A predictor-corrector interior-point algorithm using a special power function for $P_*(\kappa)-$weighted horizontal linear complementarity problem

Xiaoni Chi[†], Lin Gan, Qian Cheng and Jein-Shan Chen[†]

Abstract: In this paper, we present a predictor-corrector interior-point algorithm (PC IPA) with new search directions to solve $P_*(\kappa)-$weighted horizontal linear complementary problem (WHLCP). $P_*(\kappa)-$WHLCP includes monotone WLCP, $P_*(\kappa)-$horizontal linear complementary problem (HLCP), $P_*(\kappa)-$linear complementary problem (LCP), monotone LCP and convex quadratic programming as special cases, and could model a wide range of equilibrium problems in scientific engineering and economic management. The main idea of our PC IPA is transforming the centering equations of the central path by the algebraic equivalent transformation (AET) technique based on a power function with an arbitrary positive integer q. Upon analyzing the effect of different q on the transformed system, we select a power function $\varphi(t) = t^{\frac{5}{2}}$ in order to get the search direction. The feasibility and global convergence of the proposed algorithm are verified under appropriate conditions. Additionally, the iteration bound of our algorithm is comparable to the best-known bounds for such available IPAs. The efficacy of the proposed algorithm is demonstrated through the presentation of numerical results.

Keywords: predictor-corrector interior-point algorithm, $P_*(\kappa)-$weighted horizontal linear complementarity problem, power function, complexity analysis

Mathematics Subject Classification: 90C33, 90C51

---

1 Introduction

The weighted complementarity problem (WCP) is pertinent to numerous practical problem within the domains of scientific engineering and economic management [5, 6, 34], such as Fisher's competitive market equilirium, the perfect price discrimination market model with production, the quadratic programming and weighted centering problem, and so on. Potra [6] first proposed the notion of WCP. He [6] demonstrated that the skew-symmetric WCP not only simplified the concept of the linear programming and weighted centering (LPWC) problem [16], but also served as a representation for the market equilibrium problem [34]. Moreover, the market equilibrium problem can be viewed as a nonlinear complementarity problem (CP) [12, 13, 33], or it can be transformed into a weighted linear complementarity problem (WLCP), with the latter being more efficient.

The weighted horizontal linear complementarity problem (WHLCP), which intends to seek out a pair $(x, s) \in \mathbb{R}^{2n}$ that satisfies

$$Qx + Rs = b, \quad xs = \omega, \quad (x, s) \geq 0, \tag{1.1}$$

where $b \in \mathbb{R}^n, Q, R \in \mathbb{R}^{n\times n}$ and $\omega \in \mathbb{R}^n_+$. The mixed weighted linear complementarity problem (MWLCP) [7] involves finding $(x, s, y) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ such that

$$Ax + Bs + Cy = d, \quad xs = \omega, \quad (x, s) \geq 0, \tag{1.2}$$

where $d \in \mathbb{R}^{n+m}$, $A, B \in \mathbb{R}^{(n+m)\times n}$, $\omega \in \mathbb{R}^n_+$, and $C \in \mathbb{R}^{(n+m)\times m}$. Note that if C becomes empty and A, B are square matrices, the MWLCP (1.2) can be transformed into WHLCP (1.1), see [7, 31]. In addition, the WHLCP (1.1) degenerates into horizontal linear complementarity problem (HLCP) when $\omega = 0$. In the case of $R = -I$ and $\omega = 0$, the WHLCP (1.1) reduces to linear complementarity problem (LCP). In this paper, we target on $P_*(\kappa)-$WHLCP, where (Q, R) in (1.1) is a $P_*(\kappa)-$pair. Here, the $P_*(\kappa)-$matrix was initially defined by Kojima et al. [20] and detailed mathematical format of $P_*(\kappa)-$WHLCP will be presented in Section 2.

Numerical algorithms for WLCP include smoothing Newton algorithm [15, 22] and interior-point algorithm (IPA)[6, 21, 32]. Among various IPAs, the predictor-corrector (PC) IPA stands out for its practical efficiency and good polynomial iteration complexity. Each iteration process of the PC IPA includes two stages: predictor step and corrector step. The step is used to reach the $\varepsilon-$approximate solution. Then, one or more corrector steps are performed to ensure that the new iterate is located in the appropriate neighborhood of the central path. Sonnevend et al. [8] presented the first PC IPA for linear optimization (LO). Mizuno, Todd and Ye [27] provided a Mizuno-Todd-Ye PC IPA for LO that utilized a single corrector step in each iteration. Thereafter, the PC IPAs have been successfully applied to a variety of optimization problems including LCP [2], $P_*(\kappa)-$LCP [14], sufficient LCP [28] and WLCP [6, 7], etc., which have the polynomial iteration complexity. For example, Miao [14] gave a PC IPA for $P_*(\kappa)-$LCP, and achieved quadratic convergence with an iteration complexity $O((1+\kappa)\sqrt{n}L)$, where L denotes the input size of the problem. Potra [6] proposed two IPAs for WLCP: an extension of so-called largest step IPA by McShane [18], and a variant of the Mizuno-Todd-Ye PC method stemmed from [27]. The Mizuno-Todd-Ye PC IPA for WLCP in [6] has $O\left(\frac{(x^0)^Ts^0/n+\|x^0s^0-\omega\|}{\min(x^0s^0)}\log\frac{(x^0)^Ts^0/n+\|x^0s^0-\omega\|}{\varepsilon}\right)$ polynomial iteration complexity. Potra [7] proposed a corrector-predictor IPA that is not hindered by the handicap of problems, and features a computational complexity proportional to $1+\kappa$.

Determining the search direction is crucial for IPAs. The algebraic equivalence transformation (AET) technique [35], provided an effective approach for deriving the search direction in IPAs. The main idea of AET technique is to transform the centering equation $xs = \mu e$ in the central path system by a continuously differentiable function $\varphi(t)$. Then Newton's method is applied to the modified system to yield new search directions. Darvay [35] used a square root function $\varphi(t) = \sqrt{t}$ to obtain the search direction of the IPA for LO. Subsequently, Kheirfam [2] extended Darvay's technique [35] for LO to $P_*(\kappa)-$HLCP. The identity function is the most widely used function, i.e., $\varphi(t) = t$, which can lead to the classical search direction [4, 29]. Moussaoui and Achache [23] developed a weighted full-Newton step path-following IPA based on $\varphi(t) = t^{\frac{3}{2}}$ for convex quadratic optimization (CQO).

Recently, Kheirfam and Nasrollahi [3] proposed a full-Newton IPA for LO that used the AET technique based on integer powers of the square root function $\varphi_q(t) = t^{\frac{q}{2}}, \forall t > 0$ with $q \in \mathbb{N} = \{1, 2, 3, \dots\}$, and derived a polynomial complexity $O\left(\sqrt{n}\log\frac{\mu^0\left(n+\frac{4q(q-2)}{((q-1)^2+2)^2}\right)}{\varepsilon}\right)$. Several existing AET functions used in IPAs can be considered as the particular case of the power function $\varphi(t) = t^{\frac{q}{2}}$. For example, Kheirfam [2] designed a PC path-following IPA using $\varphi(t) = \sqrt{t}$ for $P_*(\kappa)-$HLCP. Darvay [36] presented a PC IPA for $P_*(\kappa)-$LCP, where the search direction is derived by using $\varphi(t) = t^2$. For monotone WLCP, Asadi et al. [26] developed a full-Newton step IPA using $\varphi(t) = t$, while Kheirfam [1] gave a full-Newton step IPA based on $\varphi(t) = \sqrt{t}$.

Table 1: IPAs based on different power functions

Algorithm	$\varphi(t)$	Stopping criterion	Polynomial complexity
IPA for LCP [30]	$t^{\frac{5}{2}}$	$x^Ts \le \varepsilon$	$O\left(\sqrt{n}\log\frac{n}{\varepsilon}\right)$
IPA for monotone WLCP[1]	$\sqrt{t}$	$\\|\sqrt{\omega}-\sqrt{xs}\\| \le \varepsilon$	$O\left(\frac{6(\min(x^0s^0)+\rho+1)}{\min(x^0s^0)}\log\frac{\sqrt{\frac{\min(x^0s^0)}{2}+\beta}}{\varepsilon}\right)$
IPA for monotone WLCP[26]	t	$\\|xs-\omega\\| \le \varepsilon$	$O\left(\frac{5(\rho+\min(x^0s^0))}{\min(x^0s^0)}\log\frac{\frac{\min(x^0s^0)}{2}+\rho}{\varepsilon}\right)$
PC IPA for $P_*(\kappa)-$LCP[36]	t^2	$x^Ts \le \varepsilon$	$O\left((1+4\kappa)\sqrt{n}\log\frac{3(x^0)^Ts^0}{4\varepsilon}\right)$
PC IPA for $P_*(\kappa)-$HLCP[2]	$\sqrt{t}$	$x^Ts \le \varepsilon$	$O\left((1+2\kappa)\sqrt{n}\log\frac{(x^0)^Ts^0}{\varepsilon}\right)$
PC IPA for $P_*(\kappa)-$WHLCP	$t^{\frac{5}{2}}$	$\\|xs-\omega\\| \le \varepsilon$	$O\left((1+4\widetilde{\kappa})\sqrt{n}\log\frac{\frac{2}{5(1+8\widetilde{\kappa})}\max(x^0s^0)+\rho}{\varepsilon}\right)$

Compared to LCP and HLCP, the analysis of WLCP and WHLCP is more complex due to the existence of weight vectors. Table 1 compares our algorithm for $P_*(\kappa)-$WHLCP with existing IPAs for WLCP [1, 26], LCP [30, 36] and HLCP [2] based on different power functions, where $\rho = \|x^0s^0-\omega\|$ and $\beta = \max\left\{\|\sqrt{x^0s^0}-\omega\|, \left\|\frac{x^0s^0-\omega}{2\sqrt{\omega}-e}\right\|\right\}$ with $2\sqrt{\omega}-e > 0$, and a special $\widetilde{\kappa}$ given in (3.1). Our algorithm aims to solve $P_*(\kappa)-$WHLCP, which includes monotone WLCP [1, 26], $P_*(\kappa)-$HLCP [2], $P_*(\kappa)-$LCP [36] and monotone LCP [30] as special cases. It bears noting that the WHLCP in our paper differs from the problem in [32], which focuses on the the special case of $P_*(\kappa)-$WHLCP when $R = -I$. Moreover, although our algorithm uses the same power function $\varphi(t) = t^{\frac{5}{2}}$ as the path-following IPA for solving monotone LCP [30], whereas we propose a predictor-corrector IPA (PC IPA) specifically designed for $P_*(\kappa)-$WHLCP, which reduces to montone LCP if $R = -I, \omega = 0$ and $\kappa = 0$.

In this paper, we present a PC IPA for $P_*(\kappa)-$WHLCP based on a special AET function $\varphi(t) = t^{\frac{5}{2}}$.

i) It is a PC IPA for $P_*(\kappa)-$WHLCP using the AET function $\varphi(t) = t^{\frac{5}{2}}$, which is a special case of a class of AET functions $\varphi(t) = t^{\frac{q}{2}},\ q \in \mathbb{N}$. To our best knowledge, this is the first PC IPA for solving $P_*(\kappa)-$WHLCP, which is an extension of monotone WLCP, $P_*(\kappa)-$HLCP, $P_*(\kappa)-$LCP and monotone LCP.

ii) In each main iteration, our algorithm first executes a full-Newton corrector step, which does not require carrying out any line search. The damped predictor step is then performed, which is intended to closely approximate the solution of $P_*(\kappa)-$WHLCP.

iii) We provide the strict feasibility of the corrector step via function $\varphi(t) = t^{\frac{q}{2}},\ q \in \mathbb{N}$. According to the influence of different values of q on the search direction, we choose the AET function $\varphi(t) = t^{\frac{5}{2}}$ to obtain the search direction of the corrector step.

iv) Complexity analysis of the proposed algorithm becomes more challenging for $P_*(\kappa)-$WHLCP due to the presence of the weight vector. In spite of this fact, with appropriate choices of default parameter, the global convergence and best iteration complexity of our algorithm are proved and derived, respectively.

v) Finally, numerical experiments are conducted on some instances to demonstrate the effectiveness of our algorithm.

The outline of the paper is given below. We briefly review the concept of central path of $P_*(\kappa)-$WHLCP in Section 2. Moreover, we discuss the search direction obtained by a class of power functions, and present a PC IPA for $P_*(\kappa)-$WHLCP utilizing a specific power function. The polynomial iteration complexity of the presented algorithm is established in Section 3. Numerical results of our algorithm are given in Section 4.

Conventions Let $\mathbb{N}$ be the set of positive integers, defined as $\mathbb{N} = \{1, 2, 3, \dots\}$. The set of real numbers is indicated by $\mathbb{R}$, nonnegative real numbers by $\mathbb{R}_+$, and positive real numbers by $\mathbb{R}_\oplus$. Define $\mathbb{R}^n$ as the n-dimensional vector space with the 1-norm $\|\cdot\|_1$, 2-norm $\|\cdot\|$ and the infinite norm $\|\cdot\|_\infty$. For $u, r \in \mathbb{R}^n$, ur denotes the componentwise product of the vectors u and r. Similarly, for $u, r \in \mathbb{R}^n$, $\sqrt{u} = (\sqrt{u_1}, \dots, \sqrt{u_n})^T$ and $\frac{u}{r} = \left(\frac{u_1}{r_1}, \dots, \frac{u_n}{r_n}\right)^T$, where $r_i \neq 0$ for $i = 1, 2, \dots, n$. Let $\min u = \min\{u_i : i = 1, \dots, n\}$ and $\max u = \max\{u_i : i = 1, \dots, n\}$, where u_i is the ith component of the vector u. The diagonal matrix composed of the elements of vector u is called $U = \operatorname{diag}(u)$. The rank of the matrix U is represented by $rank(U)$. Define the symbol $h(u) = (h(u_1), \dots, h(u_n))^T \in \mathbb{R}^n$, where $u \in \mathbb{R}^n$ and $h : \mathbb{R} \to \mathbb{R}$ is a univariate function. In addition, let $e = (1, \dots, 1)^T$ where the dimension is determined by the context; $I = \operatorname{diag}(e)$ is an identity matrix.

2 A PC IPA using a special power function for $P_*(\kappa)-$WHLCP

In this section, we review the concept of the central path for $P_*(\kappa)-$WHLCP and provide a new search direction based on the AET technique with a class of power functions. Then a PC IPA via a special power function is presented for solving $P_*(\kappa)-$WHLCP.

2.1 Central path of $P_*(\kappa)-$WHLCP

WHLCP (1.1) with $P_*(\kappa)-$pair is called $P_*(\kappa)-$WHLCP. As described in [20], (Q, R) is a $P_*(\kappa)-$pair if there is a constant $\kappa \geq 0$ satisfying

$$Qu + Rr = 0 \Rightarrow (1 + 4\kappa) \sum_{i \in \mathcal{J}_+} u_i r_i + \sum_{i \in \mathcal{J}_-} u_i r_i \geq 0,\ \forall u, r \in \mathbb{R}^n,$$

where $\mathcal{J} = \{1, 2, \cdots, n\},\ \mathcal{J}_- = \{i \in \mathcal{J} : u_i r_i < 0\},\ \mathcal{J}_+ = \{i \in \mathcal{J} : u_i r_i > 0\}$. The handicap of the pair (Q, R) is the smallest of these κ [9], i.e.,

$$\vartheta(Q, R) := \min\{\kappa : \kappa \geq 0,\ (Q, R) \text{ is a } P_*(\kappa) - \text{pair}\}.$$

The union of all $P_*(\kappa)-$matrices with nonnegative κ is called as the class of P_*-matrices, i.e., $P_* := \bigcup_{\kappa \geq 0} P_*(\kappa)$. Väliaho [9] proved that P_*-matrices are just sufficient. The class of P_*-matrices exhibits these properties:

i) (Submatrix)[25] If $M \in P_*$, then its every principal submatrix is also sufficient, namely $M_{RR} \in P_*$ for all $R \subseteq I$.

ii) (Principal rearrangement)[24] If $M \in P_*$, then for any permutation matrix P (in the same size as M), the rearranged matrix $P^T MP \in P_*$.

iii) (PPT)[24] If $M \in P_*$, then any principal pivotal transformation of M is also sufficient, namely $\mathcal{P}_J(M) \in P_*$ for all $J \subseteq I$.

iv) (Rank one matrix)[10] A rank one matrix A is sufficient if and only if it has a nonnegative diagonal and if $A_{ii} = 0$ then the i-th row and column of the matrix A are all zero. In other words, $A = uv^T \in P_*$ ($u, v \in \mathbb{R}^n$) if and only if $u_i v_i > 0$ or $u_i = v_i = 0$ for all i.

v) (Scaling)[20] If $M \in P_*$, then $PMQ \in P_*$ for any diagonal matrices P and Q, where $P_{ii}Q_{ii} > 0$ for all indices i.

vi) (Block diagonal)[11] If $M_1, M_2 \in P_*$ (possibly with different sizes), then

$$\begin{pmatrix} M_1 & O \\ O & M_2 \end{pmatrix} \in P_*.$$

vii) (Blowing)[11] If $M \in P_*$, then

$$\begin{pmatrix} -M & I \\ I & D \end{pmatrix} \in P_*$$

for any nonnegative diagonal matrix D.

viii) (Shifting)[11] If $M \in P_*$, then $M + D \in P_*$ for any nonnegative diagonal matrix D.

