = P_{\cap_{j=0}^{\infty} \Psi_j}(x^g)$.

Statement 5. x^* is a solution of EP(Ω, f).

The proof of the first four statements is quite similar to that of Theorem 2, so we omit it. Now, we will show that Statement 5 is true. Indeed, we have

$$\begin{aligned} 0 = f(z^k, z^k) &= f(z^k, (1-\theta_k)x^k + \theta_k y^k) \\ &\le (1-\theta_k) f(z^k, x^k) + \theta_k f(z^k, y^k), \end{aligned} \tag{26}$$

From (25) and (26) we get

$$f(z^k, x^k) \ge \theta_k [f(z^k, x^k) - f(z^k, y^k)] \ge \frac{\theta_k \mu}{\gamma_k} \|y^k - x^k\|^2 \ge \frac{\theta_k \mu}{\beta} \|y^k - x^k\|^2.$$

Combining with $\lim_{k\to\infty} f(z^k, x^k) = 0$, we have

$$\lim_{k\to\infty} \theta_k \|y^k - x^k\|^2 = 0. \tag{27}$$

Now we consider two distinct cases.

Case 1. $\limsup_{k\to\infty} \theta_k > 0$. Then there exists $\bar{\theta} > 0$ and a subsequence $\{\theta_{k_l}\} \subset \{\theta_k\}$ such that $\theta_{k_l} > \bar{\theta}$ for all l. From (27), we get

$$\lim_{l\to\infty} \|x^{k_l} - y^{k_l}\| = 0.$$

Using the same argument as in the proof of Statement 5 in Theorem 2, we get that x^* is a solution of EP(Ω, f). *Case 2.* $\lim_{k\to\infty} \theta_k = 0$. From the boundedness of $\{y^k\}$, it deduces that there exists $\{y^{k_l}\} \subset \{y^k\}$ such that $y^{k_l} \rightharpoonup \bar{y}$ as $l \to \infty$. Replacing y by x^{k_l} in (18), we get

$$f(x^{k_l}, y^{k_l}) \le -\frac{1}{\gamma_{k_l}} \|y^{k_l} - x^{k_l}\|^2.$$

Suppose that $\theta_{k_l} = \eta_{m_{k_l}}$ satisfies the linesearch (25). Then, for $m_{k_l} - 1$, we have

$$
\begin{aligned}
&f((1-\eta_{m_{k_l}-1})x^{k_l}+\eta_{m_{k_l}-1}y^{k_l},x^{k_l})-f((1-\eta_{m_{k_l}-1})x^{k_l}+\eta_{m_{k_l}-1}y^{k_l},y^{k_l})\\
&\quad<\frac{\mu}{\gamma_{k_l}}\|y^{k_l}-x^{k_l}\|^2.
\end{aligned}
$$

Combining the two above inequalities, we get

$$
\begin{aligned}
f(x^{k_l},y^{k_l})&\le-\frac{1}{\gamma_{k_l}}\|y^{k_l}-x^{k_l}\|^2\\
&<\frac{1}{\mu}[f((1-\eta_{m_{k_l}-1})x^{k_l}+\eta_{m_{k_l}-1}y^{k_l},y^{k_l})\\
&\quad-f((1-\eta_{m_{k_l}-1})x^{k_l}+\eta_{m_{k_l}-1}y^{k_l},x^{k_l})]
\end{aligned}
\tag{28}
$$

According to the algorithm, we have $\eta_{m_{k_l}-1}\to 0$ and by Statement 4, we also have $x^{k_l}\to x^*$, $y^{k_l}\rightharpoonup\bar{y}$ as $l\to\infty$. It implies that $z^{k_l,m_{k_l}-1}\to x^*$ as $l\to\infty$. Besides that $\{\frac{1}{\gamma_{k_l}}\|y^{k_l}-x^{k_l}\|^2\}$ is bounded, without loss of generality we may assume that $\lim_{l\to\infty}\frac{1}{\gamma_{k_l}}\|y^{k_l}-x^{k_l}\|^2$ exists. Therefore, letting $l\to\infty$ in (28) and using $(A1')$, we get

$$
f(x^*,\bar{y})\le-\lim_{l\to\infty}\frac{1}{\gamma_{k_l}}\|y^{k_l}-x^{k_l}\|^2\le\frac{1}{\mu}f(x^*,\bar{y}).
$$

