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Variational analysis of circular cone programs

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This paper conducts variational analysis of circular programs, which form a new class of optimization problems in nonsymmetric conic programming, important for optimization theory and its applications. First, we derive explicit formulas in terms of the initial problem data to calculate various generalized derivatives/co-derivatives of the projection operator associated with the circular cone. Then we apply generalized differentiation and other tools of variational analysis to establish complete characterizations of full and tilt stability of locally optimal solutions to parameterized circular programs.

Keywords: variational analysis; optimization; generalized differentiation; conic programming; circular cone; second-order cone; projection operator; full and tilt stability

AMS Subject Classifications: 90C30; 90C31; 49J52

1. Introduction

The *circular cone* [1,2] is a pointed, closed, convex cone having hyperspherical sections orthogonal to its axis of revolution about which the cone is invariant to rotation. Let its half-aperture angle be $\theta \in (0, \frac{\pi}{2})$. Then the n-dimensional circular cone denoted by $\mathcal{L}_\theta$ can be expressed as follows (see Figure 1):

$$\begin{aligned}\mathcal{L}_\theta :&= \left\{x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \,\middle|\, \|x\| \cos\theta \le x_1\right\} \\ &= \left\{x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \,\middle|\, \|x_2\| \le x_1 \tan\theta\right\}. \end{aligned} \tag{1.1}$$

When $\theta = 45°$, the circular cone reduces to the well-known *second-order cone* (SOC for short, also known as the Lorentz cone and the ice-cream cone) given by

$$\begin{aligned}\mathcal{K}^n :&= \left\{x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \,\middle|\, \|x_2\| \le x_1\right\} \\ &= \left\{x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \,\middle|\, \|x\| \cos 45° \le x_1\right\}. \end{aligned} \tag{1.2}$$

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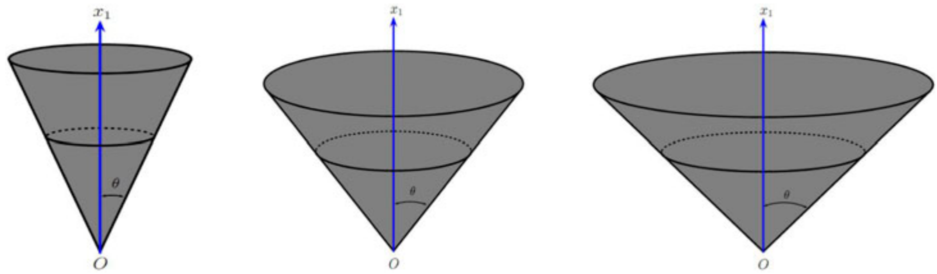

(a) $0 < \theta < 45°$ (b) $\theta = 45°$ (c) $45° < \theta < 90°$

Figure 1. The graphs of circular cones.

Concerning SOC, for any vector $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ we can decompose it as

$$x = \lambda_1(x) u_x^{(1)} + \lambda_2(x) u_x^{(2)}, \tag{1.3}$$

where $\lambda_1(x), \lambda_2(x)$ and $u_x^{(1)}, u_x^{(2)}$ are the *spectral values* and the associated *spectral vectors* of x relative to $\mathcal{K}^n$ defined by, respectively,

$$\begin{aligned} \lambda_i(x) &:= x_1 + (-1)^i \|x_2\|, \\ u_x^{(i)} &:= \begin{cases} \frac{1}{2}\left(1, (-1)^i \dfrac{x_2}{\|x_2\|}\right) & \text{if } x_2 \neq 0, \\ \frac{1}{2}\left(1, (-1)^i w\right) & \text{if } x_2 = 0, \quad i = 1, 2, \end{cases} \end{aligned}$$

with w being any unit vector in $\mathbb{R}^{n-1}$. If $x_2 \neq 0$, decomposition (1.3) is unique. Using this decomposition, for any $f : \mathbb{R} \to \mathbb{R}$ we consider [3,4] the vector function associated with $\mathcal{K}^n$, $n \geq 1$ by

$$f^{\text{soc}}(x) := f(\lambda_1(x)) u_x^{(1)} + f(\lambda_2(x)) u_x^{(2)}, \quad x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}. \tag{1.4}$$

If f is defined only on some subset of $\mathbb{R}$, then f^{soc} is defined on the corresponding subset of $\mathbb{R}^n$. Definition (1.4) is unambiguous whether $x_2 \neq 0$ or $x_2 = 0$.

Note that circular cone systems described by (1.1) with $\theta \neq 45°$ naturally arises in many real-life engineering problems. In particular, we refer the reader to the recent paper [5] and the bibliographies therein to the important class of optimal grasping manipulation problems for multi-fingered robots in which the grasping force of the ith finger is subject to a contact friction constraint given by

$$\left\|(u_{i2}, u_{i3})\right\| \leq \mu u_{i1}, \tag{1.5}$$

where u_{i1} is the normal force of the ith finger, u_{i2} and u_{i3} are the friction forces of the ith finger, $\|\cdot\|$ is the 2-norm and μ is the friction coefficient; see Figure 2.

It is easy to see that (1.5) is a circular cone constraint corresponding to the description $u_i = (u_{i1}, u_{i2}, u_{i3}) \in \mathcal{L}_\theta$ in (1.1) with the angle $\theta = \tan^{-1}\mu < 45°$.

Observe that a possible way to deal with circular cone constraints is to scale $\mathcal{L}_\theta$ as SOC by

$$\mathcal{L}_\theta = A^{-1}\mathcal{K}^n \quad \text{and} \quad \mathcal{K}^n = A\mathcal{L}_\theta \quad \text{with} \quad A = \begin{bmatrix} \tan\theta & 0 \\ 0 & I \end{bmatrix}, \tag{1.6}$$

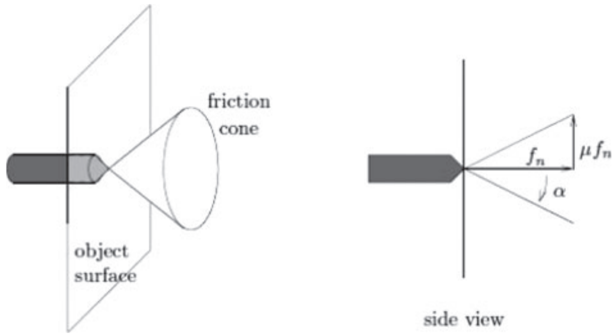

Figure 2. The grasping force forms a circular cone where $\alpha = \tan^{-1} \mu < 45°$.

which is justified in [2, Theorem 2.1]. However, this approach may not be acceptable from both theoretical and numerical viewpoints. Indeed, the ‘scaling’ step can cause undesirable numerical performance due to round-off errors in computers, which has been confirmed by experiments. Furthermore, we will see in what follows that applying (1.6) does not help to obtain some major results of the paper while being useful in deriving the other ones.

Optimization problems with both SOC and circular cone constraints belong to a broad and important class in modern optimization theory known as *conic* or *cone-constrained programming*; see, e.g. [6–8] and the references therein. However, the main difference between circular cone constraints and those given by SOC and most of the other constraint systems in conic programming is that the circular cone $\mathcal{L}_\theta$ is *non-self-dual*, i.e. *nonsymmetric*, which makes its study more challenging and rather limited.

In contrast to symmetric conic programming, we are not familiar with a variety of publications devoted to their nonsymmetric counterparts. Referring the reader to [9–12] and the bibliographies therein, observe that there is no unified way to handle nonsymmetric cone constraints, and each study uses certain specific features of the nonsymmetric cones under consideration. The previous papers [2,13] concerning the circular cone show that some properties holding in the SOC framework can be extended to the circular cone setting. At the same time, some other SOC properties fail to be satisfied for the general nonsymmetric circular cone, where the angle $\theta \neq 45°$ plays a crucial role; see [14].