When M is a $P_*(\kappa)-$matrix, the corresponding WLCP is solvable under the assumption of strict feasibility [7]. For $\kappa = 0$, $P_*(0)-$WHLCP is reduced to the monotone WHLCP. The $P_*(\kappa)-$WHLCP is a comprehensive problem, which contains monotone WLCP, $P_*(\kappa)-$HLCP, $P_*(\kappa)-$LCP, monotone LCP, and convex quadratic programming as special cases.

The notations $\mathcal{F}$ and $\mathcal{F}^+$ are used to denote the feasible region and the strictly feasible region of the $P_*(\kappa)-$WHLCP

$$\mathcal{F} := \{(x, s) \in \mathbb{R}^{2n}_+ : Qx + Rs = b\},$$

$$\mathcal{F}^+ := \{(x, s) \in \mathcal{F} : x > 0, s > 0\},$$

respectively. The solution set of the $P_*(\kappa)-$WHLCP is

$$\mathcal{F}^* := \{(x, s) \in \mathcal{F} : xs = \omega\}.$$

In this paper, we impose two assumptions:

- The strictly feasible region $\mathcal{F}^+$ is non-empty.
- The pair (Q, R) in the system (1.1) is a $P_*(\kappa)-$pair with $\text{rank}(Q, R) = n$.

These assumptions are standard in numerous theoretical studies on IPAs. Then, because $P_*(\kappa)$-WHLCP (1.1) satisfies the strict feasibility and sufficiency, its solvability can be readily established on the basis of Theorem 4 from [7]. Additionally, a full-rank $P_*(\kappa)$-pair (Q, R) ensures the uniqueness of search directions for IPAs [20].

Given a strictly feasible initial point, i.e., $(x^0, s^0) \in \mathcal{F}^+$, we set

$$\omega(t) = tx^0 s^0 + (1 - t)\omega, \; t \in (0, 1]. \tag{2.1}$$

By substituting the parameterized equation $xs = \omega(t)$ with $t \in (0, 1]$ into the second equation in (1.1), we obtain

$$Qx + Rs = b, \; xs = \omega(t), \; (x, s) \geq 0. \tag{2.2}$$

If (Q, R) is a $P_*(\kappa)-$pair and $\mathcal{F}^+ \neq \emptyset$, the parameterized system (2.2) has a unique solution for any $t \in (0, 1]$, see [7]. The central path of $P_*(\kappa)-$WHLCP is

$$\mathcal{C} = \{(x(t), s(t)) : t \in (0, 1]\},$$

where $(x(t), s(t))$ is the solution of the system (2.2), also called the $t-$center of $P_*(\kappa)-$WHLCP. As the parameter $t \to 0$, there exists the limit of the central path, which belongs to the solution set of $P_*(\kappa)-$WHLCP (1.1) [6].

2.2 Search direction obtained by AET

Let function $\varphi : (\xi, \infty) \to \mathbb{R}$ be continuously differentiable and invertible, where $\xi \in [0, 1)$. We perform an equivalent algebraic transformation by substituting $\varphi\left(\frac{xs}{\omega(t)}\right) = \varphi(e)$ with the parameterized equation $xs = \omega(t)$ in (2.2). Then, the system (2.2) is rewritten as

$$Qx + Rs = b, \quad \varphi\left(\frac{xs}{\omega(t)}\right) = \varphi(e), \quad (x, s) \geq 0. \tag{2.3}$$

By applying Newton's method to system (2.3), we get

$$\begin{aligned} Q\Delta x + R\Delta s &= 0, \\ s\Delta x + x\Delta s &= a_\varphi, \end{aligned} \tag{2.4}$$

where

$$a_\varphi = \omega(t) \frac{\varphi(e) - \varphi\left(\frac{xs}{\omega(t)}\right)}{\varphi'\left(\frac{xs}{\omega(t)}\right)}.$$

Since (Q, R) is a $P_*(\kappa)-$pair with $\kappa = \vartheta(Q, R)$ and $Q\Delta x + R\Delta s = 0$ from the Newton system (2.4), one has

$$(1 + 4\kappa) \sum_{i \in \mathcal{I}_+} \Delta x_i \Delta s_i + \sum_{i \in \mathcal{I}_-} \Delta x_i \Delta s_i \geq 0, \tag{2.5}$$

where

$$\mathcal{I} = \{1, 2, \cdots, n\}, \quad \mathcal{I}_- = \{i \in \mathcal{I} : \Delta x_i \Delta s_i < 0\}, \quad \mathcal{I}_+ = \{i \in \mathcal{I} : \Delta x_i \Delta s_i > 0\}.$$

To facilitate analysis, we define the vector v as

$$v := \sqrt{\frac{xs}{\omega(t)}}, \tag{2.6}$$

and the search directions in the v space as

$$d_x := \frac{v\Delta x}{x}, \quad d_s := \frac{v\Delta s}{s}. \tag{2.7}$$

Combining (2.6) and (2.7), one has

$$\Delta x \Delta s = d_x d_s \omega(t). \tag{2.8}$$

Then, the system (2.4) is transformed into

$$\begin{aligned} \widetilde{Q} d_x + \widetilde{R} d_s &= 0, \\ d_x + d_s &= p_v, \end{aligned} \tag{2.9}$$

where

$$\widetilde{Q} = QXV^{-1}, \ \widetilde{R} = RSV^{-1}, \ X = \text{diag}(x), \ S = \text{diag}(s), \ V = \text{diag}(v),$$

and

$$p_v = \frac{\varphi(e) - \varphi(v^2)}{v\varphi'(v^2)}.$$

Using the power function $\varphi_q(t) = t^{\frac{q}{2}}$ with $q \in \mathbb{N}$ [3], we derive specific p_v given by

$$p_v = \frac{2(v^{1-q} - v)}{q}. \tag{2.10}$$

The function $\varphi_q(t)$ and the corresponding a_φ, p_v with $q = 1, 2, 3, 4, 5$ are summarized in Table 2.

Define the norm-based proximity measure $\delta : \mathbb{R}_\oplus \to \mathbb{R}_+$ by

$$\delta(v) := \delta(x, s; t) := \frac{q}{2}\|p_v\| = \|v^{1-q} - v\|. \tag{2.11}$$

We designate $\delta(v)$ as the distance between the central path and the current iteration point (x, s). Via (2.6) and (2.11) there yields

$$\delta(v) = 0 \quad \Longleftrightarrow \quad v^{1-q} - v = 0 \quad \Longleftrightarrow \quad v = e \quad \Longleftrightarrow \quad xs = \omega(t).$$

Substituting (2.10) into system (2.9) yields

$$\begin{aligned} \widetilde{Q} d_x + \widetilde{R} d_s &= 0, \\ d_x + d_s &= \frac{2(v^{1-q} - v)}{q}. \end{aligned} \tag{2.12}$$

It is obvious that the value of q affects the solution of the system (2.12), which plays a critical role in determining the search direction.

Table 2: Power function $\varphi_q(t)$ and the corresponding a_φ, p_v

q	$\varphi_q(t)$	a_φ	p_v
1	t	$2\left(\sqrt{\omega(t)xs} - xs\right)$	$2(e-v)$
2	$\sqrt{t}$	$\omega(t) - xs$	$v^{-1} - v$
3	$t^{\frac{3}{2}}$	$\frac{2}{3}\left(\omega(t)\left(\frac{xs}{\omega(t)}\right)^{-\frac{1}{2}} - xs\right)$	$\frac{2}{3}(v^{-2} - v)$
4	t^2	$\frac{1}{2}\left(\frac{\omega^2(t)}{xs} - xs\right)$	$\frac{1}{2}(v^{-3} - v)$
5	$t^{\frac{5}{2}}$	$\frac{2}{5}\left(\omega(t)\left(\frac{xs}{\omega(t)}\right)^{-\frac{3}{2}} - xs\right)$	$\frac{2}{5}(v^{-4} - v)$

- $q = 1$, $\varphi_1(t) = \sqrt{t}$ yields $p_v = 2(e - v)$ (Darvay [35]);
- $q = 3$, $\varphi_3(t) = t^{\frac{3}{2}}$ yields $p_v = \frac{2}{3}(v^{-2} - v)$ (Moussaoui and Achache [23]);
- $q = 5$, $\varphi_5(t) = t^{\frac{5}{2}}$ yields $p_v = \frac{2}{5}(v^{-4} - v)$ (Grimes and Achache [30]).

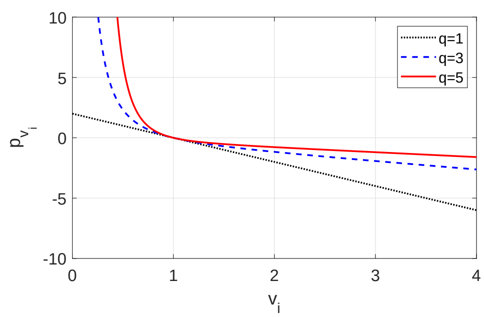

Figure 1: The values of p_{v_i} for $q = 1, 3, 5$

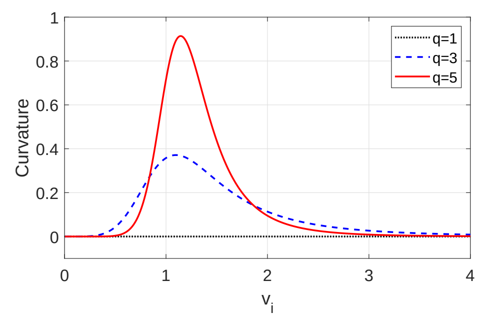

Figure 2: The curvatures of p_{v_i} for $q = 1, 3, 5$

Table 3: Comparison of different p_v for $q = 1, 3, 5$

q	p_v	Curvature	Functional form
1	$2(e - v)$	0	Linear
3	$\frac{2}{3}(v^{-2} - v)$	$\frac{\lvert 4v_i^{-4}\rvert}{\left[1+\left(\frac{-4}{3}v_i^{-3} - \frac{2}{3}\right)^2\right]^{\frac{3}{2}}}$	Non-linear
5	$\frac{2}{5}(v^{-4} - v)$	$\frac{\lvert 8v_i^{-6}\rvert}{\left[1+\left(\frac{-8}{5}v_i^{-5} - \frac{2}{5}\right)^2\right]^{\frac{3}{2}}}$	Non-linear

Let p_{v_i} be the component of $p_v = 2(v^{1-q} - v)/q$ for $q \in \mathbb{N}$, where $i = 1, 2, \ldots, n$. Comparisons of p_{v_i} for $q = 1, 3, 5$ are plotted in Figure 1. The curvature of p_{v_i} with respect to v_i for different q are depicted in Figure 2 and the curvature formulas are listed in Table 3.

Why do we choose $\varphi(t) = t^{\frac{5}{2}}$ in the corrector step? As shown in Figure 1, the value of p_{v_i} with $q = 5$ approaches to zero more quickly near $v_i = 1$ compared to $q = 1$ and $q = 3$. Furthermore, as indicated in Figure 2, the curvature peak for $q = 5$ near $v_i = 1$ is significantly higher than the cases for $q = 1$ and $q = 3$. These phenomena means that small changes in v_i can lead to more significant adjustments in the value of p_{v_i}. Overall, in each iteration, system (2.12) with $q = 5$ makes larger adjustments, which increases the probability of finding a more proper scaled search direction (d_x, d_s) and then reduces the computation time.

2.3 A PC IPA for $P_*(\kappa)-$WHLCP

In this subsection, we look into the search directions for the predictor and corrector steps, and give a new PC IPA for $P_*(\kappa)-$WHLCP. To proceed, we substitute the AET function $\varphi : \mathbb{R}_\oplus \to \mathbb{R},\ \varphi(t) = t^{\frac{5}{2}}$ in system (2.9) and thus obtain

$$p_v = \frac{2}{5}(v^{-4} - v). \tag{2.13}$$

Let us consider the proximity measure $\delta : \mathbb{R}_\oplus \to \mathbb{R}_+$

$$\delta(v) := \delta(x, s; t) := \frac{5}{2}\|d_x + d_s\| = \frac{5}{2}\|p_v\| = \|v^{-4} - v\|. \tag{2.14}$$

From (2.6) and (2.14), we have

$$\delta(v) = 0 \quad \Longleftrightarrow \quad v^{-4} - v = 0 \quad \Longleftrightarrow \quad v = e \quad \Longleftrightarrow \quad xs = \omega(t).$$

Therefore, as the value of t tends to 0, we obtain a solution of $P_*(\kappa)-$WHLCP (1.1). For δ defined by (2.14), the $\tau t-$neighbourhood of the central path is defined as

$$\mathcal{N}(\tau, t) := \{(x, s) \in \mathcal{F}^+ : \delta(x, s; t) \leq \tau t\},$$

where τ is a threshold parameter and $t \in (0, 1]$.

Our algorithm begins with a strictly feasible point $(x^0, s^0) \in \mathcal{N}(\tau, t_0)$ with $t_0 = 1$. We first execute a corrector step, followed by a predictor step. In the corrector step, we obtain (d_x, d_s) by solving system

$$\begin{aligned} \widetilde{Q}d_x + \widetilde{R}d_s &= 0, \\ d_x + d_s &= \frac{2}{5}(v^{-4} - v). \end{aligned} \tag{2.15}$$

Computing system (2.15) and applying (2.7) give the search direction $(\Delta x, \Delta s)$. Then, taking a full-Newton step, there yields an iteration point

$$(x^c, s^c) = (x + \Delta x, s + \Delta s). \tag{2.16}$$

Next, we define the vector v^c after a corrector step as

$$v^c := \sqrt{\frac{x^c s^c}{\omega(t)}}. \tag{2.17}$$

Consequently, the scaled search directions of the predictor step are defined as follows

$$d_x^p := \frac{v^c \Delta^p x}{x^c}, \quad d_s^p := \frac{v^c \Delta^p s}{s^c}. \tag{2.18}$$

In the predictor step, we calculate the predictor system

$$\begin{aligned} \widetilde{Q}_+ d_x^p + \widetilde{R}_+ d_s^p &= 0, \\ d_x^p + d_s^p &= -v^c, \end{aligned} \tag{2.19}$$

where $\widetilde{Q}_+ = QX_+V_+^{-1},\ \widetilde{R}_+ = RS_+V_+^{-1},\ X_+ = \operatorname{diag}(x^c),\ S_+ = \operatorname{diag}(s^c),\ V_+ = \operatorname{diag}(v^c)$, and obtain the scaled predictor search direction

$$(d_x^p, d_s^p) = \left(\left(\widetilde{Q}_+ - \widetilde{R}_+\right)^{-1} \widetilde{R}_+ v^c,\ \left(\widetilde{R}_+ - \widetilde{Q}_+\right)^{-1} \widetilde{Q}_+ v^c\right).$$

Thus, we get $\Delta^p x$ and $\Delta^p s$ by using (2.18). Then, we update the iteration point (x^p, s^p) after a predictor step as below

$$(x^p, s^p) = (x^c + \theta t \Delta^p x, s^c + \theta t \Delta^p s), \tag{2.20}$$

where $\theta \in (0, 1)$ is the barrier update parameter.