Since $\mu\in(0,1)$, therefore, we have $f(x^*,\bar{y})=0$ and $\lim_{l\to\infty}\frac{1}{\gamma_{k_l}}\|y^{k_l}-x^{k_l}\|^2=0$. Because $0<\alpha\le\gamma_{k_l}\le\beta$, so we obtain

$$
\lim_{l\to\infty}\|y^{k_l}-x^{k_l}\|^2=0.
$$

By the Case 1, it is immediate that x^* is a solution of EP(Ω,f). □

3 Numerical examples

In this section, we will use proposed algorithms 1, 3, and 4 to find a solution of nonmonotone equilibrium problems and compare them with Algorithm SVN. All the programs are written in Matlab R2014a and performed on a DELL laptop with an Intel(R) Core(TM) i7-10750 H CPU @ 2.60 GHz with 16 GB of RAM.

Example 1. In this first example, we consider the nonmonotone equilibrium problem $EP(\Omega,f)$ recently investigated in papers [6, 8, 26], which is arising in Nash-Cournot oligopolistic electricity market equilibrium model. The bifunction *f* here is nonsmooth and convex, preventing the transformation of the equilibrium problem into a variational inequality problem. Specifically, $f:\mathbb{R}^6\to\mathbb{R}^6$ is defined for each $x,y\in\mathbb{R}^6$ by

$$f(x,y) = [(A+B)x + By + a]^T(y-x) + c(y) - c(x),$$

where A is a nonpositive semidefinite matrix, B is a symmetric positive definite matrix, $a \in \mathbb{R}^6$, and $c(x)$ is a nonsmooth function. The values of A, B, a, and $c(x)$ can be found in [8, 24]. Note that for these values, the equilibrium function is nonmonotone and nonsmooth because the matrix A is not positive semidefinite and function $c(x)$ is nonsmooth. The constraint set of this problem is defined by $\Omega = \{x \in \mathbb{R}^6 : lb \leq x \leq ub\}$, where $lb = (0,0,0,0,0,0)^T$ and $ub = (80,80,50,55,30,40)^T$.

We have used Algorithm 4 and Algorithm SVN for solving this problem with some starting points x^g and linesearch parameters $\{\eta_k\}$ and η in Algorithm 4 and Algorithm SVN, respectively. We choose $\gamma_k = 1, \forall k$ and $\mu = 0.8$ in Algorithm 4, while $\gamma = 1$ in Algorithm SVN. To terminate the algorithms we use the stopping criteria $Err = \frac{\|x^{k+1}-x^k\|}{\max\{1,\|x^k\|\}} \leq 10^{-4}$. The computed results are reported in Table 2.

From Table 2, we can see that the computation time of Algorithm 4 is less than that of Algorithm SVN. In addition, the computation time of Algorithm 4 with the linesearch parameter sequence $\{\eta_k\} = \{\frac{1}{\sqrt{k}}\}$ is better than that with $\{\eta_k\} = \{\eta^k\}$ ($\eta = 0.99, 0.9, 0.7, 0.5, 0.1$).

Example 2. In this example, we will use three proposed algorithms and algorithm SVN to find a solution of the following equilibrium problem EP(Ω, f) where

$$\Omega = \left[-n\pi, \frac{n\pi}{2}\right]^n \text{ and } f(x,y) = \sum_{i=1}^{n}(y_i - x_i)\cos\frac{x_i}{n} + \varphi(y) - \varphi(x),$$

Table 2 Results on Example 1 with some starting points and linesearch parameters

x^g	Alg. 4			Alg. SVN		
	$\{\eta_k\}$	CPU(s)	ITER	η	CPU(s)	ITER
$(0, 0, 0, 0, 0, 0)^T$	0.99^k	20.7266	1484	0.99	326.9531	1287
	0.9^k	16.8438	946	0.9	462.9219	1384
	0.7^k	24.5703	1289	0.7	310.3125	1079
	0.5^k	31.6407	1907	0.5	368.6094	1194
	0.1^k	32.9766	1408	0.1	213.8906	949
	$\frac{1}{k}$	19.9688	1761	0.3	199.5938	811
	$\frac{1}{\sqrt{k}}$	15.5156	1487	0.8	691.5000	1658
$(30, 20, 10, 15, 10, 10)^T$	0.99^k	3.1071	325	0.99	45.9375	332
	0.9^k	2.3171	290	0.9	131.6406	617
	0.7^k	2.5796	531	0.7	53.3281	360
	0.5^k	3.7460	548	0.5	55.3438	353
	0.1^k	5.0625	330	0.1	18.9531	245
	$\frac{1}{k}$	1.7326	364	0.3	56.6563	357
	$\frac{1}{\sqrt{k}}$	1.3750	329	0.8	215.5469	812

and $\varphi(x) = \phi(x_1, x_2, ..., x_n) = \max\{0, -\frac{n\pi}{2} - x_1, -\frac{n\pi}{2} - x_2, ..., -\frac{n\pi}{2} - x_n\}$.