This paper is mainly devoted to two major interrelated issues of variational analysis and optimization for problems involving circular cone constraints. Our first goal is to calculate, entirely in terms of the initial circular cone data, some generalized differential constructions of variational analysis that have been proven to be important for various aspects of optimization. Namely, we derive explicit formulas to calculate generalized differential constructions for the (metric) *projection* operator associated with the general circular cone that are known as the *B-subdifferential, directional derivative, graphical derivative, regular derivative, regular coderivative*, and (limiting) *coderivative*. Except the B-subdifferential and the (regular and limiting) coderivatives, the results obtained are new even for the symmetric SOC case. The obtained calculations allow us, in particular, to prove the *strong semismoothness* of the projection operator onto the circular cone, which is important for many applications including those to numerical optimization. Furthermore, we establish new relationships between these generalized differential constructions for the projection operator onto the circular cone and the metric projection onto the orthogonal spaces to the spectral vectors in the circular cone representation.

The second major goal of this paper is to completely characterize the notions of *tilt stability* and *full stability* of mathematical programs with circular cone constraints. These fundamental stability concepts were introduced in optimization theory by Rockafellar and his collaborators [15,16] and then have been intensively studied by many researchers, especially in the recent years, for various classes of optimization problems; see, e.g. [7,8,16–27] and the references therein. The construction of the *second-order subdifferential/generalized Hessian* in the sense of Mordukhovich [28] (i.e. the coderivative of the first-order subgradient mapping) plays a crucial role in the characterization of tilt and full stability obtained in the literature. In this paper we establish, by using the obtained second-order calculations and the recent results of [25], complete characterizations of full and tilt stability for locally optimal solutions to mathematical programs with circular cone constraints expressed entirely in terms of the initial program data via certain second-order growth and strong sufficient optimality conditions under appropriate constraint qualifications.

The rest of the paper is organized as follows. In Section 2 we recall and briefly discuss the generalized differential constructions of variational analysis employed in deriving the main results of this paper. Section 3 is devoted to calculating the generalized derivatives listed above for the projection operator onto the circular cone. In Section 4 we represent these generalized differential constructions for the aforementioned projection operator via the orthogonal projections generated by the spectral vectors of the circular cone. Finally, Section 5 applies the second-order subdifferential of the indicator function associated with the circular cone and related to the above coderivative calculations to establish complete characterizations of full and tilt stability of mathematical programs with circular cone constraints.

Throughout the paper we use the standard notation and terminology of variational analysis; see, e.g. [29,30]. Given a set-valued mapping/multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, recall that the constructions

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \Big\{ y \in \mathbb{R}^m \Big| \ \exists \text{ sequences } x_k \to \bar{x},\ y_k \to y \text{ such that } y_k \in F(x_k) \text{ for all } k \in I\!N := \{1, 2, \ldots\} \Big\}, \tag{1.7}$$

$$\operatorname*{Lim\,inf}_{x\to\bar{x}} F(x) := \Big\{ y \in \mathbb{R}^m \Big| \ \text{for any } x_k \to \bar{x},\ \exists\, y_k \to y \text{ such that } y_k \in F(x_k) \text{ for all } k \in I\!N \Big\} \tag{1.8}$$

are known as the (Painlevé–Kuratowski) *outer limit* and *inner limit* of F as $x \to \bar{x}$, respectively. For a set $\Omega \subset \mathbb{R}^n$, the symbol $x \xrightarrow{\Omega} \bar{x}$ signifies that $x \to \Omega$ with $x \in \Omega$.

2. Tools of variational analysis

In this section we briefly review those tools of generalized differentiation in variational analysis, which are widely used in the subsequent sections. We start with geometric notions.

Given a set $\Omega \subset I\!R^n$ locally closed around $x \in \Omega$, the (Bouligand–Severi) *tangent/contingent cone* to Ω at $\bar{x} \in \Omega$ is defined by

$$T_\Omega(\bar{x}) := \operatorname*{Lim\,sup}_{t\downarrow 0} \frac{\Omega - \bar{x}}{t} = \Big\{ d \in I\!R^n \Big| \exists t_k \downarrow 0,\ d_k \to d \text{ with } \bar{x} + t_k d_k \in \Omega \Big\} \tag{2.1}$$

via the outer limit (1.7), while the (Clarke) *regular tangent cone* to Ω at $\bar{x} \in \Omega$ is given by

$$\widehat{T}_\Omega(\bar{x}) := \operatorname*{Lim\,inf}_{x \xrightarrow{\Omega} \bar{x}} T_\Omega(x) \tag{2.2}$$

via the inner limit (1.8). The (Fréchet) *regular normal cone* to Ω at $\bar{x} \in \Omega$ is

$$\widehat{N}_\Omega(\bar{x}) := \Big\{ z \in \mathbb{R}^n \Big| \, \langle z, x - \bar{x} \rangle \le o\big(\|x - \bar{x}\|\big) \ \text{ for all } \ x \in \Omega \Big\}, \tag{2.3}$$

and the (Mordukhovich, limiting) *normal cone* to Ω at $\bar{x} \in \Omega$ can be equivalently defined by

$$N_\Omega(\bar{x}) := \operatorname*{Lim\,sup}_{x \xrightarrow{\Omega} \bar{x}} \widehat{N}_\Omega(x) = \operatorname*{Lim\,sup}_{x \to \bar{x}} \Big\{ \text{cone}\big[x - \Pi_\Omega(x)\big] \Big\}, \tag{2.4}$$

where Π_Ω denotes the (Euclidean) *projection* operator onto Ω, and where 'cone' stands for the conic (may not be convex) hull of the set in question.

Consider next a set-valued mapping $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with its *graph* and *domain* given by

$$\text{gph}\, H := \big\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^m \big| \, y \in H(x) \big\} \quad \text{and} \quad \text{dom}\, H := \big\{ x \in \mathbb{R}^n \big| \, H(x) \ne \emptyset \big\},$$

respectively. The *graphical derivative* of H at $(\bar{x}, \bar{y}) \in \text{gph}\, H$ is defined by

$$DH(\bar{x}, \bar{y})(w) := \big\{ z \in \mathbb{R}^m \big| \, (w, z) \in T_{\text{gph}\, H}(\bar{x}, \bar{y}) \big\}, \quad w \in \mathbb{R}^n, \tag{2.5}$$

via the tangent cone (2.1), while the (limiting) *coderivative* is defined via the normal cone (2.4) by

$$D^* H(\bar{x}, \bar{y})(y^*) := \big\{ x^* \in \mathbb{R}^n \big| \, (x^*, -y^*) \in N_{\text{gph}\, H}(\bar{x}, \bar{y}) \big\}, \quad y^* \in \mathbb{R}^m, \tag{2.6}$$

where we drop $\bar{y}$ in the derivative/coderivative notion if H is single-valued at $\bar{x}$. Similarly, the *regular derivative* and the *regular coderivative* of H at $(\bar{x}, \bar{y})$ are defined via, respectively, (2.2) and (2.3) by

$$\widehat{D}H(\bar{x}, \bar{y})(w) := \big\{ z \in \mathbb{R}^m \big| \, (w, z) \in \widehat{T}_{\text{gph}\, H}(\bar{x}, \bar{y}) \big\}, \quad w \in \mathbb{R}^n, \tag{2.7}$$

$$\widehat{D}^* H(\bar{x}, \bar{y})(y^*) := \big\{ x^* \in \mathbb{R}^n \big| \, (x^*, -y^*) \in \widehat{N}_{\text{gph}\, H}(\bar{x}, \bar{y}) \big\}, \quad y^* \in \mathbb{R}^m. \tag{2.8}$$

Now let $f : \mathbb{R}^n \to \overline{\mathbb{R}} := (-\infty, \infty]$ be an extended real-valued function finite at $\bar{x} \in \mathbb{R}^n$. To define the second-order subdifferential construction needed in what follows, we proceed in the way of [28,29] and begin with the first-order (limiting) *subdifferential* of f at $\bar{x}$ given by

$$\partial f(\bar{x}) := \big\{ v \in \mathbb{R}^n \big| \, (v, -1) \in N_{\text{epi}\, f}\big(\bar{x}, f(\bar{x})\big) \big\} \tag{2.9}$$

via the normal cone (2.4) of the *epigraph* $\{(x, \mu) \in \mathbb{R}^n \times \mathbb{R} |\ \mu \ge f(x)\}$ of f. Observe the representation $N_\Omega(\bar{x}) = \partial \delta_\Omega(\bar{x})$ the normal cone (2.4) via the subdifferential (2.9) of the *indicator function* $\delta_\Omega(x)$ of Ω equal to 0 if $x \in \Omega$ and ∞ otherwise. The *second-order subdifferential* (or generalized Hessian) of f at $\bar{x}$ relative to $\bar{y} \in \partial f(\bar{x})$ is defined as the coderivative (2.6) of the first-order subdifferential (2.9) by

$$\partial^2 f(\bar{x}, \bar{y})(u) := (D^* \partial f)(\bar{x}, \bar{y})(u), \quad u \in \mathbb{R}^n. \tag{2.10}$$