Now, we briefly describe the PC IPA for $P_*(\kappa)-$WHLCP (1.1). The PC IPA starts by selecting a strictly feasible initial point (x^0, s^0) that satisfies $x^0 s^0 \geq \omega$ and $(x^0, s^0) \in \mathcal{N}(\tau, t_0)$ with $t_0 = 1$. From system (2.15) and (2.7), the search direction $(\Delta x, \Delta s)$ is obtained, and then the iteration point after the corrector step $(x^c, s^c) = (x + \Delta x, s + \Delta s)$ is updated. Subsequently, the search direction $(\Delta^p x, \Delta^p s)$ is determined by system (2.19) and (2.18), and the iteration point after the predictor step $(x^p, s^p) = (x^c + \theta t \Delta^p x, s^c + \theta t \Delta^p s)$ is updated, where $\theta \in (0, 1)$ and $t \in (0, 1]$. The algorithm continues to run until $\|x^k s^k - \omega\|$ is no longer greater than ε, with ε being the predetermined accuracy threshold. The pseudocode of our PC IPA for $P_*(\kappa)-$WHLCP is shown below.

Algorithm 1 PC IPA for $P_*(\kappa)-$WHLCP

$k = 0.$
$x = x^0, s = s^0, t = t_0.$
while $\|xs - \omega\| > \varepsilon$ **do**
  $(\Delta x, \Delta s)$ is obtained by solving (2.15) and using (2.7);
  $(x^c, s^c) = (x, s) + (\Delta x, \Delta s);$
  $(\Delta^p x, \Delta^p s)$ is obtained by solving (2.19) and using (2.18);
  $(x^p, s^p) = (x^c, s^c) + \theta t(\Delta^p x, \Delta^p s);$
  $\omega(t_p) = (1 - \theta t)\omega(t), t_p = (1 - \theta)t;$
  $x = x^p, s = s^p;$
  $\omega(t) = \omega(t_p), t = t_p;$
  $k = k + 1.$
end while

3 Analysis of the algorithm

In this section, we will analyze the corrector and predictor steps of the proposed PC IPA for $P_*(\kappa)-$WHLCP. First, we give the following lemmas in the setting of $\delta := \delta(v) = \|v^{1-q} - v\|$ with $q \in \mathbb{N}$.

Lemma 3.1. *Let $x^0 s^0 \geq \omega$, $v > 0$ and $\delta := \delta(v)$ be given by (2.11) with $q \in \mathbb{N}$. Then, the system (2.12) has a unique solution for which*

$$\|d_x d_s\|_\infty \leq \frac{1 + 4\widetilde{\kappa}}{q^2}\delta^2, \quad \|d_x d_s\| \leq \frac{3 + 8\widetilde{\kappa}}{2q^2}\delta^2,$$

where

$$\widetilde{\kappa} := \left(\frac{1}{4} + \kappa\right)\frac{\max(x^0 s^0)}{\min \omega} - \frac{1}{4}. \tag{3.1}$$

Proof. For the sake of convenience, we denote

$$\sigma_+ := \sum_{i \in \mathcal{I}_+} d_{x_i} d_{s_i}, \quad \sigma_- := \sum_{i \in \mathcal{I}_-} d_{x_i} d_{s_i}.$$

To determine the upper bound for $\|d_x d_s\|_\infty$ and $\|d_x d_s\|$, we estimate the upper bound for σ_+ and the lower bound for σ_-. In light of the second equation in system (2.9) and (2.11), one has

$$\sigma_+ \leq \frac{\sum_{i \in \mathcal{I}_+}(d_{x_i} + d_{s_i})^2}{4} \leq \frac{\sum_{i \in \mathcal{I}}(d_{x_i} + d_{s_i})^2}{4} = \frac{\|p_v\|^2}{4} = \frac{\delta^2}{q^2}. \tag{3.2}$$

Next, it follows from (2.1), (2.5), (2.8), (2.11) and the second equation in system (2.12) that

$$\begin{aligned}
\sigma_- &= \sum_{i \in \mathcal{I}_-} \frac{\Delta x_i \Delta s_i}{\omega_i(t)} \geq \frac{\sum_{i \in \mathcal{I}_-} \Delta x_i \Delta s_i}{\min \omega} \geq -\frac{(1 + 4\kappa)\sum_{i \in \mathcal{I}_+} \Delta x_i \Delta s_i}{\min \omega} \\
&= -\frac{1 + 4\kappa}{\min \omega} \cdot \sum_{i \in \mathcal{I}_+} d_{x_i} d_{s_i} \omega_i(t) \geq -\frac{1 + 4\kappa}{4 \min \omega} \cdot \sum_{i \in \mathcal{I}_+} (d_{x_i} + d_{s_i})^2 \omega_i(t) \\
&\geq -\frac{1 + 4\kappa}{4 \min \omega} \cdot \|\sqrt{\omega(t)} p_v\|^2 \geq -\frac{(1 + 4\kappa)\max(x^0 s^0)\|p_v\|^2}{4 \min \omega} \\
&= -\frac{(1 + 4\kappa)\max(x^0 s^0)\delta^2}{q^2 \min \omega} = -\frac{1 + 4\widetilde{\kappa}}{q^2}\delta^2,
\end{aligned} \tag{3.3}$$

where $\widetilde{\kappa}$ is given in (3.1). Thus, using (3.2) and (3.3), we calculate

$$\|d_x d_s\|_\infty = \max_{1 \leq i \leq n} |d_x d_s|_i \leq \max\{\sigma_+, -\sigma_-\} \leq \max\left\{\frac{\delta^2}{q^2}, \frac{1 + 4\widetilde{\kappa}}{q^2}\delta^2\right\} \leq \frac{1 + 4\widetilde{\kappa}}{q^2}\delta^2. \tag{3.4}$$

Then, combining (3.2), (3.3) and (3.4) yields

$$\begin{aligned}
\|d_x d_s\| &\leq \sqrt{\|d_x d_s\|_1 \cdot \|d_x d_s\|_\infty} = \sqrt{\sum_{i \in I} |d_{x_i} d_{s_i}| \cdot \|d_x d_s\|_\infty} \\
&\leq \sqrt{(\sigma_+ - \sigma_-) \cdot \|d_x d_s\|_\infty} \leq \sqrt{\left(\frac{\delta^2}{q^2} + \frac{1 + 4\widetilde{\kappa}}{q^2}\delta^2\right) \cdot \frac{1 + 4\widetilde{\kappa}}{q^2}\delta^2} \leq \frac{3 + 8\widetilde{\kappa}}{2q^2}\delta^2.
\end{aligned}$$

Thus, the proof is completed. □

Lemma 3.2. *Let $\delta := \delta(v)$ be given by (2.11) with $q \in \mathbb{N}$. For any $v \in \mathbb{R}^n_{\oplus}$, the following inequality holds*

$$1 - \delta(v) \leq v_i \leq 1 + \delta(v), \ i = 1, 2, \ldots, n.$$

Proof. From (2.11), for any $v \in \mathbb{R}^n_{\oplus}$, we have

$$\delta(v) = \|v^{1-q} - v\| = \left\| \frac{e - v^q}{v^{q-1}} \right\| = \left\| (e - v)\left(e + \frac{e}{v} + \cdots + \frac{e}{v^{q-1}} \right) \right\| \geq \|e - v\|,$$

which indicates that $|1 - v_i| \leq \delta(v)$. This yields the desired result. □

With $q = 5$ in Lemma 3.1, a specific corollary is achieved.

Corollary 3.3. *Let $x^0 s^0 \geq \omega$, $v > 0$ and $\delta := \delta(v)$ be given by (2.14) and $\widetilde{\kappa}$ is derived from (3.1). Then, the system (2.15) yields a unique solution satisfying*

$$\|d_x d_s\|_\infty \leq \frac{1 + 4\widetilde{\kappa}}{25}\delta^2, \quad \|d_x d_s\| \leq \frac{3 + 8\widetilde{\kappa}}{50}\delta^2.$$

3.1 The corrector step

The feasibility and convergence of Algorithm 1 for $P_*(\kappa)-$WHLCP after a corrector step will be proved in what follows. To this end, we propose a sufficient condition on the proximity measure such that the new iterate $(x^c, s^c) = (x + \Delta x, s + \Delta s)$ remains strictly feasible after a single corrector step.

Lemma 3.4. *Let $x^0 s^0 \geq \omega$ and $v > 0$. Suppose that $\delta := \delta(v) := \|v^{1-q} - v\| < \dfrac{q}{\sqrt{2^\lambda + 4\widetilde{\kappa}}}$ with $q \in \mathbb{N}$, where*

$$\lambda := \begin{cases} 1, & \text{if } q = 1; \\ 0, & \text{if } q \geq 2 \text{ and } q \in \mathbb{N}. \end{cases} \tag{3.5}$$

Then, the corrector step is strictly feasible, i.e., $(x^c, s^c) \in \mathcal{F}^+$.

Proof. For $(x, s) \in \mathcal{F}^+$ and each $\alpha \in [0, 1]$, we set

$$x(\alpha) = x + \alpha\Delta x, \ s(\alpha) = s + \alpha\Delta s.$$

From (2.7), we obtain

$$x(\alpha) = \frac{x}{v}(v + \alpha d_x), \quad s(\alpha) = \frac{s}{v}(v + \alpha d_s).$$

Then, by the second equality of system (2.9) and (2.10), we have

$$\begin{aligned} \frac{x(\alpha)s(\alpha)}{\omega(t)} &= (v + \alpha d_x)(v + \alpha d_s) \\ &= v^2 + \alpha v(d_x + d_s) + \alpha^2 d_x d_s \\ &= v^2 + \frac{2\alpha}{q}\left(v^{2-q} - v^2\right) + \alpha^2 d_x d_s \\ &= (1 - \alpha)v^2 + \alpha\left(\frac{2}{q}v^{2-q} + \frac{q-2}{q}v^2 + \alpha d_x d_s\right). \end{aligned} \tag{3.6}$$

Since $\omega(t) > 0$, the inequality $x(\alpha)s(\alpha) > 0$ holds for any $\alpha \in [0, 1]$ if

$$\frac{2}{q}v^{2-q} + \frac{q-2}{q}v^2 + \alpha d_x d_s > 0.$$

By Corollary 3.3, one has

$$\begin{aligned} \frac{2}{q}v^{2-q} + \frac{q-2}{q}v^2 + \alpha d_x d_s &\geq \frac{2}{q}v^{2-q} + \frac{q-2}{q}v^2 - \alpha\|d_x d_s\|_\infty e \\ &\geq \frac{2}{q}v^{2-q} + \frac{q-2}{q}v^2 - \|d_x d_s\|_\infty e \\ &\geq \frac{2}{q}v^{2-q} + \frac{q-2}{q}v^2 - \frac{(1 + 4\widetilde{\kappa})\delta^2}{q^2}e \\ &:= G, \end{aligned}$$

which together with (3.6) implies that $x(\alpha)s(\alpha) > 0$ holds if $G > 0$. Now we prove $G > 0$ by considering the following three cases.

(i) For the case of $q = 1$, if $\delta(v) < \frac{1}{\sqrt{2 + 4\widetilde{\kappa}}}$, then

$$\begin{aligned} G &= 2v - v^2 - (1 + 4\widetilde{\kappa})\delta^2 e \\ &= -(v - e)^2 + e - (1 + 4\widetilde{\kappa})\delta^2 e \\ &\geq e - (2 + 4\widetilde{\kappa})\delta^2 e > 0, \end{aligned}$$

where the first inequality is derived from Lemma 3.2.

(ii) For the case of $q = 2$, if $\delta(v) < \frac{2}{\sqrt{1 + 4\widetilde{\kappa}}}$, then $G = e - \frac{(1 + 4\widetilde{\kappa})\delta^2}{4} e > 0$.

(iii) For $q > 2$ and $q \in \mathbb{N}$, if $\delta(v) < \frac{q}{\sqrt{1 + 4\widetilde{\kappa}}}$, then $G > \frac{2}{q} v^{2-q} + \frac{q-2}{q} v^2 - e$. Consider the function $f : (0, +\infty) \to \mathbb{R}$, $f(\varsigma) := \frac{2}{q}\varsigma^{2-q} + \frac{q-2}{q}\varsigma^2 - 1$, where $q > 2$ and $q \in \mathbb{N}$. The first derivative of f is

$$f'(\varsigma) = \frac{2(q-2)\left(\varsigma - \varsigma^{1-q}\right)}{q},$$

and its second derivative is

$$f''(\varsigma) = \frac{2(q-2)\left(1 + (q-1)\varsigma^{-q}\right)}{q}.$$

Since $f'(1) = 0$ and $f''(\varsigma) > 0$ for all $\varsigma > 0$, the convex function f has the minimum $f(1)$ on $(0, +\infty)$, see

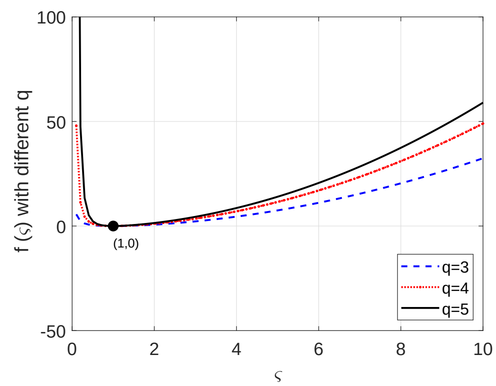

Figure 3: The graph of $f(\varsigma)$ with $q \in \{3, 4, 5\}$

Figure 3. Thus,

$$f(v_i) \geq f(1) = 0, \ \forall v_i > 0, \ i = 1, 2, \ldots, n, \tag{3.7}$$

which proves $G > 0$ for any $q > 2$ and $q \in \mathbb{N}$.

In a word, $G > 0$ holds if $\delta < \frac{q}{\sqrt{2^\lambda + 4\widetilde{\kappa}}}$ with $q \in \mathbb{N}$, where λ is given in (3.5). Then, we have $x(\alpha)s(\alpha) > 0$ for all $\alpha \in [0, 1]$. Given that $x(\alpha)$ and $s(\alpha)$ are linear functions of α, and $x(0) = x^0 > 0$, $s(0) = s^0 > 0$, we can deduce that $x(\alpha) > 0$, $s(\alpha) > 0$. Therefore, $x = x(1) > 0$ and $s = s(1) > 0$. This finalizes the proof. □

For notational simplicity, we define $v^c = \sqrt{\frac{x^c s^c}{\omega(t)}}$. The following lemma gives the upper bound on $\|e - (v^c)^2\|$ and the lower bound on $\min(v^c)$.