By direct computation, we have that $\varphi(\bar{x}) = \varphi(\bar{y}) = 0,\ f(\bar{x}, \bar{y}) = \frac{n\pi}{24} > 0$ but $f(\bar{y}, \bar{x}) = \frac{(\sqrt{2}-1)n\pi}{12\sqrt{2}} > 0$. Where, $\bar{x} = \left(-\frac{n\pi}{3}, \frac{n\pi}{2}, \frac{n\pi}{2}, ..., \frac{n\pi}{2}\right)^T$ and $\bar{y} = \left(-\frac{n\pi}{4}, \frac{n\pi}{3}, \frac{n\pi}{2}, ..., \frac{n\pi}{2}\right)^T$. This implies that *f* is not quasimonotone on Ω.

We have,

$$
\begin{aligned}
f(x,z) - f(x,y) - f(y,z) &= \sum_{i=1}^{n}(z_i - x_i)\cos\frac{x_i}{n} + \varphi(z) - \varphi(x) \\
&\quad - \sum_{i=1}^{n}(y_i - x_i)\cos\frac{x_i}{n} - \varphi(y) + \varphi(x) \\
&\quad - \sum_{i=1}^{n}(z_i - y_i)\cos\frac{y_i}{n} - \varphi(z) + \varphi(y) \\
&= \sum_{i=1}^{n}(z_i - y_i)\left(\cos\frac{x_i}{n} - \cos\frac{y_i}{n}\right) \\
&\leq \sqrt{\sum_{i=1}^{n}(z_i - y_i)^2}\sqrt{\sum_{i=1}^{n}\left(\cos\frac{x_i}{n} - \cos\frac{y_i}{n}\right)^2} \\
&= \|z - y\|\sqrt{\sum_{i=1}^{n}4\left(\sin\frac{x_i - y_i}{2n}\sin\frac{x_i + y_i}{2n}\right)^2} \\
&\leq \|z - y\|\sqrt{\sum_{i=1}^{n}4\sin^2\frac{x_i - y_i}{2n}} \\
&\leq \frac{1}{n}\|z - y\|\|x - y\| \\
&\leq \frac{1}{2n^2}\|x - y\|^2 + \frac{1}{2}\|z - y\|^2.
\end{aligned}
$$

Hence *f* satisfies the Lipschitz-type with constants $L_1 = \frac{1}{2n^2}$ and $L_2 = \frac{1}{2}$.

Next, we compute the subdifferential $\partial\varphi(x)$. Indeed, we consider the following cases.

Case 1.

Case 1.1. If $-\frac{n\pi}{2} < x_i \leq \frac{n\pi}{2}$ for all $i = 1, ..., n$, then $\varphi(x) = 0$ and $\partial\varphi(x) = \{(0, 0, ..., 0)^T\}$.

Case 1.2. If there is a set $I_a = \{i_1, i_2, ..., i_k\}$, $(k \geq 1)$ such that

$$\varphi(x) = 0 = -\frac{n\pi}{2} - x_{i_1} = -\frac{n\pi}{2} - x_{i_2} = ... = -\frac{n\pi}{2} - x_{i_k},$$

then $\partial\varphi(x) = \{-\sum_{i\in I_a}\alpha_i e_i : \sum_{i\in I_a}\alpha_i \le 1, 0 \le \alpha_i \le 1, \ \forall i \in I_a\}$, in which $e_i = (0, ..., 1, ..., 0)^T$ is the i^{th} unit vector.