Finally in this section, consider a single-valued mapping $F\colon \mathbb{R}^n \to \mathbb{R}^m$ locally *Lipschitzian* around $\bar{x}$ and recall that F is almost everywhere differentiable in a neighbourhood of $\bar{x}$ with the derivative $\nabla F(x)$ by the classical Rademacher theorem; see [30]. Then the *B-subdifferential* of F at $\bar{x}$ is defined by

$$\partial_B F(\bar{x}) := \left\{ \lim_{x_k \to x} \nabla F(x_k) \,\middle|\, F \text{ is differentiable at } x_k \right\}. \tag{2.11}$$

Recall also that F is *directionally differentiable* at $\bar{x}$ if the limit

$$F'(x;h) := \lim_{t\to 0^+} \frac{F(x+th)-F(x)}{t} \quad \text{exists for all } h \in \mathbb{R}^n. \tag{2.12}$$

Having this, F is said to be *semismooth* at $\bar{x}$ if F is locally Lipschitzian around $\bar{x}$, directionally differentiable at this point, and satisfies the relationship

$$Vh - F'(x;h) = o\big(\|h\|\big) \quad \text{for any } V \in \operatorname{co}\partial_B F(x+h) \text{ as } h \to 0. \tag{2.13}$$

Furthermore, F is *ρ-order semismooth* at x with $0<\rho<\infty$ if (2.13) is replaced above by

$$Vh - F'(x;h) = O\big(\|h\|^{1+\rho}\big) \quad \text{for any } V \in \operatorname{co}\partial_B F(x+h) \text{ as } h \to 0. \tag{2.14}$$

The case of $\rho = 1$ in (2.13) corresponds to *strongly semismooth* mappings.

3. Generalized differentiation of the projection operator onto the circular cone

In this section we derive precise formulas for calculating the above generalized derivatives of the projection operator onto the circular cone (1.1). First we recall the following *spectral decomposition* from [2, Theorem 3.1] of any vector $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ relative to the circular cone $\mathcal{L}_\theta$:

$$x = \lambda_1(x)u_x^1 + \lambda_2(x)u_x^2, \tag{3.1}$$

where the *spectral values* $\lambda_1(x)$ and $\lambda_2(x)$ are defined by

$$\lambda_1(x) := x_1 - \|x_2\|\text{ctan}\theta, \quad \lambda_2(x) := x_1 + \|x_2\|\tan\theta, \tag{3.2}$$

and where the *spectral vectors* u_x^1 and u_x^2 are written as

$$u_x^1 := \frac{1}{1+\text{ctan}^2\theta}\begin{bmatrix} 1 & 0 \\ 0 & \text{ctan}\theta \end{bmatrix}\begin{bmatrix} 1 \\ -\bar{x}_2 \end{bmatrix}, \quad u_x^2 := \frac{1}{1+\tan^2\theta}\begin{bmatrix} 1 & 0 \\ 0 & \tan\theta \end{bmatrix}\begin{bmatrix} 1 \\ \bar{x}_2 \end{bmatrix} \tag{3.3}$$

with $\bar{x}_2 := x_2/\|x_2\|$ if $x_2 \neq 0$ and $\bar{x}_2$ equal to any unit vector $w \in \mathbb{R}^{n-1}$ otherwise. Given any $f\colon \mathbb{R} \to \mathbb{R}$ we construct the vector function

$$f^{\mathcal{L}_\theta}(x) := f\big(\lambda_1(x)\big)u_x^1 + f\big(\lambda_2(x)\big)u_x^2. \tag{3.4}$$

associated with circular cone. It follows from [2] that the projection $\Pi_{\mathcal{L}_\theta}(x)$ of x onto $\mathcal{L}_\theta$, which is a single-valued and Lipschitzian operator, corresponds to $f(t) := (t)_+ = \max\{t, 0\}$ in (3.4), i.e. we have

$$\Pi_{\mathcal{L}_\theta}(x) = \big(x_1 - \|x_2\|\text{ctan}\theta\big)_+ u_x^1 + \big(x_1 + \|x_2\|\tan\theta\big)_+ u_x^2. \tag{3.5}$$

Our first result in this section provides a complete calculation of the *B-subdifferential* (2.11) of the projection operator (3.5) entirely in terms of the initial data of the general circular cone (1.1). This result is widely used in what follows.

LEMMA 3.1 (calculating the B-subdifferential of the projection operator) *For any $x \in \mathbb{R}^n$ with the spectral decomposition* (3.1)*, the B-subdifferential of the projection operator $\Pi_{\mathcal{L}_\theta}$ is calculated as follows:*

(a) *If $\lambda_1(x)\lambda_2(x) \neq 0$, then $\Pi_{\mathcal{L}_\theta}$ is differentiable at x and $\partial_B(\Pi_{\mathcal{L}_\theta})(x) = \{\nabla\Pi_{\mathcal{L}_\theta}(x)\}$.*

(b) *If $\lambda_1(x) = 0$ and $\lambda_2(x) > 0$, then*

$$\partial_B(\Pi_{\mathcal{L}_\theta})(x) = \left\{I,\, I + \frac{1}{\tan\theta + \text{ctan}\theta}\begin{bmatrix} -\tan\theta & \bar{x}_2 \\ \bar{x}_2 & -\text{ctan}\theta\,\bar{x}_2\bar{x}_2^T \end{bmatrix}\right\}.$$

(c) *If $\lambda_1(x) < 0$ and $\lambda_2(x) = 0$, then*

$$\partial_B(\Pi_{\mathcal{L}_\theta})(x) = \left\{0,\, \frac{1}{\tan\theta + \text{ctan}\theta}\begin{bmatrix} \text{ctan}\theta & \bar{x}_2^T \\ \bar{x}_2 & \tan\theta\,\bar{x}_2\bar{x}_2^T \end{bmatrix}\right\}.$$

(d) *If $\lambda_1(x) = \lambda_2(x) = 0$, then*

$$\begin{aligned}\partial_B(\Pi_{\mathcal{L}_\theta})(x) = \Bigg\{&\frac{1}{\tan\theta + \text{ctan}\theta}\\ &\times\begin{bmatrix} \text{ctan}\theta & w^T \\ w & \Big(\tan\theta + \text{ctan}\theta\Big)aI + \Big[\tan\theta - a(\text{ctan}\theta + \tan\theta)\Big]ww^T \end{bmatrix}\\ &\times\left|\begin{array}{c} a \in [0,1] \\ \|w\| = 1 \end{array}\right.\Bigg\}\bigcup\Big\{0, I\Big\}.\end{aligned}$$

Proof In case (a) the function $f(t) = (t)_+$ is differentiable at $\lambda_i(x)$ for $i = 1, 2$. Hence, it follows from [13, Theorem 2.3] that $\Pi_{\mathcal{L}_\theta}$ is also differentiable at x. Furthermore, in this case we have by (3.5) that

$$\Pi_{\mathcal{L}_\theta}(x) = \begin{cases} x & \text{if} \;\; \lambda_1(x) > 0 \text{ and } \lambda_2(x) > 0, \\ 0 & \text{if} \;\; \lambda_1(x) < 0 \text{ and } \lambda_2(x) < 0, \\ (x_1 + \|x_2\|\tan\theta)u_x^2 & \text{if} \;\; \lambda_1(x) < 0 \text{ and } \lambda_2(x) > 0. \end{cases}$$