Lemma 3.5. *Let $x^0s^0 \geq \omega$ and $v > 0$. Suppose that $\delta = \|v^{1-q} - v\| < \frac{q}{\sqrt{2^\lambda + 4\widetilde{\kappa}}}$ with $q \in \mathbb{N}$, where λ is given in* (3.5)*. Then the following inequalities hold:*

(i) $\min(v^c) \geq \tilde{\eta}(\delta)$, *where*

$$\tilde{\eta}(\delta) := \begin{cases} \sqrt{1 - (2 + 4\widetilde{\kappa})\delta^2}, & \text{if } q = 1; \\ \sqrt{1 - \dfrac{1 + 4\widetilde{\kappa}}{q^2}\delta^2}, & \text{if } q \geq 2 \text{ and } q \in \mathbb{N}. \end{cases} \tag{3.8}$$

(ii) $\|e - (v^c)^2\| \leq \zeta(q, \delta)$, *where*

$$\zeta(q, \delta) := \begin{cases} \dfrac{5 + 8\widetilde{\kappa}}{2}\delta^2, & \text{if } q = 1; \\ \dfrac{2q^2 - 4q + 3 + 8\widetilde{\kappa}}{2q^2}\delta^2, & \text{if } q \geq 2 \text{ and } q \in \mathbb{N}. \end{cases}$$

Proof. (i) Using (3.6) with $\alpha = 1$, we have

$$(v^c)^2 = \frac{2}{q}v^{2-q} + \frac{q-2}{q}v^2 + d_x d_s, \text{ with } q \in \mathbb{N}. \tag{3.9}$$

Now, we discuss the minimum value of v^c in three cases of different values of q. For $q = 1$, by (3.9), Lemma 3.2 and Corollary 3.3, we obtain

$$\begin{aligned} \min(v^c) &= \min\left\{\sqrt{2v - v^2 + d_x d_s}\right\} \geq \min\left\{\sqrt{e - \delta^2 e + d_x d_s}\right\} \\ &\geq \sqrt{1 - \delta^2 - \|d_x d_s\|_\infty} \geq \sqrt{1 - (2 + 4\widetilde{\kappa})\delta^2}. \end{aligned} \tag{3.10}$$

For $q = 2$, from (3.9) and Lemma 3.1, we have

$$\min(v^c) = \min\left\{\sqrt{e + d_x d_s}\right\} \geq \sqrt{1 - \|d_x d_s\|_\infty} \geq \sqrt{1 - \frac{1 + 4\widetilde{\kappa}}{4}\delta^2}. \tag{3.11}$$

For $q > 2$ and $q \in \mathbb{N}$, it follows from (3.7), (3.9) and Corollary 3.3 that

$$\begin{aligned} \min(v^c) &= \min\left\{\sqrt{\frac{2}{q}v^{2-q} + \frac{q-2}{q}v^2 + d_x d_s}\right\} = \min\left\{\sqrt{f(v) + e + d_x d_s}\right\} \\ &\geq \sqrt{f(v_i) + 1 - \|d_x d_s\|_\infty} \geq \sqrt{1 - \|d_x d_s\|_\infty} \geq \sqrt{1 - \frac{1 + 4\widetilde{\kappa}}{q^2}\delta^2}. \end{aligned} \tag{3.12}$$

Hence, in light of (3.10), (3.11) and (3.12), we have

$$\min(v^c) \geq \tilde{\eta}(\delta) := \begin{cases} \sqrt{1 - (2 + 4\widetilde{\kappa})\delta^2}, & \text{if } q = 1; \\ \sqrt{1 - \dfrac{1 + 4\widetilde{\kappa}}{q^2}\delta^2}, & \text{if } q \geq 2 \text{ and } q \in \mathbb{N}. \end{cases}$$

(ii) By (3.9) and Corollary 3.3, we know that

$$\begin{aligned} \|e - (v^c)^2\| &\leq \left\|e - \frac{2}{q}v^{2-q} - \frac{q-2}{q}v^2\right\| + \|d_x d_s\| \\ &\leq \left\|e - \frac{2}{q}v^{2-q} - \frac{q-2}{q}v^2\right\| + \frac{3 + 8\widetilde{\kappa}}{2q^2}\delta^2. \end{aligned} \tag{3.13}$$

In the case of $q = 1$, we obtain from Lemma 3.2

$$\left\|e - \frac{2}{q}v^{2-q} - \frac{q-2}{q}v^2\right\| = \left\|(v - e)^2\right\| \leq \|v - e\|^2 \leq \delta^2. \tag{3.14}$$

In the case of $q = 2$, there holds

$$\left\|e - \frac{2}{q}v^{2-q} - \frac{q-2}{q}v^2\right\| = \|e - e\| = 0. \tag{3.15}$$

For $q > 2$ and $q \in \mathbb{N}$, it follows from (2.11) that

$$\left\| e - \frac{2}{q}v^{2-q} - \frac{q-2}{q}v^2 \right\| \leq \left\| \frac{e - \frac{2}{q}v^{2-q} - \frac{q-2}{q}v^2}{(v^{1-q}-v)^2} \right\|_\infty \|v^{1-q} - v\|^2$$
$$\leq \left\| \frac{(q-2)v^{2q} - qv^{2q-2} + 2v^q}{q(v^{2q} - 2v^q + e)} \right\|_\infty \delta^2. \tag{3.16}$$

Thus, we define a function $\xi : (0, +\infty) \to \mathbb{R}$ by

$$\xi(\varsigma) = \frac{(q-2)\varsigma^{2q} - q\varsigma^{2q-2} + 2\varsigma^q}{q(\varsigma^{2q} - 2\varsigma^q + 1)},$$

which is strictly increasing and positive for any $\varsigma > 0$, see Figure 4. Then

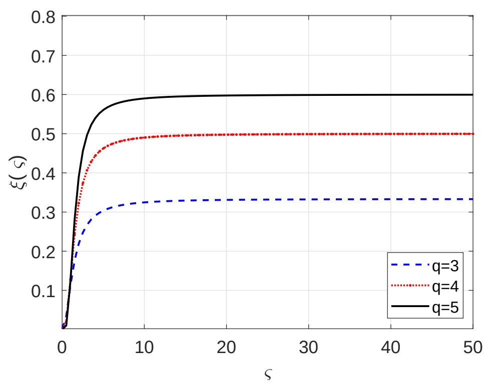

Figure 4: The graph of $\xi(\varsigma)$

$$0 < \xi(\varsigma) \leq \lim_{\varsigma \to +\infty} \xi(\varsigma) = \frac{q-2}{q}, \ \forall \varsigma \in (0, +\infty).$$

Using (3.16) and the fact that $\xi(v_i) \leq \frac{q-2}{q}$ for any $v_i > 0$, $q > 2$ and $q \in \mathbb{N}$, we have

$$\left\| e - \frac{2}{q}v^{2-q} - \frac{q-2}{q}v^2 \right\| \leq \frac{(q-2)\delta^2}{q}. \tag{3.17}$$

Substituting inequalities (3.14), (3.15) and (3.17) into (3.13), we obtain

$$\|e - (v^c)^2\| \leq \zeta(q, \delta) := \begin{cases} \dfrac{5 + 8\widetilde{\kappa}}{2}\delta^2, & \text{if } q = 1; \\ \dfrac{2q^2 - 4q + 3 + 8\widetilde{\kappa}}{2q^2}\delta^2, & \text{if } q \geq 2 \text{ and } q \in \mathbb{N}. \end{cases}$$

This completes the proof. □

By taking $q = 5$ in Lemma 3.4 and Lemma 3.5, we arrive at the following corollary.

Corollary 3.6. *Let* $\delta := \delta(v) < \dfrac{5}{\sqrt{1 + 4\widetilde{\kappa}}}$, $x^0 s^0 \geq \omega$, *and* $v > 0$. *Then, the corrector step is strictly feasible, i.e.,* $(x^c, s^c) \in \mathcal{F}^+$. *Besides, we have*

(i) $\min(v^c) \geq \eta(\delta)$, *where*

$$\eta(\delta) := \sqrt{1 - \frac{1 + 4\widetilde{\kappa}}{25}\delta^2}; \tag{3.18}$$

(ii)

$$\|e - (v^c)^2\| \leq \frac{33 + 8\widetilde{\kappa}}{50}\delta^2. \tag{3.19}$$

Now we establish an upper bound for the proximity measure $\delta(v^c) = \delta(x^c, s^c; t)$ of the iterate (x^c, s^c) after one corrector step.

Lemma 3.7. *Let $x^0 s^0 \geq \omega$ and $v > 0$. Suppose that $\delta = \|v^{1-q} - v\| < \dfrac{q}{\sqrt{2^\lambda + 4\widetilde{\kappa}}}$ with $q \in \mathbb{N}$, where λ is given in (3.5). Then, the following results hold.*

(i) For $q = 1$, we have

$$\delta_c := \delta(v^c) \leq \frac{5 + 8\widetilde{\kappa}}{2(1 + \tilde{\eta}(\delta))}\delta^2;$$

(ii) For $q \geq 2$ and $q \in \mathbb{N}$, we have

$$\delta_c := \delta(v^c) \leq \left(\frac{\varrho}{1 + \tilde{\eta}(\delta)} + \frac{1}{\tilde{\eta}^{1+\varrho}(\delta)} + \frac{1}{\tilde{\eta}^{3+\varrho}(\delta)} + \cdots + \frac{1}{\tilde{\eta}^{q-1}(\delta)}\right) \times \frac{2q^2 - 4q + 3 + 8\widetilde{\kappa}}{2q^2}\delta^2.$$

Here $\tilde{\eta}(\delta)$ is defined by (3.8) and

$$\varrho := \begin{cases} 1, & \text{if } q = 2j + 1,\ j \in \mathbb{N}; \\ 0, & \text{if } q = 2j,\ j \in \mathbb{N}. \end{cases} \tag{3.20}$$

Proof. First, from (2.11), we know that

$$\begin{aligned} \delta(v^c) &= \left\|(v^c)^{1-q} - v^c\right\| \\ &= \left\|\frac{e + v^c + (v^c)^2 + \cdots + (v^c)^{q-2} + (v^c)^{q-1}}{(e + v^c)(v^c)^{q-1}}(e - (v^c)^2)\right\|. \end{aligned} \tag{3.21}$$

Then, we define the function $h : (0, +\infty) \to \mathbb{R}$

$$\begin{aligned} h(\varsigma) &:= \frac{1 + \varsigma + \varsigma^2 + \cdots + \varsigma^{q-2} + \varsigma^{q-1}}{(1 + \varsigma)\varsigma^{q-1}} \\ &= \begin{cases} \dfrac{1}{1 + \varsigma}, & \text{if } q = 1; \\ \dfrac{\varrho}{1 + \varsigma} + \dfrac{1}{\varsigma^{1+\varrho}} + \dfrac{1}{\varsigma^{3+\varrho}} + \cdots + \dfrac{1}{\varsigma^{q-1}}, & \text{if } q \geq 2 \text{ and } q \in \mathbb{N}, \end{cases} \end{aligned}$$

where ϱ is given by (3.20). Since $h(\varsigma)$ is monotonically decreasing with respect to ς, by combining Lemma 3.5 (ii) and (3.21), we have

$$\begin{aligned} \delta(v^c) &= \left\|h(v^c)(e - (v^c)^2)\right\| \leq \|h(v^c)\|_\infty \left\|e - (v^c)^2\right\| \\ &\leq h(\tilde{\eta}(\delta)) \left\|e - (v^c)^2\right\| \leq h(\tilde{\eta}(\delta))\zeta(q, \delta) \\ &= \begin{cases} \dfrac{(5 + 8\widetilde{\kappa})h(\tilde{\eta}(\delta))}{2}\delta^2, & \text{if } q = 1; \\ \dfrac{(2q^2 - 4q + 3 + 8\widetilde{\kappa})h(\tilde{\eta}(\delta))}{2q^2}\delta^2, & \text{if } q \geq 2 \text{ and } q \in \mathbb{N}, \end{cases} \end{aligned}$$

which is the desired result. □

By considering the case of $q = 5$ in Lemma 3.7, we obtain an upper bound for the proximity measure $\delta(v^c) = \delta(x^c, s^c; t)$ and demonstrate the local quadratic convergence of the corrector step.

Corollary 3.8. *Let $\delta := \delta(v) < \dfrac{5}{\sqrt{1 + 4\widetilde{\kappa}}}$, $v > 0$ and $x^0 s^0 \geq \omega$. Then, there holds*

$$\delta_c := \delta(v^c) \leq \left(\frac{1}{\eta^4(\delta)} + \frac{1}{\eta^2(\delta)} + \frac{1}{1 + \eta(\delta)}\right)\frac{33 + 8\widetilde{\kappa}}{50}\delta^2,$$

where $\eta(\delta)$ is given in (3.18). Furthermore, if $\delta < \dfrac{4}{\sqrt{1 + 4\widetilde{\kappa}}}$, then $\delta_c \leq 2(4 + \widetilde{\kappa})\delta^2$, which implies the quadratic convergence of the corrector step.

Proof. Setting $q = 5$ in Lemma 3.7, we obtain

$$\delta(v^c) \le \left(\frac{1}{\eta^4(\delta)} + \frac{1}{\eta^2(\delta)} + \frac{1}{1+\eta(\delta)}\right)\frac{33+8\widetilde{\kappa}}{50}\delta^2.$$

Moreover, assuming $\delta < \dfrac{4}{\sqrt{1+4\widetilde{\kappa}}}$, it follows from (3.18) that $\eta(\delta) \ge \frac{3}{5}$. Then, one has

$$\begin{aligned}\delta(v^c) &\le \left(\frac{1}{\eta^4(\delta)} + \frac{1}{\eta^2(\delta)} + \frac{1}{1+\eta(\delta)}\right)\frac{33+8\widetilde{\kappa}}{50}\delta^2 \\ &\le \left(\frac{25}{162} + \frac{1}{18} + \frac{1}{80}\right)\cdot(33+8\widetilde{\kappa})\delta^2 \\ &< 0.2224\cdot(33+8\widetilde{\kappa})\delta^2 = 7.3392 + 1.7792\widetilde{\kappa}\delta^2 < 2(4+\widetilde{\kappa})\delta^2,\end{aligned}$$

which indicates the local quadratic convergence of the corrector step. □

3.2 The predictor step

In this subsection, we shall discuss the upper bound for the proximity measure $\delta(v^p) = \delta(x^p, s^p; t_p)$ after one predictor step (2.20) and prove the convergence of Algorithm 1. Let

$$v^p := \sqrt{\frac{x^p s^p}{\omega(t_p)}}, \tag{3.22}$$

where $\omega(t_p) = (1-\theta t)\omega(t),\ t_p = (1-\theta)t$ with $\theta \in (0,1)$. Since (Q,R) is a $P_*(\kappa)-$pair and $Q\Delta x + R\Delta s = 0$, there holds

$$(1+4\kappa)\sum_{i\in\mathcal{I}_+}\Delta^p x_i \Delta^p s_i + \sum_{i\in\mathcal{I}_-}\Delta^p x_i\Delta^p s_i \ge 0, \tag{3.23}$$

where

$$\mathcal{I} = \{1,2,\cdots,n\},\quad \mathcal{I}_- = \{i\in\mathcal{I} : \Delta^p x_i\Delta^p s_i < 0\},\quad \mathcal{I}_+ = \{i\in\mathcal{I} : \Delta^p x_i\Delta^p s_i > 0\}.$$

For subsequent analysis, we estimate the upper bound for $\|d_x^p d_s^p\|$.

Lemma 3.9. *Suppose that* $\delta := \delta(v) < \dfrac{5}{\sqrt{1+4\widetilde{\kappa}}}$*,* $v > 0$ *and* $x^0 s^0 \ge \omega$*. Then, one has*

$$\|d_x^p d_s^p\| \le \frac{n(1+2\widetilde{\kappa})}{2}\left[1 + \left(\frac{1}{\eta^4(\delta)} + \frac{1}{\eta^2(\delta)} + \frac{1}{1+\eta(\delta)}\right)\frac{33+8\widetilde{\kappa}}{50}\delta^2\right]^2,$$

where $\eta(\delta)$ *is given as in* (3.18).

Proof. In light of (2.18), the inequality (3.23) can be written as

$$J := (1+4\kappa)\sum_{i\in\mathcal{I}_+}\omega_i(t) d^p_{x_i} d^p_{s_i} + \sum_{i\in\mathcal{I}_-}\omega_i(t) d^p_{x_i} d^p_{s_i} \ge 0. \tag{3.24}$$

Let $x^0 s^0 \ge \omega$. It follows from (2.1), (3.1) and (3.24) that

$$\begin{aligned}0 \le J &\le \max(x^0 s^0)(1+4\kappa)\sum_{i\in\mathcal{I}_+} d^p_{x_i} d^p_{s_i} + \min\omega \sum_{i\in\mathcal{I}_-} d^p_{x_i} d^p_{s_i} \\ &= \min\omega\cdot\left[(d_x^p)^T d_s^p + \left(\frac{\max(x^0 s^0)(1+4\kappa)}{\min\omega} - 1\right)\sum_{i\in\mathcal{I}_+} d^p_{x_i} d^p_{s_i}\right] \\ &= \min\omega\cdot\left((d_x^p)^T d_s^p + 4\widetilde{\kappa}\sum_{i\in\mathcal{I}_+} d^p_{x_i} d^p_{s_i}\right) \\ &\le \min\omega\cdot\left((d_x^p)^T d_s^p + \widetilde{\kappa}\|d_x^p + d_s^p\|^2\right),\end{aligned}$$

from which we obtain

$$(d_x^p)^T d_s^p \geq -\widetilde{\kappa}\|d_x^p + d_s^p\|^2. \tag{3.25}$$

Next, using the second equation in (2.19) and (3.25), we achieve

$$\|d_x^p\|^2 + \|d_s^p\|^2 = \|d_x^p + d_s^p\|^2 - 2(d_x^p)^T d_s^p \leq (1+2\widetilde{\kappa})\|v^c\|^2. \tag{3.26}$$

Under the assumption that $\delta < \dfrac{5}{\sqrt{1+4\widetilde{\kappa}}}$, $v > 0$ and $x^0 s^0 \geq \omega$, it can be concluded from Lemma 3.2, Corollary 3.8 and (3.26) that

$$\begin{aligned}\|d_x^p d_s^p\| &\leq \frac{\|d_x^p\|^2 + \|d_s^p\|^2}{2} \leq \frac{1+2\widetilde{\kappa}}{2}\|v^c\|^2 \leq \frac{n(1+2\widetilde{\kappa})}{2}(1+\delta_c)^2 \\ &\leq \frac{n(1+2\widetilde{\kappa})}{2}\left[1 + \left(\frac{1}{\eta^4(\delta)} + \frac{1}{\eta^2(\delta)} + \frac{1}{1+\eta(\delta)}\right)\frac{33+8\widetilde{\kappa}}{50}\delta^2\right]^2.\end{aligned}$$

Then, the proof is complete. □

The next lemma guarantees that the predictor step remains strictly feasible if certain conditions are satisfied.