Case 2. If there is a set $I_a = \{i_1, i_2, ..., i_k\}$, $(k \ge 1)$ such that

$$\varphi(x) = -\frac{n\pi}{2} - x_{i_1} = -\frac{n\pi}{2} - x_{i_2} = ... = -\frac{n\pi}{2} - x_{i_k},$$

then $\partial\varphi(x) = \{-\sum_{i\in I_a}\alpha_i e_i : \sum_{i\in I_a}\alpha_i = 1, 0 \le \alpha_i \le 1, \ \forall i \in I_a\}$, in which $e_i = (0, 0, ..., 1, ..., 0)^T$ is the i^{th} unit vector. Therefore,

$$\partial_2 f(z^k, x^k) = \left\{ \left(\cos\frac{z_1^k}{n}, ..., \cos\frac{z_n^k}{n} \right)^T + u^k : u^k \in \partial\varphi(x^k) \right\}.$$

Set $a = \left(-\frac{n\pi}{8}, ..., -\frac{n\pi}{8}\right)^T$, $b = (1, 1, ..., 1)^T$. The parameters and starting points are given as follows:

- Algorithm 1: $\theta_k = \xi_k = 1/2, \gamma_{-1} = 0.5, \{\delta_k\} = \{\frac{1}{k+1}\}_{k\ge 1}$, $x^{-1} = x^g = x^0 = y^0 = a; x^{-1} = x^g = x^0 = y^0 = b.$
- Algorithm 3: $\mu = 0.5, \beta = 0.45, \lambda = \sqrt{\frac{1-2\beta L_2}{2\beta L_1}}, x^g = x^0 = a; x^g = x^0 = b.$
- Algorithm 4: $\gamma_k = 1, \forall k, \{\eta_k\} = \{0.99^k\}, \mu = 0.5, x^g = x^0 = a; x^g = x^0 = b.$
- Algorithm SVN: $\gamma = 1, \theta = 0.99, x^g = x^0 = a; x^g = x^0 = b.$

The solution sets of EP(Ω, f) and MEP(Ω, f) are given as follow

$$S_E = \left\{ \left(-\frac{n\pi}{2}, -\frac{n\pi}{2}, ..., -\frac{n\pi}{2}\right)^T ; \left(\frac{n\pi}{2}, \frac{n\pi}{2}, ..., \frac{n\pi}{2}\right)^T \right\} \ and\ S_M = \left\{ \left(-\frac{n\pi}{2}, -\frac{n\pi}{2}, ..., -\frac{n\pi}{2}\right)^T \right\}.$$

Using above algorithms to compute we see that they are convergent to the solution $x^* = \left(-\frac{n\pi}{2}, -\frac{n\pi}{2}, ..., -\frac{n\pi}{2}\right)$. Hence, to terminate algorithms, we use the stopping criteria $Err = \min\left\{ \frac{\|x^{k+1}-x^*\|}{\|x^*\|}, \frac{\|y^k - x^*\|}{\|x^*\|} \right\} \le 10^{-4}$. The computed results are reported in Table 3 and Figs. 1 and 2.

From computational results above, we can see that if the bifunction *f* is not too complicated and the Lipschitz constant is large, then the computation time and the number of iterations of the Algorithm 3 are larger than those of other algorithms. While if the Lipschitz constants of *f* are small then the computation time and the number of iterations of the Algorithm 3 are much smaller than those of the rest algorithms.

Table 3 Results for Example 2 with some starting points

x^g, n	Alg. 1		Alg. 3		Alg. 4		Alg. SVN	
	CPU(s)	ITER	CPU(s)	ITER	CPU(s)	ITER	CPU(s)	ITER
a								
5	7.7500	293	2.7813	189	14.0156	1758	79.1094	1409.0
10	30.1875	751	6.2813	218	65.3594	3658	420.1563	2580
20	126.1406	1618	7.9063	233	79.5156	4087	895.4375	3675
30	340.0000	2493	51.7969	432	675.2031	4276	4012.4219	4175
50	742.6406	4149	81.5469	547	727.2500	4192	8067.2813	4473
b								
5	7.5156	538	4.5781	336	39.2500	2664	110.7344	1590
10	16.5000	949	9.0156	278	38.6094	2691	154.3125	1628
20	208.2188	1999	19.3906	298	95.7813	5200	3810.1563	4300
30	561.8750	3526	42.1406	412	1086.6094	5378	7315.5625	5031
50	1553.4531	6159	72.4844	499	1259.5000	5465	8184.0000	5490