In particular, $\|x_2\| \neq 0$ when $\lambda_1(x) < 0$ and $\lambda_2(x) > 0$, and thus $\nabla\|x_2\| = \bar{x}_2$. This gives us

$$\partial_B\Pi_{\mathcal{L}_\theta}(x) = \Big\{\nabla\Pi_{\mathcal{L}_\theta}(x)\Big\},$$

where the derivative of $\Pi_{\mathcal{L}_\theta}$ at x is calculated by

$$\begin{aligned}&\nabla\Pi_{\mathcal{L}_\theta}(x)\\ &= \begin{cases} I & \text{if} \;\; \lambda_1(x) > 0 \text{ and } \lambda_2(x) > 0, \\ 0 & \text{if} \;\; \lambda_1(x) < 0 \text{ and } \lambda_2(x) < 0, \\ \dfrac{1}{\tan\theta + \text{ctan}\theta}\begin{bmatrix} \text{ctan}\theta & \bar{x}_2^T \\ \bar{x}_2 & \dfrac{x_1 + \|x_2\|\tan\theta}{\|x_2\|}I - \dfrac{x_1}{\|x_2\|}\bar{x}_2\bar{x}_2^T \end{bmatrix} & \text{if} \;\; \lambda_1(x) < 0 \text{ and } \lambda_2(x) > 0. \end{cases}\end{aligned} \tag{3.6}$$

In case (b) we have $\|x_2\| \neq 0$, and so it follows from [13, Theorem 3.1] that

$$\partial_B\big(f^{\mathcal{L}_\theta}\big)(x) = \left\{ \begin{bmatrix} \xi & \varrho \bar{x}_2^T \\ \varrho \bar{x}_2 & aI + (\eta - a)\bar{x}_2\bar{x}_2^T \end{bmatrix} \;\middle|\; \begin{array}{l} a = \dfrac{f\big(\lambda_2(x)\big) - f\big(\lambda_1(x)\big)}{\lambda_2(x) - \lambda_1(x)} \\ \xi - \varrho \mathrm{ctan}\theta \in \partial_B f(\lambda_1(x)) \\ \xi + \varrho \tan\theta \in \partial_B f(\lambda_2(x)) \\ \eta = \xi - \varrho(\mathrm{ctan}\theta - \tan\theta) \end{array} \right\}.$$

This implies by the obvious calculation

$$\partial_B(t)_+ = \begin{cases} 1 & \text{for } t > 0, \\ \{0, 1\} & \text{for } t = 0, \\ 0 & \text{for } t < 0 \end{cases}$$

that the B-subdifferential of the projection operator is represented as

$$\partial_B(\Pi_{\mathcal{L}_\theta})(x) = \left\{ \begin{bmatrix} \xi & \varrho \bar{x}_2^T \\ \varrho \bar{x}_2 & aI + (\eta - a)\bar{x}_2\bar{x}_2^T \end{bmatrix} \;\middle|\; \begin{array}{l} a = 1 \\ \xi - \varrho \mathrm{ctan}\theta \in \{0, 1\} \\ \xi + \varrho \tan\theta = 1 \\ \eta = \xi - \varrho(\mathrm{ctan}\theta - \tan\theta) \end{array} \right\}. \tag{3.7}$$

Analysing (3.7) in the case of $\xi - \varrho\mathrm{ctan}\theta = 1$ and $\xi + \varrho\tan\theta = 1$ shows that $\xi = 1$, $\varrho = 0$, and $\eta = 1$. Hence (3.7) reduces in this case to I. For $\xi - \varrho\mathrm{ctan}\theta = 0$ we know that

$$\xi = \frac{\mathrm{ctan}\theta}{\tan\theta + \mathrm{ctan}\theta}, \quad \varrho = \frac{1}{\tan\theta + \mathrm{ctan}\theta}, \quad \text{and} \quad \eta = \frac{\tan\theta}{\tan\theta + \mathrm{ctan}\theta},$$

and so Equation (3.7) in this case takes the form of

$$\begin{aligned} &\begin{bmatrix} \dfrac{\mathrm{ctan}\theta}{\tan\theta + \mathrm{ctan}\theta} & \dfrac{1}{\tan\theta + \mathrm{ctan}\theta}\bar{x}_2^T \\ \dfrac{1}{\tan\theta + \mathrm{ctan}\theta}\bar{x}_2 & I + \left(\dfrac{\tan\theta}{\tan\theta + \mathrm{ctan}\theta} - 1\right)\bar{x}_2\bar{x}_2^T \end{bmatrix} \\ &\quad = I + \frac{1}{\tan\theta + \mathrm{ctan}\theta}\begin{bmatrix} -\tan\theta & \bar{x}_2^T \\ \bar{x}_2 & -\mathrm{ctan}\theta\bar{x}_2\bar{x}_2^T \end{bmatrix}, \end{aligned}$$

which gives us the B-subdifferential representation

$$\partial_B\big(\Pi_{\mathcal{L}_\theta}\big)(x) = \left\{ I, I + \frac{1}{\tan\theta + \mathrm{ctan}\theta}\begin{bmatrix} -\tan\theta & \bar{x}_2 \\ \bar{x}_2 & -\mathrm{ctan}\theta\bar{x}_2\bar{x}_2^T \end{bmatrix} \right\}.$$

In case (c) we also have $x_2 \neq 0$. Similarly to case (b), it is not hard to verify that

$$\partial_B\big(\Pi_{\mathcal{L}_\theta}\big)(x) = \left\{ 0, \frac{1}{\tan\theta + \mathrm{ctan}\theta}\begin{bmatrix} \mathrm{ctan}\theta & \bar{x}_2^T \\ \bar{x}_2 & \tan\theta\bar{x}_2\bar{x}_2^T \end{bmatrix} \right\}.$$

It remains to consider case (d) when $x = 0$. Then the result of [13, Theorem 3.4] tells us that

$$\partial_B\big(\Pi_{\mathcal{L}_\theta}\big)(x) = \left\{ \begin{bmatrix} \xi & \varrho w^T \\ \varrho w & aI + (\eta - a)ww^T \end{bmatrix} \;\middle|\; \begin{array}{l} \text{either } \ a = \xi \in \{0,1\}, \ \varrho = 0 \\ \text{or } \ a \in [0,1] \\ \xi - \varrho \mathrm{ctan}\theta = 0 \\ \xi + \varrho \tan\theta = 1 \\ \eta = \xi - \varrho(\mathrm{ctan}\theta - \tan\theta) \\ \|w\| = 1 \end{array} \right\}$$
$$= \left\{ \frac{1}{\tan\theta + \mathrm{ctan}\theta} \begin{bmatrix} \mathrm{ctan}\theta & w^T \\ w & \big(\tan\theta + \mathrm{ctan}\theta\big)aI + \Big(\tan\theta - a(\mathrm{ctan}\theta + \tan\theta)\Big)ww^T \end{bmatrix} \times \;\middle|\; \begin{array}{l} a \in [0,1] \\ \|w\| = 1 \end{array} \right\} \bigcup \Big\{0, I\Big\},$$

which thus completes the proof of the lemma. □

Our next goal is to verify the *directional differentiability* of the projection operator (3.5) and derive formulas for calculating its directional derivative (2.12). Observe to this end that the result of [13, Theorem 2.2] tells us that the vector function $f^{\mathcal{L}_\theta}$ from (3.4) is directionally differentiable at x provided that f is directionally differentiable at $\lambda_i(x)$ for $i = 1, 2$. Moreover, for $x_2 = 0$ we have

$$\begin{aligned} \Big(f^{\mathcal{L}_\theta}\Big)'(x;h) &= \frac{1}{1+\mathrm{ctan}^2\theta} f'\Big(x_1; h_1 - \|h_2\|\mathrm{ctan}\theta\Big) \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{ctan}\theta \end{bmatrix} \begin{bmatrix} 1 \\ -\bar{h}_2 \end{bmatrix} \\ &\quad + \frac{1}{1+\tan^2\theta} f'\Big(x_1; h_1 + \|h_2\|\tan\theta\Big) \begin{bmatrix} 1 & 0 \\ 0 & \tan\theta \end{bmatrix} \begin{bmatrix} 1 \\ \bar{h}_2 \end{bmatrix} \\ &= f'\Big(x_1; h_1 - \|h_2\|\mathrm{ctan}\theta\Big)u_h^1 + f'\Big(x_1; h_1 + \|h_2\|\tan\theta\Big)u_h^2. \end{aligned} \tag{3.8}$$