Lemma 3.10. *Let $(x^c, s^c) \in \mathcal{F}^+$, $x^0 s^0 \geq \omega$. Suppose that $\delta < \dfrac{1}{\sqrt{3(1+4\widetilde{\kappa})}}$ and $\theta < \dfrac{1}{\sqrt{3n(1+2\widetilde{\kappa})}}$ with $n \geq 2$. Then, the predictor step is strictly feasible, i.e.,*

$$(x^p, s^p) \in \mathcal{F}^+.$$

Proof. Define

$$x^p(\beta) = x^c + \beta\theta t \Delta^p x, \quad s^p(\beta) = s^c + \beta\theta t \Delta^p s, \quad \forall \beta \in [0,1].$$

Using (2.17), (2.18) and the second equation in system (2.19), we obtain

$$\begin{aligned}x^p(\beta)s^p(\beta) &= x^c s^c + \beta\theta t(s\Delta^p x + x\Delta^p s) + \beta^2\theta^2 t^2 \Delta^p x \Delta^p s \\ &= \omega(t)[(v^c)^2 + \beta\theta t v^c(d_x^p + d_s^p) + \beta^2\theta^2 t^2 d_x^p d_s^p] \\ &= \omega(t)[(1-\beta\theta t)(v^c)^2 + \beta^2\theta^2 t^2 d_x^p d_s^p],\end{aligned}$$

which can be rewritten as

$$\frac{x^p(\beta)s^p(\beta)}{\omega(t)(1-\beta\theta t)} = (v^c)^2 + \frac{\beta^2\theta^2 t^2}{1-\beta\theta t}\, d_x^p d_s^p. \tag{3.27}$$

Clearly, if the minimal component of (3.27) is greater than 0, the feasibility of the predictor step is proved. Therefore, by (3.27), Corollary 3.6 (i) and Lemma 3.9, we have

$$\begin{aligned}\min\left\{\frac{x^p(\beta)s^p(\beta)}{\omega(t)(1-\beta\theta t)}\right\} &= \min\left\{(v^c)^2 + \frac{\beta^2\theta^2 t^2}{1-\beta\theta t}\, d_x^p d_s^p\right\} \\ &\geq \eta^2(\delta) - \frac{\beta^2\theta^2 t^2}{1-\beta\theta t}\ \|d_x^p d_s^p\|_\infty \\ &\geq \eta^2(\delta) - \frac{\theta^2 t^2}{1-\theta t}\ \|d_x^p d_s^p\| \\ &\geq \eta^2(\delta) - \frac{\theta^2 t^2}{1-\theta t}\frac{n(1+2\widetilde{\kappa})}{2} \\ &\quad \times \left[1 + \left(\frac{1}{\eta^4(\delta)} + \frac{1}{\eta^2(\delta)} + \frac{1}{1+\eta(\delta)}\right)\frac{33+8\widetilde{\kappa}}{50}\delta^2\right]^2 \\ &:= z(\delta, \theta, n).\end{aligned} \tag{3.28}$$

Now, we establish the strict positivity of $z(\delta,\theta,n)$ by considering $\delta < \dfrac{1}{\sqrt{3(1+4\widetilde{\kappa})}}$ and $\theta < \dfrac{1}{\sqrt{3n(1+2\widetilde{\kappa})}}$

with $n \geq 2$. Indeed, by (3.18) and (3.28), there holds

$$
\begin{aligned}
z(\delta,\theta,n) &> \frac{24}{25} - \frac{n(1+2\widetilde{\kappa})\theta^2 t^2}{2(1-\theta t)}\left[1+\left(\frac{25^2}{24^2}+\frac{25}{24}+\frac{5}{5+2\sqrt{6}}\right)\frac{33+8\widetilde{\kappa}}{150(1+4\widetilde{\kappa})}\right]^2 \\
&> \frac{24}{25} - \frac{n(1+2\widetilde{\kappa})\theta^2 t^2}{2(1-\theta t)}\left[1+\left(\frac{25^2}{24^2}+\frac{25}{24}+\frac{5}{5+2\sqrt{6}}\right)\times\frac{11}{50}\right]^2 \\
&> \frac{24}{25} - \frac{n(1+2\widetilde{\kappa})\theta^2 t^2}{2(1-\theta t)}\left(1+\frac{2}{3}\right)^2 \\
&\geq \frac{24}{25} - \frac{25n(1+2\widetilde{\kappa})\theta^2}{18(1-\theta)} \\
&> \frac{24}{25} - \frac{25\sqrt{3n(1+2\widetilde{\kappa})}}{54(\sqrt{3n(1+2\widetilde{\kappa})}-1)} \\
&> \frac{24}{25} - \frac{5(6+\sqrt{6})}{54} \\
&> \frac{24}{25} - \frac{5(6+3)}{54} \\
&= \frac{24}{25} - \frac{5}{6} \\
&= \frac{19}{150} \\
&> 0,
\end{aligned}
$$

where the first inequality is due to $\eta(\delta) \geq \eta\left(\frac{1}{\sqrt{3(1+4\widetilde{\kappa})}}\right) \geq \sqrt{1-\frac{1}{75}} > \frac{2\sqrt{6}}{5}$, and the third inequality is derived from

$$
\begin{aligned}
\left(\frac{25^2}{24^2}+\frac{25}{24}+\frac{5}{5+2\sqrt{6}}\right)\times\frac{11}{50} &= \frac{275}{1152}+\frac{11}{48}+\frac{11}{50+20\sqrt{6}} \\
&< \frac{276}{1150}+\frac{12}{48}+\frac{11}{50+38} \\
&= \frac{6}{25}+\frac{1}{4}+\frac{1}{8} = \frac{123}{200} < \frac{132}{198} = \frac{2}{3}.
\end{aligned}
$$

Hence, if $\theta \in \left(0, \frac{1}{\sqrt{3n(1+2\widetilde{\kappa})}}\right)$, then $x^p(\beta)s^p(\beta) > 0$ for $\beta \in [0,1]$. Because $x^p(\beta)$ and $s^p(\beta)$ are continuous functions of β, $x^p(0) = x^c > 0$ and $s^p(0) = s^c > 0$, we get $(x^p, s^p) = (x^p(1), s^p(1)) \in \mathcal{F}^+$. This completes the proof. □

After a predictor step, the following lemma provides an upper bound for the proximity measure $\delta(v^p) = \delta(x^p, s^p; t_p)$.

Lemma 3.11. *Let $x^0 s^0 \geq \omega$, $\delta < \dfrac{1}{\sqrt{3(1+4\widetilde{\kappa})}}$ and $\theta < \dfrac{1}{\sqrt{3n(1+2\widetilde{\kappa})}}$ with $n \geq 2$. Then, there holds*

$$
\delta_p := \delta(v^p) \leq \left[\frac{1}{z^2(\delta,\theta,n)}+\frac{1}{z(\delta,\theta,n)}+\frac{1}{1+\sqrt{z(\delta,\theta,n)}}\right]\times\left[\frac{31}{50}\delta^2+1-z(\delta,\theta,n)\right],
$$

where $z(\delta,\theta,n)$ is given by (3.28).

Proof. From (3.27) with $\beta = 1$, we have

$$
(v^p)^2 = \frac{x^p s^p}{\omega(t)(1-\theta t)} = (v^c)^2 + \frac{\theta^2 t^2}{1-\theta t} d_x^p d_s^p. \tag{3.29}
$$

Using (3.28) and (3.29), we deduce

$$
\min(v^p) \geq \sqrt{z(\delta,\theta,n)}. \tag{3.30}
$$

Moreover, it follows from (3.18), (3.19), (3.28), (3.29) and Lemma 3.9 that

$$
\begin{aligned}
\left\|e-(v^p)^2\right\| &= \left\|e-(v^c)^2-\frac{\theta^2t^2}{1-\theta t}d_x^p d_s^p\right\| \\
&\le \|e-(v^c)^2\|+\frac{\theta^2t^2}{1-\theta t}\|d_x^p d_s^p\| \\
&\le \frac{33+8\widetilde{\kappa}}{50}\delta^2+\frac{\theta^2t^2}{1-\theta t}\frac{n(1+2\widetilde{\kappa})}{2} \\
&\quad \times\left[1+\left(\frac{1}{\eta^4(\delta)}+\frac{1}{\eta^2(\delta)}+\frac{1}{1+\eta(\delta)}\right)\frac{33+8\widetilde{\kappa}}{50}\delta^2\right]^2 \\
&= \frac{33+8\widetilde{\kappa}}{50}\delta^2+\eta^2(\delta)-z(\delta,\theta,n) \\
&= \frac{33+8\widetilde{\kappa}}{50}\delta^2+1-\frac{1+4\widetilde{\kappa}}{25}\delta^2-z(\delta,\theta,n) \\
&= \frac{31}{50}\delta^2+1-z(\delta,\theta,n). \qquad (3.31)
\end{aligned}
$$

Combining (2.14), (3.30) and (3.31) yields

$$
\begin{aligned}
\delta(v_p) &= \|(v^p)^{-4}-v^p\| \\
&\le \left[\frac{1}{\min(v^p)^4}+\frac{1}{\min(v^p)^2}+\frac{1}{1+\min(v^p)}\right]\cdot\|e-(v^p)^2\| \\
&\le \left[\frac{1}{z^2(\delta,\theta,n)}+\frac{1}{z(\delta,\theta,n)}+\frac{1}{1+\sqrt{z(\delta,\theta,n)}}\right] \\
&\quad \times\left[\frac{31}{50}\delta^2+1-z(\delta,\theta,n)\right],
\end{aligned}
$$

which completes the proof. □

Lemma 3.12. *Let $x^0s^0\ge\omega$, $t_p=(1-\theta)t$ and $\theta\le\frac{1+t}{4\sqrt{10n}(1+4\widetilde{\kappa})}$ with $n\ge2$. If $\delta\le\frac{t}{\sqrt{6}(1+4\widetilde{\kappa})}$, then $\delta_p\le\frac{t_p}{\sqrt{6}(1+4\widetilde{\kappa})}$.*

Proof. For $\delta\le\frac{t}{\sqrt{6}(1+4\widetilde{\kappa})}$, by Lemma 3.11, one has

$$
\begin{aligned}
\delta_p &\le g\left(\sqrt{z(\delta,\theta,n)}\right)\times\left[\frac{31}{50}\delta^2+1-\sqrt{z(\delta,\theta,n)}\right] \\
&\le g\left(\sqrt{z\left(\frac{t}{\sqrt{6}(1+4\widetilde{\kappa})},\theta,n\right)}\right)\times\left[\frac{31}{50}\delta^2+1-z\left(\frac{t}{\sqrt{6}(1+4\widetilde{\kappa})},\theta,n\right)\right],
\end{aligned}
$$

where the monotonically decreasing function g is defined by

$$
g(\varsigma):=\frac{1}{\varsigma^4}+\frac{1}{\varsigma^2}+\frac{1}{1+\varsigma},\qquad \text{for } \varsigma\in(0,+\infty).
$$

Then, the inequality $\delta_p\le\frac{t_p}{\sqrt{6}(1+4\widetilde{\kappa})}$ holds , if

$$
g\left(\sqrt{z\left(\frac{t}{\sqrt{6}(1+4\widetilde{\kappa})},\theta,n\right)}\right)\times\left[\frac{31}{50}\delta^2+1-z\left(\frac{t}{\sqrt{6}(1+4\widetilde{\kappa})},\theta,n\right)\right]\le\frac{(1-\theta)t}{\sqrt{6}(1+4\widetilde{\kappa})},
$$

i.e.,

$$
\begin{aligned}
H:=g&\left(\sqrt{z\left(\frac{t}{\sqrt{6}(1+4\widetilde{\kappa})},\theta,n\right)}\right) \\
&\times\left[\frac{31}{50}\delta^2+1-z\left(\frac{t}{\sqrt{6}(1+4\widetilde{\kappa})},\theta,n\right)\right]+\frac{(\theta-1)t}{\sqrt{6}(1+4\widetilde{\kappa})}\le 0. \qquad (3.32)
\end{aligned}
$$

To prove (3.32), we compute the lower bound for $z\left(\frac{t}{\sqrt{6}(1+4\widetilde{\kappa})}, \theta, n\right)$. By (3.18) and (3.28), we get

$$
\begin{aligned}
& z\left(\frac{t}{\sqrt{6}(1+4\widetilde{\kappa})}, \theta, n\right) \\
&= \eta^2\left(\frac{t}{\sqrt{6}(1+4\widetilde{\kappa})}\right) - \frac{n(1+2\widetilde{\kappa})\theta^2 t^2}{2(1-\theta t)}\left[1+g\left(\eta\left(\frac{t}{\sqrt{6}(1+4\widetilde{\kappa})}\right)\right)\frac{(33+8\widetilde{\kappa})t^2}{300(1+4\widetilde{\kappa})^2}\right]^2 \\
&\geq 1 - \frac{t^2}{150(1+4\widetilde{\kappa})} - \frac{n(1+2\widetilde{\kappa})\theta^2 t^2}{2(1-\theta t)}\left[1+g\left(\sqrt{1-\frac{t^2}{150(1+4\widetilde{\kappa})}}\right)\frac{(33+8\widetilde{\kappa})t^2}{300(1+4\widetilde{\kappa})^2}\right]^2 \\
&\geq 1 - \left[\frac{1}{150(1+4\widetilde{\kappa})} + \frac{n(1+2\widetilde{\kappa})\theta^2}{2(1-\theta t)}\left[1+g\left(\sqrt{1-\frac{1}{150(1+4\widetilde{\kappa})}}\right)\frac{33+8\widetilde{\kappa}}{300(1+4\widetilde{\kappa})^2}\right]^2\right] t^2 \\
&\geq 1 - \left[\frac{1}{150(1+4\widetilde{\kappa})} + \frac{n(1+2\widetilde{\kappa})\theta^2}{2(1-\theta t)}\left[1+\frac{33+8\widetilde{\kappa}}{118(1+4\widetilde{\kappa})^2}\right]^2\right] t^2 \\
&\geq 1 - \left[\frac{1}{150(1+4\widetilde{\kappa})} + \frac{n(1+2\widetilde{\kappa})\theta^2}{2(1-\theta t)}\left(1+\frac{33}{118}\right)^2\right] t^2 \\
&= 1 - \left[\frac{1}{150(1+4\widetilde{\kappa})} + \left(\frac{151}{118}\right)^2\frac{n(1+2\widetilde{\kappa})\theta^2}{2(1-\theta t)}\right] t^2 \\
&> 1 - \left[\frac{1}{150(1+4\widetilde{\kappa})} + \left(\frac{4}{3}\right)^2\frac{n(1+2\widetilde{\kappa})\theta^2}{2(1-\theta t)}\right] t^2 \\
&= 1 - \left[\frac{1}{150(1+4\widetilde{\kappa})} + \frac{8n(1+2\widetilde{\kappa})\theta^2}{9(1-\theta t)}\right] t^2, \qquad (3.33)
\end{aligned}
$$

where the third inequality is due to the fact

$$
\begin{aligned}
& g\left(\sqrt{1-\frac{1}{150(1+4\widetilde{\kappa})}}\right) \\
&\leq \frac{1}{\left(1-\frac{1}{150(1+4\widetilde{\kappa})}\right)^2} + \frac{1}{1-\frac{1}{150(1+4\widetilde{\kappa})}} + \frac{1}{1+\sqrt{1-\frac{1}{150(1+4\widetilde{\kappa})}}} \\
&\leq \frac{150^2}{149^2} + \frac{150}{149} + \frac{1}{1+\sqrt{\frac{149}{150}}} = \frac{150\times 299}{149^2} + \frac{1}{1+\sqrt{\frac{149}{150}}} \\
&< 2.0202 + 0.5009 = 2.5211 < \frac{150}{59}.
\end{aligned}
$$