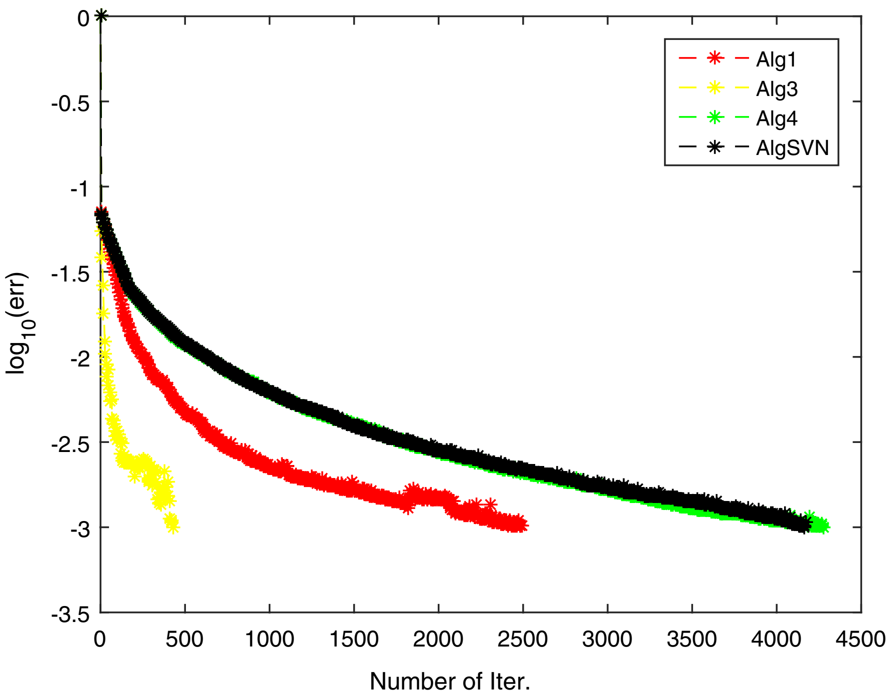

Fig. 1 Number of iterations in the case of $n = 30$ and $x^g = a$

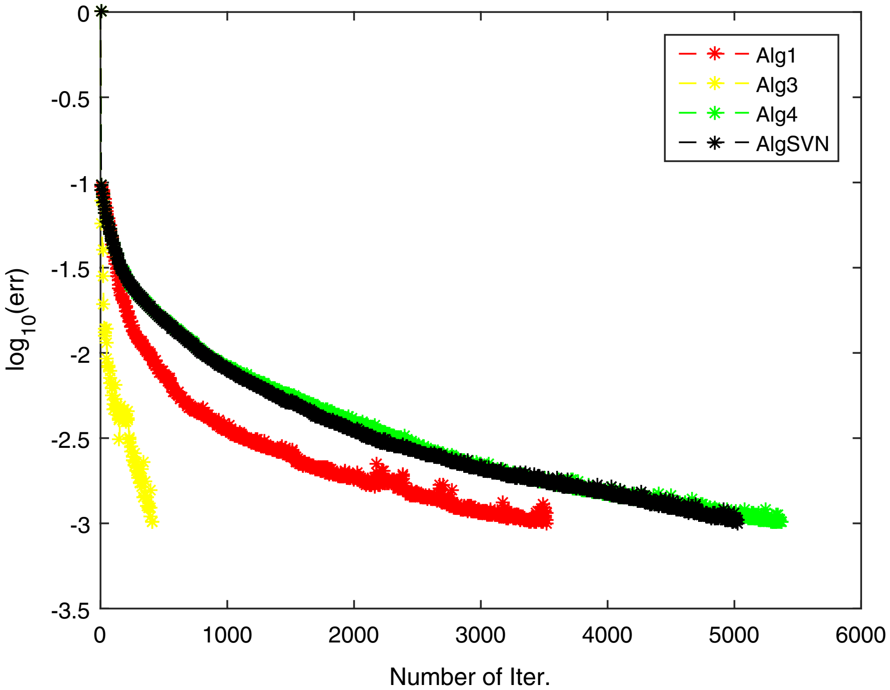

Fig. 2 Number of iterations in the case of $n = 30$ and $x^g = b$

4 Conclusion

We have proposed three variants of projection extragradient algorithms for solving nonmonotone equilibrium problems in Hilbert spaces, applicable both when the bifunction is of Lipschitz-type and when no such condition is assumed. Unlike many existing methods, our approach avoids the use of shrinking projection techniques. Strong convergence of all proposed algorithms has been established under mild assumptions, and numerical examples have been presented to demonstrate their efficiency and practical performance.

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Data Availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study. The readers who are interested in MATLAB codes for examples should contact Bui Van Dinh.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Ethical Approval Not applicable.

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