On the other hand, for $x_2 \neq 0$ we denote

$$M_{x_2} := \begin{bmatrix} 0 & 0 \\ 0 & I - \dfrac{x_2 x_2^T}{\|x_2\|^2} \end{bmatrix}$$

and arrive at the following relationships:

$$\begin{aligned} &\Big(f^{\mathcal{L}_\theta}\Big)'(x;h) \\ &\quad = \frac{1}{1+\mathrm{ctan}^2\theta} f'\left(\lambda_1(x); h_1 - \frac{x_2^T h_2}{\|x_2\|}\mathrm{ctan}\theta\right) \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{ctan}\theta \end{bmatrix} \begin{bmatrix} 1 \\ -\bar{x}_2 \end{bmatrix} \\ &\qquad - \frac{\mathrm{ctan}\theta}{1+\mathrm{ctan}^2\theta} \frac{f(\lambda_1(x))}{\|x_2\|} M_{x_2} h \\ &\qquad + \frac{1}{1+\tan^2\theta} f'\left(\lambda_2(x); h_1 + \frac{x_2^T h_2}{\|x_2\|}\tan\theta\right) \begin{bmatrix} 1 & 0 \\ 0 & \tan\theta \end{bmatrix} \begin{bmatrix} 1 \\ \bar{x}_2 \end{bmatrix} \\ &\qquad + \frac{\tan\theta}{1+\tan^2\theta} \frac{f(\lambda_2(x))}{\|x_2\|} M_{x_2} h \end{aligned}$$

$$= f'\left(\lambda_1(x); h_1 - \frac{x_2^T h_2}{\|x_2\|}\mathrm{ctan}\theta\right) u_x^1 + f'\left(\lambda_2(x); h_1 + \frac{x_2^T h_2}{\|x_2\|}\tan\theta\right) u_x^2 + \frac{f(\lambda_2(x)) - f(\lambda_1(x))}{\lambda_2(x) - \lambda_1(x)} M_{x_2} h. \tag{3.9}$$

This leads us to calculate the *directional derivative* (2.12) of the projection operator (3.5).

LEMMA 3.2 (calculating the directional derivative of the projection operator) *The projector operator* (3.5) *is directionally differentiable at any point* $x \in \mathbb{R}^n$ *with the spectral decomposition* (3.1)*, and its directional derivative at* x *in any direction* $h \in \mathbb{R}^n$ *is calculated as follows:*

(a) *If* $\lambda_1(x)\lambda_2(x) \neq 0$*, then* $\Pi'_{\mathcal{L}_\theta}(x; h) = \nabla\Pi_{\mathcal{L}_\theta}(x)h$.
(b) *If* $\lambda_1(x) = 0$ *and* $\lambda_2(x) > 0$*, then* $\Pi'_{\mathcal{L}_\theta}(x; h) = h - (1 + \mathrm{ctan}^2\theta)\big((u_x^1)^T h\big)_- u_x^1$.
(c) *If* $\lambda_1(x) < 0$ *and* $\lambda_2(x) = 0$*, then* $\Pi'_{\mathcal{L}_\theta}(x; h) = (1 + \tan^2\theta)\big((u_x^2)^T h\big)_+ u_x^2$.
(d) *If* $\lambda_1(x) = \lambda_2(x) = 0$*, then* $\Pi'_{\mathcal{L}_\theta}(x; h) = \Pi_{\mathcal{L}_\theta}(h)$.

Proof The directional differentiability of (3.5) at x follows from the discussions above. Moreover, in case (a), corresponding to $f(t) = (t)_+$ in (3.4), we get the differentiability of $\Pi_{\mathcal{L}_\theta}$ at this point, and hence $\Pi'_{\mathcal{L}_\theta}(x; h) = \nabla\Pi_{\mathcal{L}_\theta}(x)h$ for all $h \in \mathbb{R}^n$.

In case (b) we have $x_2 \neq 0$. It follows from (3.9) that

$$\begin{aligned}
\Pi'_{\mathcal{L}_\theta}(x; h) &= \left(h_1 - \frac{x_2^T h_2}{\|x_2\|}\mathrm{ctan}\theta\right)_+ u_x^1 + \left(h_1 + \frac{x_2^T h_2}{\|x_2\|}\tan\theta\right) u_x^2 + M_{x_2} h \\
&= (1 + \mathrm{ctan}^2\theta)\big((u_x^1)^T h\big)_+ u_x^1 + h \\
&\quad + \begin{pmatrix} \dfrac{\tan^2\theta}{1 + \tan^2\theta}\left(-h_1 + \dfrac{x_2^T h_2}{\|x_2\|}\mathrm{ctan}\theta\right) \\ \left[\dfrac{\tan\theta}{1 + \tan^2\theta}\left(h_1 + \dfrac{x_2^T h_2}{\|x_2\|}\tan\theta\right) - \dfrac{x_2^T h_2}{\|x_2\|}\right]\bar{x}_2 \end{pmatrix} \\
&= (1 + \mathrm{ctan}^2\theta)\big((u_x^1)^T h\big)_+ u_x^1 + h \\
&\quad - \begin{pmatrix} \dfrac{\tan^2\theta}{1 + \tan^2\theta}\left(h_1 - \dfrac{x_2^T h_2}{\|x_2\|}\mathrm{ctan}\theta\right) \\ \dfrac{\tan^2\theta}{1 + \tan^2\theta}\left(h_1 - \bar{x}_2^T h_2\mathrm{ctan}\theta\right)\left(-\mathrm{ctan}\theta\bar{x}_2\right) \end{pmatrix} \\
&= (1 + \mathrm{ctan}^2\theta)\big((u_x^1)^T h\big)_+ u_x^1 + h \\
&\quad - \frac{\tan^2\theta}{1 + \tan^2\theta}(1 + \mathrm{ctan}^2\theta)\big((u_x^1)^T h\big) \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{ctan}\theta \end{bmatrix} \begin{bmatrix} 1 \\ -\bar{x}_2 \end{bmatrix} \\
&= (1 + \mathrm{ctan}^2\theta)\big((u_x^1)^T h\big)_+ u_x^1 + h - (1 + \mathrm{ctan}^2\theta)\big((u_x^1)^T h\big) u_x^1 \\
&= h - (1 + \mathrm{ctan}^2\theta)\big((u_x^1)^T h\big)_- u_x^1,
\end{aligned}$$

where the representations $t = (t)_+ + (t)_-$ for all $t \in \mathbb{R}$ are used together with

$$\frac{\tan\theta}{1+\tan^2\theta}\left(h_1 + \frac{x_2^T h_2}{\|x_2\|}\tan\theta\right) - \frac{x_2^T h_2}{\|x_2\|}$$

$$= \frac{\tan\theta}{1+\tan^2\theta}\left(h_1 + \frac{x_2^T h_2}{\|x_2\|}\tan\theta - \frac{1+\tan^2\theta}{\tan\theta}\frac{x_2^T h_2}{\|x_2\|}\right)$$

$$= \frac{\tan\theta}{1+\tan^2\theta}\left(h_1 - \frac{x_2^T h_2}{\|x_2\|}\mathrm{ctan}\theta\right), \quad \text{and}$$

$$\big((u_x^1)^T h\big)_+ - \big((u_x^1)^T h\big) = -\big((u_x^1)^T h\big)_-.$$

In case (c) we employ (3.9) again to get the conclusion claimed. The final case (d) yields $x = 0$, and hence representation (3.8) gives us the equalities

$$\Pi'_{\mathcal{L}_\theta}(x; h) = (h_1 - \|h_2\|\mathrm{ctan}\theta)_+ u_h^1 + (h_1 + \|h_2\|\tan\theta)_+ u_h^2 = \Pi_{\mathcal{L}_\theta}(h),$$

which therefore complete the proof of the lemma. □

The following theorem uses the previous considerations to establish the *strongly semismoothness* property of the projection operator $\Pi_{\mathcal{L}_\theta}$. It has been well recognized the importance of this property of Lipschitzian mappings in many aspects of variational analysis and optimization; in particular, to establish the *quadratic rate* of convergence of the so-called *semismooth Newton method*; see [31,32].