Given that $\theta \le \dfrac{1+t}{4\sqrt{10n}(1+4\widetilde{\kappa})}$ with $n \ge 2$, the inequality (3.33) becomes

$$
\begin{aligned}
z\left(\frac{t}{\sqrt{6}(1+4\widetilde{\kappa})},\theta,n\right) &\ge 1-\left[\frac{1}{150(1+4\widetilde{\kappa})}+\frac{(1+2\widetilde{\kappa})(1+t)^2}{180(1-\theta t)(1+4\widetilde{\kappa})^2}\right]t^2\\
&\ge 1-\left[\frac{1}{150(1+4\widetilde{\kappa})}+\frac{1+2\widetilde{\kappa}}{45(1-\theta)(1+4\widetilde{\kappa})^2}\right]t^2\\
&\ge 1-\left[\frac{1}{150(1+4\widetilde{\kappa})}+\frac{1+2\widetilde{\kappa}}{45\left(1-\frac{1}{2\sqrt{10n}(1+4\widetilde{\kappa})}\right)(1+4\widetilde{\kappa})^2}\right]t^2\\
&\ge 1-\left[\frac{1}{150(1+4\widetilde{\kappa})}+\frac{1+2\widetilde{\kappa}}{45\left(1-\frac{1}{2\sqrt{10n}}\right)(1+4\widetilde{\kappa})^2}\right]t^2\\
&\ge 1-\left[\frac{1}{150(1+4\widetilde{\kappa})}+\frac{1+2\widetilde{\kappa}}{45\left(1-\frac{1}{4\sqrt{5}}\right)(1+4\widetilde{\kappa})^2}\right]t^2\\
&> 1-\left[\frac{1}{150(1+4\widetilde{\kappa})}+\frac{1+2\widetilde{\kappa}}{45\left(1-\frac{1}{6}\right)(1+4\widetilde{\kappa})^2}\right]t^2\\
&= 1-\frac{(5+12\widetilde{\kappa})t^2}{150(1+4\widetilde{\kappa})^2}. \qquad (3.34)
\end{aligned}
$$

It follows from (3.34) that the upper bound of H can be estimated as

$$
\begin{aligned}
H &\le g\left(\sqrt{1-\frac{(5+12\widetilde{\kappa})t^2}{150(1+4\widetilde{\kappa})^2}}\right)\times\left(\frac{31t^2}{300(1+4\widetilde{\kappa})^2}+\frac{(5+12\widetilde{\kappa})t^2}{150(1+4\widetilde{\kappa})^2}\right)+\frac{\left(\frac{1}{2\sqrt{10n}(1+2\widetilde{\kappa})}-1\right)t}{\sqrt{6}(1+4\widetilde{\kappa})}\\
&\le g\left(\sqrt{1-\frac{5+12\widetilde{\kappa}}{150(1+4\widetilde{\kappa})^2}}\right)\times\left(\frac{31}{300(1+4\widetilde{\kappa})^2}+\frac{5+12\widetilde{\kappa}}{150(1+4\widetilde{\kappa})^2}\right)+\frac{\left(\frac{1}{2\sqrt{10n}(1+2\widetilde{\kappa})}-1\right)}{\sqrt{6}(1+4\widetilde{\kappa})}\\
&\le g\left(\sqrt{\frac{29}{30}}\right)\times\left(\frac{31}{300}+\frac{1}{30}\right)+\frac{\frac{1}{2\sqrt{10n}}-1}{\sqrt{6}}\\
&< \left(\left(\frac{30}{29}\right)^2+\frac{30}{29}+\frac{1}{1+\sqrt{\frac{29}{30}}}\right)\times\frac{41}{300}+\frac{1-4\sqrt{5}}{4\sqrt{30}}\\
&< 2.6089\times 0.1367-0.3626\\
&= 0.3566-0.3626\\
&= -0.0060<0,
\end{aligned}
$$

which yields the desired result. □

3.3 Iteration bound

In this subsection, we analyze the polynomial iteration complexity of Algorithm 1. An upper bound for $\|x^p s^p-\omega\|$ after one main iteration is given by the below lemma.

Lemma 3.13. *Suppose that the initial point $(x^0,s^0)\in\mathcal{F}^+$ satisfies $x^0s^0\ge\omega$. Let $\delta\le\dfrac{t}{\sqrt{6}(1+4\widetilde{\kappa})}$ and $\theta\le\dfrac{1+t}{4\sqrt{10n}(1+4\widetilde{\kappa})}$ with $n\ge 2$. Then, there holds*

$$\|x^ps^p-\omega\|\le\Big(\phi(\widetilde{\kappa})\max(x^0s^0)+\|x^0s^0-\omega\|\Big)t,$$

where $\phi(\widetilde{\kappa}) = \dfrac{89 + 214\widetilde{\kappa}}{300(1+4\widetilde{\kappa})^2}$.

Proof. By combining $x^0 s^0 \geq \omega$, (2.1), (3.22) and (3.31), we get

$$\begin{aligned}
\|x^p s^p - \omega\| &\leq \|x^p s^p - \omega(t_p)\| + \|\omega(t_p) - \omega(t)\| + \|\omega(t) - \omega\| \\
&\leq \|e - (v^p)^2\| \, \|\omega(t)\|_\infty + \theta t \|\omega(t)\| + \|x^0 s^0 - \omega\| t \\
&\leq \|e - (v^p)^2\| \max(x^0 s^0) + \theta\sqrt{n} t \max(x^0 s^0) + \|x^0 s^0 - \omega\| t \\
&\leq \left(\frac{31}{50}\delta^2 + 1 - z(\delta, \theta, n) + \theta\sqrt{n} t \right) \max(x^0 s^0) + \|x^0 s^0 - \omega\| t. \qquad (3.35)
\end{aligned}$$

For $\delta(v) \leq \dfrac{t}{\sqrt{6}(1+4\widetilde{\kappa})}$ and $\theta \leq \frac{1+t}{4\sqrt{10n}(1+4\widetilde{\kappa})}$ with $t \in (0,1]$, $n > 2$, it follows from (3.34) that

$$\begin{aligned}
&\frac{31}{50}\delta^2 + 1 - z(\delta, \theta, n) + \theta\sqrt{n} t \\
&\leq \frac{31 t^2}{300(1+4\widetilde{\kappa})^2} + 1 - z\left(\frac{t}{\sqrt{6}(1+4\widetilde{\kappa})}, \theta, n \right) + \theta\sqrt{n} t \\
&\leq \frac{31 t^2}{300(1+4\widetilde{\kappa})^2} + \frac{(5+12\widetilde{\kappa}) t^2}{150(1+4\widetilde{\kappa})^2} + \frac{(1+t) t}{4\sqrt{10}(1+4\widetilde{\kappa})} \\
&\leq \left[\frac{41 + 24\widetilde{\kappa}}{300(1+4\widetilde{\kappa})^2} + \frac{1}{2\sqrt{10}(1+4\widetilde{\kappa})} \right] t \\
&= \frac{\left[(41 + 15\sqrt{10}) + (24 + 60\sqrt{10})\widetilde{\kappa} \right] t}{300(1+4\widetilde{\kappa})^2} \\
&< \frac{(89 + 214\widetilde{\kappa}) t}{300(1+4\widetilde{\kappa})^2} \\
&:= \phi(\widetilde{\kappa})\, t. \qquad (3.36)
\end{aligned}$$

Substituting (3.36) into (3.35) leads to

$$\|x^p s^p - \omega\| \leq \big(\phi(\widetilde{\kappa}) \max(x^0 s^0) + \|x^0 s^0 - \omega\| \big)\, t,$$

which is the desired result. □

As a consequence of Lemma 3.13, we establish the polynomial iteration bound for Algorithm 1.

Theorem 3.14. *Let* $(x^0, s^0) \in \mathcal{F}^+$ *satisfy* $x^0 s^0 \geq \omega$, $\theta = \dfrac{1+t}{4\sqrt{10n}(1+4\widetilde{\kappa})}$, *and* $\tau = \dfrac{1}{\sqrt{6}(1+4\widetilde{\kappa})}$. *If* (Q, R) *is a* $P_*(\kappa)-$*pair, then Algorithm 1 generates an* $\varepsilon-$*solution of* $P_*(\kappa)-$*WHLCP (1.1) with the number of iterations not exceeding*

$$O\left((1+4\widetilde{\kappa})\sqrt{n} \log \frac{\frac{2}{5(1+8\widetilde{\kappa})} \max(x^0 s^0) + \|x^0 s^0 - \omega\|}{\varepsilon} \right).$$

Proof. Let x^k and s^k be the iterates obtained after k iterations. According to Lemma 3.13, there holds

$$\begin{aligned}
\|x^k s^k - \omega\| &\leq \Big(\phi(\widetilde{\kappa}) \max(x^0 s^0) + \|x^0 s^0 - \omega\| \Big) t_{k-1} \\
&\leq \Big(\phi(\widetilde{\kappa}) \max(x^0 s^0) + \|x^0 s^0 - \omega\| \Big) (1 - \theta_{\min})^{k-1},
\end{aligned}$$

which implies that $\|x^p s^p - \omega\| \leq \varepsilon$ is satisfied, if

$$\Big(\phi(\widetilde{\kappa}) \max(x^0 s^0) + \|x^0 s^0 - \omega\| \Big) (1 - \theta_{\min})^{k-1} \leq \varepsilon. \qquad (3.37)$$

Taking the logarithm of both sides of (3.37), we have

$$(k-1) \log(1 - \theta_{\min}) \leq \log \frac{\varepsilon}{\Big(\phi(\widetilde{\kappa}) \max(x^0 s^0) + \|x^0 s^0 - \omega\| \Big)}. \qquad (3.38)$$

Due to $\log(1-\varsigma) \le -\varsigma$ for $\varsigma \in (0,1)$, the inequality (3.38) holds if

$$k \ge \left\lceil \frac{1}{\theta_{\min}} \log \frac{\phi(\widetilde{\kappa}) \max(x^0 s^0) + \|x^0 s^0 - \omega\|}{\varepsilon} \right\rceil + 1. \tag{3.39}$$

Besides, by (3.36), we obtain

$$\begin{aligned} \phi(\widetilde{\kappa}) &= \frac{89 + 214\widetilde{\kappa}}{300(1+4\widetilde{\kappa})^2} < \frac{3+8\widetilde{\kappa}}{10(1+4\widetilde{\kappa})^2} = \frac{3+8\widetilde{\kappa}}{5(32\widetilde{\kappa}^2 + 16\widetilde{\kappa} + 2)} \\ &< \frac{3+8\widetilde{\kappa}}{5\left(32\widetilde{\kappa}^2 + 16\widetilde{\kappa} + \frac{3}{2}\right)} = \frac{1}{5\left(\frac{1}{2} + 4\widetilde{\kappa}\right)} = \frac{2}{5(1+8\widetilde{\kappa})}. \end{aligned} \tag{3.40}$$

If $\theta = \dfrac{1+t}{4\sqrt{10n}(1+4\widetilde{\kappa})}$ with $t \in (0,1]$ and $n \ge 2$, we further have

$$\theta = \frac{1+t}{4\sqrt{10n}(1+4\widetilde{\kappa})} > \frac{1}{2\sqrt{10n}(1+4\widetilde{\kappa})} = \theta_{\min}.$$

Consequently, applying $\theta_{\min} = \frac{1}{2\sqrt{10n}(1+4\widetilde{\kappa})}$ and (3.36) to (3.39), we prove the desired result. □

4 Numerical tests

In this section, we conduct some numerical experiments to verify the performance of Algorithm 1 for $P_*(\kappa)-$WHLCP. All implementations are written in MATLAB R2021(a) under Windows 11 and executed on AMD Ryzen 9 6900HX with Radeon Graphics CPU @3.30 GHz 32.0 GB RAM platform. We compute the number of iterations (Iter) and the running time in seconds (Cpu) of Algorithm 1 after reaching the termination condition Gap$= \|xs - \omega\| \le \varepsilon$, where $\varepsilon = 10^{-4}$ is the accuracy parameter.

Problem 1 Consider the $P_*(\kappa)-$WHLCPs in $\mathbb{R}^{n\times n}$, where

$$Q = \begin{pmatrix} 4 & -2 & 0 & \cdots & 0 \\ 1 & 4 & -2 & \cdots & 0 \\ 0 & 1 & 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 4 \end{pmatrix},\ R = -I,\ b = Qx^0 + Rs^0,\ \omega = rand(n,1).$$

In the special case where $R = -I$, the pair (Q,R) is a $P_*(\kappa)$-pair if and only if Q is a $P_*(\kappa)$-matrix. Here $\frac{Q+Q^T}{2}$ is a positive semidefinite (PSD) matrix, and thus Q is a $P_*(0)$-matrix [20]. Therefore, (Q,R) is a $P_*(0)$-pair, with the corresponding handicap $\vartheta(Q,R) = 0$.

Problem 2 Consider the randomly generated $P_*(\kappa)-$WHLCPs in $\mathbb{R}^{n\times n}$, where

$$Q = A^T A,\ R = rand(n,n),\ b = Qx^0 + Rs^0,\ \omega = rand(n,1),$$

with the elements of $A \in \mathbb{R}^{n\times n}$ randomly generated in the interval $[-1,1]$. Since $Q = A^T A$ is PSD, it is a $P_*(0)$-matrix. From the definition of a $P_*(\kappa)$-matrix [20], $R \in P_*(\kappa)$ if and only if

$$(1+4\kappa)\sum_{i\in J_+} r_i(Rr)_i + \sum_{i\in \mathcal{J}_-} r_i(Rr)_i \ge 0, \quad \forall r \in \mathbb{R}^n,$$

where $\mathcal{J}_+ = \{i : r_i(Rr)_i > 0\}$ and $\mathcal{J}_- = \{i : r_i(Rr)_i < 0\}$. We calculate the handicap κ of R via solving

$$\kappa = \max_{r\in\mathbb{R}^n\setminus\{0\}} \frac{-\sum_{i\in J_+} r_i(Rr)_i - \sum_{i\in\mathcal{J}_-} r_i(Rr)_i}{4\sum_{i\in\mathcal{J}_+} r_i(Rr)_i}.$$

Therefore, (Q,R) constitutes a $P_*(\kappa)$-pair, and its handicap $\vartheta(Q,R)$ is equal to κ. We provide the formula for the handicap κ here. While its existence is guaranteed within the $P_*(\kappa)$ framework, computing its exact value may not be straightforward in general. Since the proposed algorithm does not require the explicit value of κ, the validity of our numerical experiments remains fully justified.

We test Algorithm 1 by Problem 1 and Problem 2 with different dimensions and parameters. For Problem 1, the dimensions $n \in \{60,\ 100,\ 200,\ 300,\ 500,\ 800,\ 1200\}$, while for Problem 2, the dimensions $n \in \{50,\ 80,\ 100,\ 300,\ 600,\ 1000,\ 1500\}$. In both cases, we choose $\theta \in \{0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6\}$ and initial

Table 4: Numerical results for Problem 1 with different dimensions and θ

$\theta \rightarrow$	0.2		0.3		0.4		0.5		0.6	
$n \downarrow$	Iter	Cpu	Iter	Cpu	Iter	Cpu	Iter	Cpu	Iter	Cpu
Initial point $(x^0, s^0) = (e, 0.5e)$										
60	48	0.0252	30	0.0087	22	0.0064	16	0.0053	13	0.0043
100	49	0.0415	31	0.0218	22	0.0153	17	0.0131	13	0.0098
200	50	0.2332	32	0.1295	23	0.0945	17	0.0778	13	0.0602
300	51	0.5363	33	0.3095	23	0.2141	17	0.1931	13	0.1482
500	52	2.2632	33	1.3587	23	1.0049	18	0.8166	14	0.6332
800	53	7.8743	34	5.0482	24	3.7200	18	2.8972	14	2.2164
1200	54	20.8918	35	12.9567	24	10.2609	18	7.5917	14	5.9767
Initial point $(x^0, s^0) = (e, e)$										
60	51	0.0165	33	0.0133	23	0.0075	18	0.0060	14	0.0045
100	53	0.0366	33	0.0340	24	0.0182	18	0.0136	14	0.0106
200	54	0.3063	34	0.1846	24	0.1090	18	0.0828	14	0.0637
300	55	0.6347	35	0.4363	25	0.2795	19	0.1954	14	0.1410
500	56	2.5529	36	1.9448	25	1.1451	19	0.8426	15	0.6792
800	57	9.2489	36	6.3270	26	4.2984	19	3.1594	15	2.3359
1200	58	24.5187	37	15.9176	26	11.0607	20	8.4811	15	6.3646
Initial point $(x^0, s^0) = (e, 2e)$										
60	55	0.0175	35	0.0114	25	0.0080	19	0.0057	14	0.0040
100	56	0.0396	36	0.0260	25	0.0180	19	0.0146	15	0.0124
200	57	0.2540	37	0.1643	26	0.1263	19	0.0955	15	0.0723
300	58	0.5978	37	0.4144	26	0.2894	20	0.2335	15	0.1713
500	60	2.8395	38	1.7822	27	1.3426	20	0.9959	16	0.8152
800	61	9.8005	38	6.1977	27	4.5819	20	3.3815	16	2.5892
1200	61	26.0036	39	18.0860	28	12.8741	21	9.9205	16	6.9140

points $(x^0, s^0) = (e, 0.5e)$, (e, e), $(e, 2e)$, where $e \in \mathbb{R}^n$. The numerical results for Problem 1 and Problem 2 are provided in Table 4 and Table 5, respectively. Besides, Figures 5-8 depict the numerical results for $(x^0, s^0) = (e, e)$ and $\theta = 0.1$. Specifically, Figure 5 illustrates the values of Gap obtained from solving Problem 1, while Figure 6 presents the values of $\delta(v)$ generated by the same problem. Similarly, the values of Gap and $\delta(v)$ for Problem 2 are visualized in Figure 7 and Figure 8, respectively.

We compare Algorithm 1 based on $\varphi(t) = t^{\frac{5}{2}}$ with the two variants of PC IPA, which use $\varphi(t) = \sqrt{t}$ and $\varphi(t) = t^{\frac{3}{2}}$ in the AET technique characterized by system (2.3). Note that all the three functions $\varphi(t) = \sqrt{t}$, $\varphi(t) = t^{\frac{3}{2}}$, and $\varphi(t) = t^{\frac{5}{2}}$ are special cases of $\varphi(t) = t^{\frac{q}{2}}$ for $q \in \{1,\ 3,\ 5\}$, respectively, as mentioned in Section 2.2.

For Problem 1 with dimensions $n \in \{80,\ 120,\ 160,\ 200,\ 240,\ 280,\ 320,\ 360,\ 400,\ 440\}$ and Problem 2 with dimensions $n \in \{60,\ 110,\ 160,\ 210,\ 260,\ 310,\ 360,\ 410,\ 460,\ 510\}$, we use the same initial points $x^0 = s^0 = e$ and fixed parameter $\theta \in \{0.2, 0.6\}$. The comparative results are recorded in Table 6. The symbols "Ave. Cpu" and "Ave. Iter" stand for the average running time and the average iteration numbers for 10 given $P_*(\kappa)-$WHLCPs for each size n listed in Table 6, respectively. When $\theta = 0.2$, Figure 9 and Figure 10 visualize the iteration numbers and running time with different values of q for Problem 1 and Problem 2 in Table 6, respectively.

From Tables 4-6 and Figures 5-10, we summarize the following numerical findings. From which, we conclude that Algorithm 1 is a feasible and effective method for $P_*(\kappa)-$WHLCP.

i) Table 4 and Table 5 show that both problem size and the parameter θ affect the performance of the algorithm. Given a problem size, increasing θ generally speeds up convergence. The running time of Algorithm 1 escalates markedly but the iteration numbers remain relatively stable as the problem dimension increases, in the case of a fixed θ.

ii) Figure 5 and Figure 7 reveal that Gap decreases as t move towards 0, which indicates that the distance between xs and ω is decreasing. Similarly, Fig. 6 and Fig. 8 show that $\delta(v)$ trends toward 0 after peaking, confirming the algorithm's global convergence.

iii) Table 6 lists the computational results of the PC IPA based on three AET functions for Problems 1 and 2 with varying problem sizes. As depicted in Figure 9 and Figure 10, the PC IPA with $\varphi(t) = t^{\frac{5}{2}}$ slightly outperforms those with $\varphi(t) = \sqrt{t}$ and $\varphi(t) = t^{\frac{3}{2}}$, especially for larger problems $(n > 300)$. For the same problem size, the algorithm with $q = 5$ requires significantly fewer iterations

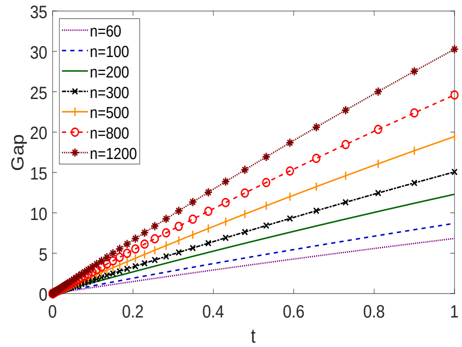

Figure 5: The values of Gap for Problem 1

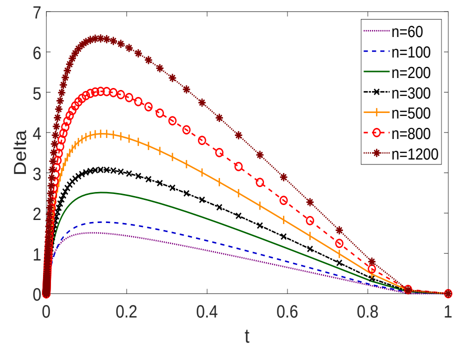

Figure 6: The values of $\delta(v)$ for Problem 1

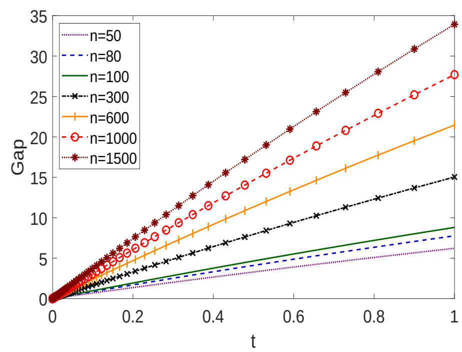

Figure 7: The values of Gap for Problem 2

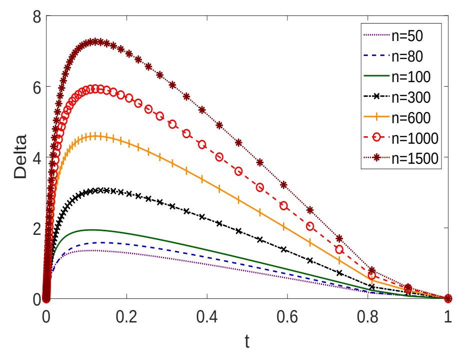

Figure 8: The values of $\delta(v)$ for Problem 2

than the algorithms with $q = 1$ and $q = 3$. However, the overall running times of the algorithms with the three q values exhibit relatively close. This is because when $q = 5$, the search direction $p_v = \frac{2}{5}\left(v^{-4} - v\right)$ derived from the function $\varphi(t) = t^{\frac{5}{2}}$ involves more complex computations, which consequently lengthens the time required for each iteration.

Problem 3

The Arrow-Debreu competitive market equilibrium problem with a linear utility function [34] may be modeled as a WHLCP , which could be efficiently solved by Algorithm 1. In 1874, Walras [19] first formulated the Arrow-Debreu competitive market equilibrium problem. In this problem, each trader is both a consumer and a producer, and the initial endowment ω_i of consumer i is not given. Instead, it is the price allocated to other commodities j. Traders come into the market having initial commodity endowments and utility functions, and they achieve the maximization of their individual utilities through the purchase and sale of commodities at market clearing prices. The linear utility function is defined as $u_i(x_i) = u(x_{i1}, \ldots, x_{in}) = \sum\limits_j u_{ij}x_{ij}$, where u_{ij} is a given utility coefficient, representing the utility of consumer i for commodity j, and x_{ij} represents the quantity of commodities sold by the producer to consumer i. Every consumer has a positive utility for at least one commodity, and every commodity provides positive utility to at least one consumer. Arrow and Debreu [17] have proven that when the utility function is a concave function and commodities are divisible, the existence of the equilibrium price holds. The price equilibrium refers to the price allocation of commodities, such that when each consumer purchases the commodity combination that maximizes their utility, the market is cleared. This means that all money is fully utilized and all products are completely traded.

The Arrow-Debreu model with a linear utility function can be written as a parameterized convex opti-

Table 5: Numerical results for Problem 2 with different dimensions and θ

$\theta \rightarrow$	0.2		0.3		0.4		0.5		0.6	
$n \downarrow$	Iter	Cpu	Iter	Cpu	Iter	Cpu	Iter	Cpu	Iter	Cpu
Initial point $(x^0, s^0) = (e, 0.5e)$										
50	48	0.0103	31	0.0117	22	0.0064	17	0.0147	13	0.0033
80	49	0.0237	32	0.0190	23	0.0134	17	0.0129	13	0.0117
100	50	0.0421	32	0.0236	23	0.0167	18	0.0147	14	0.0155
300	52	0.8298	34	0.5115	24	0.3094	17	0.1307	14	0.1123
600	54	3.9281	35	2.2977	25	1.4659	18	1.3781	14	1.806
1000	55	10.9471	35	6.8926	25	2.9998	18	2.4185	15	3.3671
1500	56	31.1532	36	19.3579	26	16.8978	20	10.8744	15	9.2319
Initial point $(x^0, s^0) = (e, e)$										
50	37	0.0064	24	0.0042	19	0.0034	15	0.0025	12	0.0023
80	38	0.0116	24	0.0097	20	0.0072	16	0.0072	13	0.0069
100	39	0.0205	25	0.0142	20	0.0103	16	0.0111	13	0.0105
300	55	0.4051	35	0.2599	25	0.2076	19	0.1431	14	0.1332
600	57	2.0088	36	1.2811	25	0.9284	19	0.6943	14	0.2996
1000	58	7.3200	37	4.7123	26	3.3209	20	2.6304	14	1.8660
1500	60	18.1300	40	11.6970	28	8.8062	21	6.1394	14	4.2936
Initial point $(x^0, s^0) = (e, 2e)$										
50	52	0.0081	33	0.0054	23	0.0039	17	0.0272	13	0.0035
80	53	0.0211	33	0.0114	24	0.0080	18	0.0059	14	0.0874
100	53	0.0239	34	0.0168	24	0.0126	18	0.0087	14	0.0127
300	58	0.4609	37	0.2820	26	0.1910	20	0.1514	14	0.2809
600	60	2.2703	38	1.3506	27	0.9763	20	0.7107	15	0.4071
1000	61	7.7045	39	4.9700	28	3.5828	21	2.6223	15	2.5918
1500	63	20.0136	41	12.6279	29	9.0536	22	6.6642	15	5.5872

mization model

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} \omega_i \log u_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_{ij} = 1, \quad \forall j \\ & u_i - \sum_{j=1}^{n} u_{ij} x_{ij} = 0, \quad \forall i \\ & u_{ij}, x_{ij} \geq 0. \quad \forall i, j \end{aligned}$$

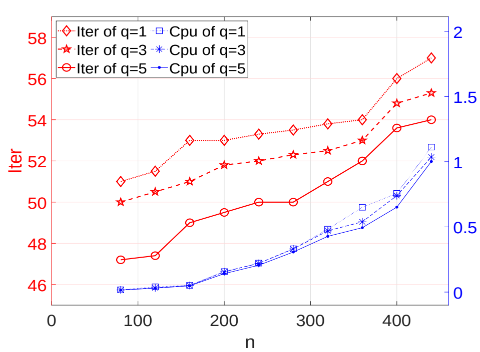

Figure 9: Numerical results for Problem 1 in the case of $q \in \{1,\ 3,\ 5\}$

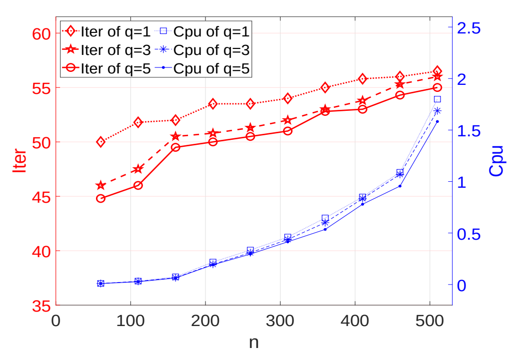

Figure 10: Numerical results for Problem 2 in the case of $q \in \{1,\ 3,\ 5\}$

Table 6: Numerical results of PC IPAs with three AET functions

		$\varphi(t)=t^{\frac{1}{2}}$		$\varphi(t)=t^{\frac{3}{2}}$		$\varphi(t)=t^{\frac{5}{2}}$	
n	θ	Ave.Iter	Ave.Cpu	Ave.Iter	Ave.Cpu	Ave.Iter	Ave.Cpu
				Problem 1			
80	0.2	51.0	0.0172	50.0	0.0165	47.2	0.0154
	0.6	14.0	0.0069	14.0	0.0066	14.0	0.0064
120	0.2	51.5	0.0403	50.5	0.0322	47.4	0.0300
	0.6	14.0	0.0117	14.0	0.0117	14.0	0.0117
160	0.2	53.0	0.0516	51.0	0.0508	49.0	0.0482
	0.6	14.0	0.0208	14.0	0.0205	14.0	0.0201
200	0.2	53.0	0.1573	51.8	0.1557	49.5	0.1415
	0.6	14.5	0.0577	14.5	0.0573	14.5	0.0572
240	0.2	53.3	0.2221	52.0	0.2201	50.0	0.2061
	0.6	14.0	0.0856	14.0	0.0849	14.0	0.0847
280	0.2	53.5	0.3321	52.3	0.3310	50.0	0.3072
	0.6	14.0	0.1252	14.0	0.1272	14.0	0.1260
320	0.2	53.8	0.4823	52.5	0.4700	51.0	0.4275
	0.6	14.0	0.1689	14.0	0.1680	14.0	0.1680
360	0.2	54.0	0.6504	53.0	0.5390	52.0	0.4940
	0.6	14.0	0.2177	14.0	0.2190	14.0	0.2112
400	0.2	56.0	0.7569	54.8	0.7397	53.6	0.6514
	0.6	14.0	0.3079	14.0	0.3108	14.0	0.3076
440	0.2	57.0	1.1119	55.3	1.0357	54.0	1.0018
	0.6	14.0	0.4152	14.0	0.4167	14.0	0.4159
				Problem 2			
60	0.2	50.0	0.0109	46.0	0.0101	44.8	0.0091
	0.6	14.0	0.0041	14.0	0.0041	14.0	0.0039
110	0.2	51.8	0.0331	47.5	0.0310	46.0	0.0266
	0.6	14.0	0.0103	14.0	0.0106	14.0	0.0101
160	0.2	52.0	0.0744	50.5	0.0660	49.5	0.0630
	0.6	14.0	0.0212	14.0	0.0219	14.0	0.0204
210	0.2	53.5	0.2181	50.8	0.1964	50.0	0.1915
	0.6	14.0	0.0646	14.0	0.0647	14.0	0.0646
260	0.2	53.5	0.3346	51.3	0.3090	50.5	0.2952
	0.6	14.8	0.1104	14.8	0.1104	14.8	0.1104
310	0.2	54.0	0.4609	52.0	0.4375	51.0	0.4145
	0.6	15.0	0.1516	15.0	0.1540	15.0	0.1541
360	0.2	55.0	0.6463	53.0	0.6007	52.8	0.5350
	0.6	15.0	0.2215	15.0	0.2244	15.0	0.2230
410	0.2	55.8	0.8499	53.8	0.8341	53.0	0.7795
	0.6	15.0	0.3375	15.0	0.3334	15.0	0.3329
460	0.2	56.0	1.0896	55.3	1.0690	54.3	0.9556
	0.6	15.0	0.4790	15.0	0.4748	15.0	0.4768
510	0.2	56.5	1.7997	56.0	1.6873	55.0	1.5827
	0.6	15.0	0.6758	15.0	0.6764	15.0	0.6755

For a given ω, the optimality conditions of this model are as follows

$$\begin{aligned} u_i\pi_i &= \omega_i, \quad \forall i \\ x_{ij}(p_j - u_{ij}\pi_i) &= 0, \quad \forall i,j \\ p_j - u_{ij}\pi_i &\geq 0, \quad \forall i,j \\ \sum_{i=1}^{n} x_{ij} &= 1, \quad \forall j \\ u_i - \sum_{j=1}^{n} u_{ij}x_{ij} &= 0, \quad \forall i \\ u_{ij}, x_{ij}, \pi_i &\geq 0. \quad \forall i,j \end{aligned} \tag{4.1}$$