THEOREM 3.3 (strong semismoothness of the projection operator) *The projection operator $\Pi_{\mathcal{L}_\theta}$ in (3.5) is strongly semismooth over $\mathbb{R}^n$.*

Proof The proof is inspired by [33, Proposition 4.5]. Note first that the directional differentiability of the Lipschitz continuous projection operator $\Pi_{\mathcal{L}_\theta}$ from Lemma 3.2, and thus it remains to show that representation (2.14) holds for it with $\rho = 1$.

To verify our claim, deduce from the proof of Lemma 3.1 that

$$\Pi_{\mathcal{L}_\theta}(x) = \begin{cases} x & \text{if } \lambda_1(x) \ge 0 \text{ and } \lambda_2(x) \ge 0, \\ \dfrac{1}{1+\tan^2\theta}(x_1 + \|x_2\|\tan\theta)\begin{bmatrix}1 & 0\\ 0 & \tan\theta\end{bmatrix}\begin{bmatrix}1\\ \bar{x}_2\end{bmatrix} & \text{if } \lambda_1(x) < 0 \text{ and } \lambda_2(x) > 0, \\ 0 & \text{if } \lambda_1(x) \le 0 \text{ and } \lambda_2(x) \le 0. \end{cases} \tag{3.10}$$

Then we split the subsequent proof into two cases: $x_2 \neq 0$ and $x_2 = 0$.

Case 1 When $x_2 \neq 0$, we can easily observe that in all the formulas from (3.10) corresponding to this case, the projection operator $\Pi_{\mathcal{L}_\theta}$ is a piecewise $\mathcal{C}^2$-smooth mapping whose strong semismoothness is well known in optimization.[34] It verifies the claim in this case.

Case 2 For $x_2 = 0$, suppose first that $x_1 \neq 0$. Then $\lambda_i(x) = x_1 \neq 0$, $i = 1, 2$. Since $\lambda_i(y)$ is Lipschitz continuous by [13, Lemma 2.1], we get from (3.10) that $\Pi_{\mathcal{L}_\theta}(y)$ is either 0 or y when y is in a neighbourhood of x. Thus $\Pi_{\mathcal{L}_\theta}$ is surely strongly semismooth at x in this setting. In the remaining setting of $x_1 = 0$ we have $x = 0$. Note that the projection operator $\Pi_{\mathcal{L}_\theta}$ is obviously positively homogeneous, i.e. $\Pi_{\mathcal{L}_\theta}(tz) = t\Pi_{\mathcal{L}_\theta}(z)$ for $z \in \mathbb{R}^n$

and $t \geq 0$. This implies that $\Pi'_{\mathcal{L}_\theta}(h; h) = \Pi_{\mathcal{L}_\theta}(h)$ and $\Pi'_{\mathcal{L}_\theta}(0; h) = \Pi_{\mathcal{L}_\theta}(h)$. Hence $\nabla\Pi_{\mathcal{L}_\theta}(h')(h') = \Pi_{\mathcal{L}_\theta}(h')$ as $h' \in D_{\Pi_{\mathcal{L}_\theta}}$. Since $D_{\Pi_{\mathcal{L}_\theta}}$ is a dense subset of $\mathbb{R}^n$, for any fixed $h \neq 0$ and $V \in \partial_B \Pi_{\mathcal{L}_\theta}(h)$, there exists $h' \in D_{\Pi_{\mathcal{L}_\theta}}$ such that $\|h' - h\| \leq \|h\|^2$ and $\|V - \nabla\Pi_{\mathcal{L}_\theta}(h')\| \leq \|h\|$. Hence for h sufficiently close to 0 we have

$$
\begin{aligned}
\|Vh - \Pi'_{\mathcal{L}_\theta}(0; h)\| &= \|Vh - \nabla\Pi_{\mathcal{L}_\theta}(h')(h') + \Pi_{\mathcal{L}_\theta}(h') - \Pi'_{\mathcal{L}_\theta}(0; h)\| \\
&= \|Vh - \nabla\Pi_{\mathcal{L}_\theta}(h')(h) + \nabla\Pi_{\mathcal{L}_\theta}(h')(h) - \nabla\Pi_{\mathcal{L}_\theta}(h')(h') \\
&\quad + \Pi_{\mathcal{L}_\theta}(h') - \Pi_{\mathcal{L}_\theta}(h)\| \\
&\leq \|V - \nabla\Pi_{\mathcal{L}_\theta}(h')\|\|h\| + \|\nabla\Pi_{\mathcal{L}_\theta}(h')\|\|h - h'\| + \|h - h'\| \\
&\leq (r+2)\|h\|^2,
\end{aligned}
$$

where r is a bounded from above of $\|\partial_B \Pi_{\mathcal{L}_\theta}(\cdot)\|$ near 0 since $\Pi_{\mathcal{L}_\theta}$ is Lipschitz. Thus

$$
L := \limsup_{h \to 0} \frac{\|Vh - \Pi'_{\mathcal{L}_\theta}(0; h)\|}{\|h\|^2} < \infty, \quad \text{i.e.} \tag{3.11}
$$

$$
Vh - \Pi'_{\mathcal{L}_\theta}(0; h) = O(\|h\|^2) \quad \text{for all } V \in \partial_B \Pi_{\mathcal{L}_\theta}(h).
$$

Now let us show that $Vh - \Pi'_{\mathcal{L}_\theta}(0; h) = O(\|h\|^2)$ for any $V \in \operatorname{co} \partial_B \Pi_{\mathcal{L}_\theta}(h)$, i.e. for any $h_k \to 0$ and $V_k \in \operatorname{co} \partial_B \Pi_{\mathcal{L}_\theta}(h_k)$ we have $V_k h_k - \Pi'_{\mathcal{L}_\theta}(0; h_k) = O(\|h_k\|^2)$. Since $V_k \in \operatorname{co} \partial_B \Pi_{\mathcal{L}_\theta}(h_k)$, it follows from the Carathéodory theorem that there are $V_k^i \in \partial_B \Pi_{\mathcal{L}_\theta}(h_k)$ and $\lambda_k^i \geq 0$ for $i = 1, \ldots, n+1$ such that

$$
V_k = \sum_{i=1}^{n+1} \lambda_k^i V_k^i \quad \text{and} \quad \sum_{i=1}^{n+1} \lambda_k^i = 1.
$$

Since $V_k^i \in \partial_B \Pi_{\mathcal{L}_\theta}(h_k)$, it follows from (3.11) that

$$
\limsup_{k \to 0} \frac{\|V_k^i h_k - \Pi'_{\mathcal{L}_\theta}(0; h_k)\|}{\|h_k\|^2} \leq L.
$$

Due to the boundedness of $\{\lambda_k^i\}$, we can assume without loss of generality that $\{\lambda_k^i\}$ converge to some $\bar{\lambda}_i$ for $i = 1, \ldots, n+1$. Hence

$$
\begin{aligned}
\limsup_{k \to 0} \frac{\|V_k h_k - \Pi'_{\mathcal{L}_\theta}(0; h_k)\|}{\|h_k\|^2} &= \limsup_{k \to 0} \frac{\left\|\sum_{i=1}^{n+1} \lambda_k^i V_k^i h_k - \Pi'_{\mathcal{L}_\theta}(0; h_k)\right\|}{\|h_k\|^2} \\
&\leq \limsup_{k \to 0} \sum_{i=1}^{n+1} \lambda_k^i \frac{\|V_k^i h_k - \Pi'_{\mathcal{L}_\theta}(0; h_k)\|}{\|h_k\|^2} \\
&\leq \sum_{i=1}^{n+1} \bar{\lambda}^i L = L.
\end{aligned}
$$

Thus $Vh - \Pi'_{\mathcal{L}_\theta}(0; h) = O(\|h\|^2)$ for any $V \in \operatorname{co} \partial_B \Pi_{\mathcal{L}_\theta}(h)$, i.e. $\Pi_{\mathcal{L}_\theta}$ is strongly semismooth at 0. $\square$

The next result, which easily follows from Lemma 3.2, provides the calculation of the *graphical derivative* (2.5) for the projection operator onto the circular cone.