It is show that there exists $\omega \geq 0$ such that under these conditions, $p_i = \omega_i$, that is, there exist (u, x) and

(p, π) such that

$$u_i \pi_i = p_i, \quad \forall i \tag{4.2}$$

$$x_{ij}(p_j - u_{ij}\pi_i) = 0, \quad \forall i, j \tag{4.3}$$

$$p_j - u_{ij}\pi_i \geq 0, \quad \forall i, j \tag{4.4}$$

$$\sum_{i=1}^{n} x_{ij} = 1, \quad \forall j \tag{4.5}$$

$$u_i - \sum_{j=1}^{n} u_{ij} x_{ij} = 0, \quad \forall i \tag{4.6}$$

$$u_{ij}, x_{ij}, \pi_i \geq 0. \quad \forall i, j \tag{4.7}$$

Define

$$x = \left(x_{11}, x_{12}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, \ldots, x_{n1}, \ldots, x_{mn}, u_1, \ldots, u_n\right)^T \in \mathbb{R}^{n^2+n},$$

$$s = \left(s_{11}, s_{12}, \ldots, s_{1n}, s_{21}, \ldots, s_{2n}, \ldots, s_{n1}, \ldots, s_{mn}, \pi_1, \ldots, \pi_n\right)^T \in \mathbb{R}^{n^2+n},$$

$$y = \left(p_1, \ldots, p_n, \pi_1, \ldots, \pi_n\right)^T \in \mathbb{R}^{2n}.$$

Let $u_i = x_j$, $p_i = y_j$, $\pi_i = s_j$. Substituting these variables into (4.2) and (4.3), we obtain

$$x_j s_j = y_j.$$

By introducing slack variables

$$s_{ij} = p_j - u_{ij}\pi_i \geq 0,$$

(4.4) could be written as $s = A^T y$, where

$$A^T = \begin{pmatrix}
1 & 0 & \cdots & 0 & -u_{11} & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & -u_{12} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & -u_{1n} & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 0 & -u_{21} & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & -u_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 & -u_{2n} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & -u_{n1} \\
0 & 1 & \cdots & 0 & 0 & 0 & \cdots & -u_{n2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & -u_{nn} \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1
\end{pmatrix} \in \mathbb{R}^{(n^2+n)\times 2n}.$$

Then (4.5) and (4.6) could be written as $Ax = b$, where

$$b = \begin{pmatrix} e \\ 0 \end{pmatrix} \in \mathbb{R}^{2n}.$$

Thus, the optimality conditions of Arrow-Debreu model (4.1) may be reformulated as

$$\begin{aligned}
x_j s_j &= y_j, \\
s - A^T y &= 0, \\
Ax &= b, \\
x, s &\geq 0.
\end{aligned} \tag{4.8}$$

For a more general convex optimization problem

$$\begin{aligned}
\max \quad & \sum_{j=1} \omega_j \log u_{ij} \\
\text{s.t.} \quad & Ax = b, \\
& x \geq 0,
\end{aligned} \tag{4.9}$$

where A is a full row rank $m \times n$ matrix, and

$$b = \begin{pmatrix} e \\ 0 \end{pmatrix} \in \mathbb{R}^m.$$

Ye [34] proved the optimality condition of the self-dual weighted analytic center of the feasible set (4.9) also corresponds to (4.8), if the feasible set of (4.9) is bounded with nonempty interior and the dual feasibility $A^T y \geq 0$ is satisfied, with $y_1, y_2, \ldots, y_l \geq 0$ $(l \leq m)$. This equivalence reveals that the process of determining the equilibrium solution for the Arrow-Debreu model fundamentally reduces to computing the self-dual weighted analytic center of (4.9). In addition, by Kakutani fixed point theorem, there must be a " fixed point " ω such that the weights and prices are completely matched. In other words, there holds

$$\begin{aligned} x_j s_j &= \omega_j, \\ s - A^T y &= 0, \\ Ax &= b, \\ x, s &\geq 0, \end{aligned}$$

which is a special case of the WLCP

$$\begin{aligned} xs &= \omega, \\ Bx + Cs + Dy &= a, \\ x, s &\geq 0, \end{aligned} \tag{4.10}$$

where

$$B = \begin{pmatrix} A \\ 0 \end{pmatrix} \in \mathbb{R}^{(n^2+3n)\times(n^2+n)}, \quad C = \begin{pmatrix} 0 \\ I \end{pmatrix} \in \mathbb{R}^{(n^2+3n)\times(n^2+n)},$$
$$D = \begin{pmatrix} 0 \\ -A^T \end{pmatrix} \in \mathbb{R}^{(n^2+3n)\times(n^2+n)}, \quad a = \begin{pmatrix} b \\ 0 \end{pmatrix} \in \mathbb{R}^{2n(n^2+n)}.$$

If (x, s, y) is a solution of (4.10), then (x, s) satisfies WHLCP

$$\begin{aligned} xs &= \omega, \\ Qx + Rs &= d, \\ x, s &\geq 0, \end{aligned}$$

where

$$K \in \ker D^T,\ Q = K^T B \in \mathbb{R}^{(n^2+n)\times(n^2+n)},$$
$$R = K^T C \in \mathbb{R}^{(n^2+n)\times(n^2+n)},\ d = K^T a \in \mathbb{R}^{n^2+n}.$$

We use Algorithm 1 to solve the randomly generated general Arrow-Debreu market equilibrium problem with $n \in \{50, 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000, 1100, 1200, 1300, 1400, 1500, 1600, 1700, 1800, 1900, 2000\}$. Choose the strict initial point $(x^0, s^0) = (e, 0.1e)$, and the weight $\omega = 0.05e$. When $\theta \in \{0.2, 0.3, 0.4, 0.5, 0.6\}$, the results of Algorithm 1 for solving the Arrow-Debreu equilibrium market of different dimensions are listed in Table 7.

Table 7: Numerical results for Problem 3 with different dimensions and θ

$\theta \rightarrow$	0.2		0.3		0.4		0.5		0.6	
$n \downarrow$	Iter	Cpu	Iter	Cpu	Iter	Cpu	Iter	Cpu	Iter	Cpu
50	37	0.0123	23	0.0079	16	0.0099	12	0.0084	9	0.0197
100	39	0.0308	24	0.0190	17	0.0223	13	0.0223	10	0.0211
200	40	0.1812	25	0.1157	18	0.0838	13	0.0651	10	0.0634
300	41	0.4186	26	0.2816	18	0.1984	14	0.1440	10	0.1083
400	42	0.9786	26	0.5978	19	0.4638	14	0.3033	11	0.2617
500	42	1.4503	27	0.9267	19	0.6580	14	0.4660	11	0.4223
600	43	2.4764	27	1.5614	19	1.1236	14	0.8203	11	0.6844
700	43	3.4899	27	2.1489	19	1.5639	14	1.1464	11	1.0672
800	43	4.5077	27	2.8644	19	2.0394	14	1.4906	11	1.2248
900	44	5.7317	27	3.6413	19	2.5322	14	1.8272	11	1.5477
1000	44	7.4640	28	4.7466	19	3.5418	14	2.3532	11	2.0651
1100	44	10.4969	28	11.0890	20	4.6589	15	3.5666	11	2.7128
1200	44	14.8522	28	8.0615	20	5.8452	15	4.4541	11	3.3759
1300	44	14.2556	28	9.6827	20	6.5981	15	4.9588	11	4.1907
1400	45	17.9057	28	11.0895	20	9.0522	15	6.8046	11	5.5447
1500	45	21.8155	28	14.1280	20	9.8693	15	5.0644	11	6.7494
1600	45	24.9060	28	15.7998	20	11.4919	15	9.7651	11	8.1993
1700	45	29.3023	28	20.2550	20	13.0583	15	11.6210	11	8.0898
1800	45	34.0388	28	21.2031	20	15.1370	15	13.8330	11	10.5106
1900	45	37.8866	29	24.7797	20	17.0764	15	14.4699	11	9.8670
2000	45	52.2749	29	33.0634	20	25.5592	15	16.0896	11	14.1616

From Table 7, we observe that both runtime and iteration counts are influenced by n and θ. When θ remains fixed, the runtime of Algorithm 1 rises notably with an increase in the problem dimension n, while the iterations counts grows slightly with increasing n. Moreover, when the problem dimension is fixed, the runtime and iteration counts of Algorithm 1 decrease as the parameter θ increases. Anyway, all the examples of Arrow-Debreu market equilibrium problems are efficiently solved by Algorithm 1.

Acknowledgements The first author's work is supported by the National Natural Science Foundation of China (No. 12361064), Guangxi Natural Science Foundation (No. 2021GXNSFAA220034) and the Science and Technology Project of Guangxi (Guike AD25069086), China. The fourth author's work is supported by National Science and Technology Council, Taiwan (NSTC 112-2115-M-003-012-MY2, NSTC 113-2124-M-003-002).

References

[1] B. Kheirfam, Complexity analysis of a full-Newton step interior-point method for monotone weighted linear complementarity problems, *J. Optim. Theory Appl.*, 202 (2024) 133–145.

[2] B. Kheirfam, A predictor-corrector interior-point algorithm for $P_*(\kappa)-$horizontal linear complementarity problem, *Numer. Algorithms*, 66 (2014) 349–361.

[3] B. Kheirfam and A. Nasrollahi, A full-Newton step interior-point method based on a class of specific algebra transformation, *Fundam. Inform.*, 163 (2018) 325–337.

[4] C. Roos, T. Terlaky and J.P. Vial, Theory and algorithms for linear optimization: an interior point approach, *Wiley, Chichester*, (1997).

[5] D. Jalota, M. Pavone, Q. Qi and Y.Y. Ye, Fisher markets with linear constraints: Equilibrium properties and efficient distributed algorithms, *Games Econ. Behav.*, 141 (2023) 223–260.

[6] F.A. Potra, Weighted complementarity problems-a new paradigm for computing equilibria, *SIAM J. Optim.*, 22 (2012) 1634–1654.

[7] F.A. Potra, Sufficient weighted complementarity problems, *Comput. Optim. Appl.*, 64 (2016) 467–488.

[8] G. Sonnevend, J. Stoer and G.Y. Zhao, On the complexity of following the central path of linear programs by linear extrapolation II, *Math. Program.*, 52 (1991) 527–553.

[9] H. Väliaho, P_*-matrices are just sufficient, *Linear Algebra Appl.*, 239 (1996) 103–108.

[10] H. Väliaho, Criteria for sufficient matrices, *Linear Algebra Appl.*, 233 (1996) 109–129.

[11] H. Väliaho, Determining the handicap of a sufficient matrix, *Linear Algebra Appl.*, 253 (1997) 279–298.

[12] J.S. Chen, J.J. Ye, J. Zhang and J.C. Zhou, Exact formula for the second-order tangent set of the second-order cone complementarity set, *SIAM J. Optim.*, 29 (2019) 2986–3011.

[13] J.Y. Han, N.H. Xiu and H.D. Qi, Nonlinear complementarity theory and algorithm, *Shanghai Science and Technology Press, Shanghai*, (2006).

[14] J. Miao, A quadratically convergent-iteration algorithm for the $P_*(\kappa)-$matrix linear complementarity problem, *Math. Program.*, 69 (1995) 355–368.

[15] J.Y. Tang, J.C. Zhou and H.C. Zhang, An accelerated smoothing newton method with cubic convergence for weighted complementarity problems, *J. Optim. Theory Appl.*, 196 (2023) 641–665.

[16] K.M. Anstreicher, Interior-point algorithms for a generalization of linear programming and weighted centering, *Optim. Methods Softw.*, 27 (2011) 605–612.

[17] K.J. Arrow and G. Debreu, Existence of a competitive equilibrium for a competitive economy, *Econometrica*, 22 (1954) 265–290.

[18] K. McShane, Superlinearly convergent $O(\sqrt{n}L)-$iteration interior-point algorithms for linear programming and the monotone linear complementarity problem, *SIAM J. Optim.*, 4 (1994) 247–261.

[19] L. Walras, Elements of Pure Economics, or the Theory of Social Wealth, *London School of Economics and Political Science*, vol. 1, Routledge, London, (1874).

[20] M. Kojima, N. Megiddo, T. Noma and A. Yoshise, A unified approach to interior point algorithms for linear complementarity problems, *Oper. Res. Lett.*, 10 (1991) 247–254.

[21] N. Karmarkar, A new polynomial-time algorithm for linear programming, *Combinatorica*, 4 (1984) 373–395.

[22] N. Lu and Z.H. Huang, A smoothing Newton algorithm for a class of non-monotonic symmetric cone linear complementarity problems, *J. Optim. Theory Appl.*, 161 (2014) 446–464.

[23] N. Moussaoui and M. Achache, A weighted-path following interior-point algorithm for convex quadratic optimization based on modified search directions, *Stat. Optim. Inf. Comput.*, 10 (2022) 873–889.

[24] R.W. Cottle, The principal pivoting method revisited, *Math. Program.*, 48 (1990) 369–385.

[25] R.W. Cottle, J.-S. Pang and V. Venkateswaran, Sufficient matrices and the linear complementarity problem, *Linear Algebra Appl.*, 114 (1989) 231–249.

[26] S. Asadi, Z. Darvay, G. Lesaja, N. Mahdavi-Amiri and F. Potra, A full-Newton step interior-point method for monotone weighted linear complementarity problems, *J. Optim. Theory Appl.*, 186 (2020) 864–878.

[27] S. Mizuno, M.J. Todd and Y. Ye, On adaptive-step primal-dual interior-point algorithms for linear programming, *Math. Oper. Res.*, 18 (1993) 964–981.

[28] T. Illés and M. Nagy, A Mizuno-Todd-Ye type predictor-corrector algorithm for sufficient linear complementarity problems, *Eur. J. Oper. Res.*, 181 (2007) 1097–1111.

[29] W. Grimes, Path-following interior-point algorithm for monotone linear complementarity problems, *Asian-Eur. J. Math.*, 15 (2022) 225–170.

[30] W. Grimes and M. Achache, A path-following interior-point algorithm for monotone LCP based on a modified newton search direction, *RAIRO-Oper. Res.*, 57 (2023) 1059–1073.

[31] X.N. Chi, M.S. Gowda and J.Y. Tao, The weighted horizontal linear complementarity problem on a Euclidean Jordan Algebra, *J. Glob. Optim.*, 73 (2019) 153–169.

[32] X.N. Chi, G.Q. Wang and G. Lesaja, Kernel-based full-Newton step feasible interior-point algorithm for $P_*(\kappa)-$weighted linear complementarity problem, *J. Optim. Theory Appl.*, 202 (2024) 108–132.

[33] X.H. Miao and J.S. Chen, On matrix characterizations for $P-$property of the linear transformation in second-order cone linear complementarity problems, *Linear Algebra Appl.*, 613 (2021) 271–294.

[34] Y. Ye, A path to the Arrow-Debreu competitive market equilibrium, *Math. Program.*, 111 (2008) 315–348.

[35] Z. Darvay, New interior point algorithms in linear programming, *Adv. Model. Optim.*, 5 (2003) 51–92.

[36] Z. Darvay, T. Illés and P.R. Rig, Predictor-corrector interior-point algorithm for $P_*(\kappa)-$linear complementarity problems based on a new type of algebraic equivalent transformation technique, *Eur. J. Oper. Res.*, 298 (2022) 25–35.

XIAONI CHI
School of Mathematics and Computing Science

Guilin University of Electronic Technology
Guilin 541004, Guangxi, China;

Guangxi Colleges and Universities Key Laboratory of
Data Analysis and Computation
Guilin University of Electronic Technology
Guilin 541004, Guangxi, China;

Center for Applied Mathematics of Guangxi (GUET)
Guilin 541004, Guangxi, China
E-mail address: chixiaoni@126.com

LIN GAN
School of Mathematics and Computing Science
Guilin University of Electronic Technology
Guilin 541004, Guangxi, China
E-mail address: ganlin163@126.com

QIAN CHENG
School of Mathematics and Computing Science
Guilin University of Electronic Technology
Guilin 541004, Guangxi, China
E-mail address: chengqian07@126.com

JEIN-SHAN CHEN
Department of Mathematics
National Taiwan Normal University
Taipei 116059, Taiwan
E-mail address: jschen@math.ntnu.edu.tw