Proposition 3.4 (calculating the graphical derivative of the projection operator) *For any $x \in \mathbb{R}^n$ with decomposition (3.1), the graphical derivative of $\Pi_{\mathcal{L}_\theta}(x)$ is calculated by*

$$D\Pi_{\mathcal{L}_\theta}(x)(w) = \left\{\Pi'_{\mathcal{L}_\theta}(x; w)\right\} \quad \textit{for any} \ \ w \in \mathbb{R}^n. \tag{3.12}$$

Proof It follows from [30, formula 8(14)] that the graphical derivative of any closed graph operator, and hence of $\Pi_{\mathcal{L}_\theta}$ in particular, can be equivalently represented as

$$D\Pi_{\mathcal{L}_\theta}(x)(w) = \operatorname*{Lim\,sup}_{\substack{\tau \searrow 0 \\ w' \to w}} \frac{\Pi_{\mathcal{L}_\theta}(x + \tau w') - \Pi_{\mathcal{L}_\theta}(x)}{\tau}. \tag{3.13}$$

By Lemma 3.2 the Lipschitzian mapping $\Pi_{\mathcal{L}_\theta}$ is directionally differentiable at x. Thus the right-hand side of (3.13) reduces to $\Pi'_{\mathcal{L}_\theta}(x; w)$, which justifies (3.12). □

Based on the calculations provided in Lemmas 3.1 and 3.2, we are now ready to establish precise formulas for computing the *regular* and *limiting coderivatives* of the projection operator $\Pi_{\mathcal{L}_\theta}$ onto the general circular cone (1.1). We proceed similarly to the proofs of the main results of the paper [35] by Outrata and Sun while using our calculations given above as well as in the proofs of the theorems. Taking into account relationships (1.6) between the circular and second-order cones, it is appealing to reduce deriving coderivative formulas for the projection onto the circular cone to those obtained for the second-order one. However, it does not seem to be possible; see more discussions in Remark 4.7.

Theorem 3.5 (calculating the regular coderivative of the projection operator) *For any $x \in \mathbb{R}^n$ with decomposition (3.1) and any $y^* \in \mathbb{R}^n$, the regular coderivative (2.8) of the projection operator $\Pi_{\mathcal{L}_\theta}(x)$ onto the circular cone (1.1) is calculated as follows:*

(a) *If $\lambda_1(x)\lambda_2(x) \neq 0$, then $\widehat{D}^*\Pi_{\mathcal{L}_\theta}(x)(y^*) = \left\{\nabla\Pi_{\mathcal{L}_\theta}(x)y^*\right\}$.*

(b) *If $\lambda_1(x) = 0$ and $\lambda_2(x) > 0$, then*

$$\widehat{D}^*\Pi_{\mathcal{L}_\theta}(x)(y^*) = \left\{x^* \in \mathbb{R}^n \,\middle|\, y^* - x^* \in \mathbb{R}_+ u_x^1,\ \langle x^*, u_x^1\rangle \geq 0\right\}.$$

(c) *If $\lambda_1(x) < 0$ and $\lambda_2(x) = 0$, then*

$$\widehat{D}^*\Pi_{\mathcal{L}_\theta}(x)(y^*) = \left\{x^* \in \mathbb{R}^n \,\middle|\, x^* \in \mathbb{R}_+ u_x^2,\ \langle y^* - x^*, u_x^2\rangle \geq 0\right\}.$$

(d) *If $\lambda_1(x) = \lambda_2(x) = 0$, then*

$$\widehat{D}^*\Pi_{\mathcal{L}_\theta}(x)(y^*) = \left\{x^* \in \mathbb{R}^n \,\middle|\, x^* \in \mathcal{L}_\theta,\ y^* - x^* \in \mathcal{L}_{\frac{\pi}{2}-\theta}\right\}.$$

Proof Due to the well-known duality between the regular coderivative and the graphical derivative of a mapping (see [30]) and by the established directional differentiability of the projection operator onto the circular cone, we have the equivalence

$$x^* \in \widehat{D}^*\Pi_{\mathcal{L}_\theta}(x)(y^*) \Longleftrightarrow \langle x^*, h\rangle \leq \left\langle y^*, \Pi'_{\mathcal{L}_\theta}(x; h)\right\rangle \quad \text{for all} \ \ h \in \mathbb{R}^n. \tag{3.14}$$

Employing (3.14) and the calculation of the directional derivative of $\Pi_{\mathcal{L}_\theta}$ in Lemma 3.2 allows us to derive the claimed formulas for the regular coderivative of $\Pi_{\mathcal{L}_\theta}$ in all the cases (a)–(d) of the theorem.

In case (a), pick any $x^* \in \widehat{D}^*\Pi_{\mathcal{L}_\theta}(x)(y^*)$ and get by using Lemma 3.2(a) and duality (3.14) that

$$\begin{aligned}\langle x^*, h\rangle \le \langle y^*, \Pi'_{\mathcal{L}_\theta}(x;h)\rangle &\iff \langle x^*, h\rangle \le \langle y^*, \nabla\Pi_{\mathcal{L}_\theta}(x)h\rangle\\ &\iff \langle x^* - \nabla\Pi_{\mathcal{L}_\theta}(x)y^*, h\rangle \le 0,\end{aligned}$$

where the last step comes from the fact that the operator $\nabla\Pi_{\mathcal{L}_\theta}$ is self-adjoint by (3.6). Hence, we have $x^* = \nabla\Pi_{\mathcal{L}_\theta}(x)y^*$, i.e. $\widehat{D}^*\Pi_{\mathcal{L}_\theta}(x)(y^*) = \{\nabla\Pi_{\mathcal{L}_\theta}(x)y^*\}$.

In case (b) we employ Lemma 3.2(b), which gives us together with (3.14) that

$$\begin{aligned}x^* \in \widehat{D}^*\Pi_{\mathcal{L}_\theta}(x)(y^*) &\iff \langle x^*, h\rangle \le \left\langle y^*, h - (1+\mathrm{ctan}^2\theta)\left((u_x^1)^T h\right)_- u_x^1\right\rangle\\ &\iff \langle x^* - y^*, h\rangle + (1+\mathrm{ctan}^2\theta)\left\langle y^*, \left((u_x^1)^T h\right)_- u_x^1\right\rangle \le 0\\ &\iff \begin{cases}\langle x^* - y^*, h\rangle \le 0, & (u_x^1)^T h \ge 0\\ \langle x^* - y^*, h\rangle + (1+\mathrm{ctan}^2\theta)(u_x^1)^T h (y^*)^T u_x^1 \le 0, & (u_x^1)^T h \le 0\end{cases}\\ &\iff \exists\,\alpha \ge 0 \text{ and } \beta \ge 0 \text{ such that } y^* - x^* = \alpha u_x^1\\ &\qquad \text{and } x^* - y^* + (1+\mathrm{ctan}^2\theta)\big((y^*)^T u_x^1\big)u_x^1 = \beta u_x^1\\ &\iff \exists\,\alpha \ge 0 \text{ and } \beta \ge 0 \text{ such that } y^* - x^* = \alpha u_x^1\\ &\qquad \text{and } (1+\mathrm{ctan}^2\theta)\big((y^*)^T u_x^1\big)u_x^1 = (\alpha+\beta)u_x^1\\ &\iff \exists\,\alpha \ge 0 \text{ such that } y^* - x^* = \alpha u_x^1\\ &\qquad \text{and } (1+\mathrm{ctan}^2\theta)\langle y^*, u_x^1\rangle \ge \alpha \qquad (3.15)\\ &\iff \exists\,\alpha \ge 0 \text{ such that } y^* - x^* = \alpha u_x^1 \text{ and } \langle x^*, u_x^1\rangle \ge 0. \qquad (3.16)\end{aligned}$$

The last equivalence above comes from the following arguments: if (3.16) holds, then

$$(1+\mathrm{ctan}^2\theta)\langle y^*, u_x^1\rangle = (1+\mathrm{ctan}^2\theta)\langle x^* + \alpha u_x^1, u_x^1\rangle \ge \alpha(1+\mathrm{ctan}^2\theta)\|u_x^1\|^2 = \alpha;$$

conversely, the validity of (3.15) implies that

$$\langle x^*, u_x^1\rangle = \langle y^*, u_x^1\rangle - \alpha\langle u_x^1, u_x^1\rangle \ge \frac{1}{1+\mathrm{ctan}^2\theta}\alpha - \frac{1}{1+\mathrm{ctan}^2\theta}\alpha = 0.$$

In case (c) we have the equivalencies by using Lemma 3.2(c) and duality (3.14):

$$\begin{aligned}x^* \in \widehat{D}^*\Pi_{\mathcal{L}_\theta}(x)(y^*) &\iff \langle x^*, h\rangle \le \left\langle y^*, (1+\tan^2\theta)\left((u_x^2)^T h\right)_+ u_x^2\right\rangle\\ &\iff \begin{cases}\langle x^*, h\rangle \le 0, & (u_x^2)^T h \le 0,\\ \left\langle x^* - (1+\tan^2\theta)\left((y^*)^T u_x^2\right)u_x^2, h\right\rangle \le 0, & (u_x^2)^T h \ge 0\end{cases}\\ &\iff \exists\,\alpha \ge 0 \text{ such that } x^* = \alpha u_x^2 \text{ and } (1+\tan^2\theta)(y^*)^T u_x^2 \ge \alpha\\ &\iff \exists\,\alpha \ge 0 \text{ such that } x^* = \alpha u_x^2 \text{ and } \langle y^* - x^*, u_x^2\rangle \ge 0,\end{aligned}$$

which readily justify the claimed result in this case.

In case (d) we have $x = 0$ and then proceed by using Lemma 3.2(d) together with (3.14). This yields

$$\begin{aligned} x^* \in \widehat{D}^*\Pi_{\mathcal{L}_\theta}(0)(y^*) &\iff \langle x^*, h\rangle \le \langle y^*, \Pi_{\mathcal{L}_\theta}(h)\rangle \quad \text{for all } h \in \mathbb{R}^n \\ &\iff \langle x^*, \Pi_{\mathcal{L}_\theta}(h) + \Pi_{\mathcal{L}_\theta^\circ}(h)\rangle \le \langle y^*, \Pi_{\mathcal{L}_\theta}(h)\rangle \quad \text{for all } h \in \mathbb{R}^n \\ &\iff \langle x^* - y^*, \Pi_{\mathcal{L}_\theta}(h)\rangle + \langle x^*, \Pi_{\mathcal{L}_\theta^\circ}(h)\rangle \le 0 \quad \text{for all } h \in \mathbb{R}^n \qquad (3.17) \\ &\iff x^* \in \mathcal{L}_\theta \text{ and } y^* - x^* \in \mathcal{L}_{\frac{\pi}{2}-\theta}, \qquad (3.18) \end{aligned}$$

where the last equivalence is justified as follows. Relationship (3.18) $\Longrightarrow$ (3.17) is implied by the inclusion $x^* - y^* \in -\mathcal{L}_{\frac{\pi}{2}-\theta} = (\mathcal{L}_\theta)^\circ$. For the converse implication, observe that the validity of (3.17) gives us $\langle x^* - y^*, h\rangle \le 0$ for all $h \in \mathcal{L}_\theta$ and $\langle x^*, h\rangle \le 0$ for all $h \in (\mathcal{L}_\theta)^\circ$, which yields in turn the fulfillment of $x^* - y^* \in (\mathcal{L}_\theta)^\circ = -\mathcal{L}_{\frac{\pi}{2}-\theta}$ and $x^* \in ((\mathcal{L}_\theta)^\circ)^\circ = \mathcal{L}_\theta$ since $\mathcal{L}_\theta$ is a closed and convex cone. $\square$

To calculate next the *coderivative* (2.6) of the projection operator $\Pi_{\mathcal{L}_\theta}$, for any $x, y^* \in \mathbb{R}^n$ we define

$$A(x, y^*) := \left\{x^* \in \mathbb{R}^n \,\middle|\, y^* - x^* \in \mathbb{R}_+ u_x^1,\ \ \langle x^*, u_x^1\rangle \ge 0\right\}, \qquad (3.19)$$

$$B(x, y^*) := \left\{x^* \in \mathbb{R}^n \,\middle|\, x^* \in \mathbb{R}_+ u_x^2,\ \langle y^* - x^*, u_x^2\rangle \ge 0\right\}. \qquad (3.20)$$

THEOREM 3.6 (calculating the coderivative of the projection operator) *For any $x \in \mathbb{R}^n$ with decomposition (3.1) and any $y^* \in \mathbb{R}^n$, the coderivative (2.8) of the projection operator $\Pi_{\mathcal{L}_\theta}(x)$ onto the circular cone (1.1) is calculated as follows:*

(a) *If $\lambda_1(x)\lambda_2(x) \ne 0$, then $D^*\Pi_{\mathcal{L}_\theta}(x)(y^*) = \left\{\nabla \Pi_{\mathcal{L}_\theta}(x) y^*\right\}$.*
(b) *If $\lambda_1(x) = 0$ and $\lambda_2(x) > 0$, then*

$$D^*\Pi_{\mathcal{L}_\theta}(x)(y^*) = \left[\partial_B(\Pi_{\mathcal{L}_\theta})(x)y^*\right] \bigcup \left\{x^* \in \mathbb{R}^n \,\middle|\, y^* - x^* \in \mathbb{R}_+ u_x^1,\ \langle x^*, u_x^1\rangle \ge 0\right\}.$$

(c) *If $\lambda_1(x) < 0$ and $\lambda_2(x) = 0$, then*

$$D^*\Pi_{\mathcal{L}_\theta}(x)(y^*) = \left[\partial_B(\Pi_{\mathcal{L}_\theta})(x)y^*\right] \bigcup \left\{x^* \in \mathbb{R}^n \,\middle|\, x^* \in \mathbb{R}_+ u_x^2,\ \langle y^* - x^*, u_x^2\rangle \ge 0\right\}.$$

(d) *If $\lambda_1(x) = \lambda_2(x) = 0$, then*

$$\begin{aligned} D^*\Pi_{\mathcal{L}_\theta}(x)(y^*) = &\left[\partial_B(\Pi_{\mathcal{L}_\theta})(x)y^*\right] \\ &\bigcup \left[\bigcup_{\xi \in \mathrm{bd}(\mathcal{L}_{\frac{\pi}{2}-\theta})/\{0\}} \left\{x^* \in \mathbb{R}^n \,\middle|\, y^* - x^* \in \mathbb{R}_+\xi,\ \langle x^*, \xi\rangle \ge 0\right\}\right] \\ &\bigcup \left[\bigcup_{\eta \in \mathrm{bd}(\mathcal{L}_\theta)/\{0\}} \left\{x^* \,\middle|\, x^* \in \mathbb{R}_+\eta,\ \langle y^* - x^*,\ \eta\rangle \ge 0\right\}\right] \\ &\bigcup \left[\left\{x^* \,\middle|\, x^* \in \mathcal{L}_\theta,\ \ y^* - x^* \in \mathcal{L}_{\frac{\pi}{2}-\theta}\right\}\right], \end{aligned}$$

where the B-subdifferential of $\Pi_{\mathcal{L}_\theta}$ at x is calculated in Lemma 3.1.

Proof Using the well-known representation of the coderivative (2.6) via the outer limit (1.7) of the regular one (see [29,30]) as well as the continuity of $\Pi_{\mathcal{L}_\theta}$, we get

$$D^*\Pi_{\mathcal{L}_\theta}(x)(y^*) = \operatorname*{Lim\,sup}_{\substack{v\to x\\ v^*\to y^*}} \widehat{D}^*\Pi_{\mathcal{L}_\theta}(v)(v^*). \tag{3.21}$$

This allows us to calculate $D^*\Pi_{\mathcal{L}_\theta}$ by passing to the limit in the relationships of Theorem 3.5.

In case (a) we easily get from (3.21) and Theorem 3.5(a) that

$$\begin{aligned} D^*\Pi_{\mathcal{L}_\theta}(x)(y^*) &= \operatorname*{Lim\,sup}_{\substack{v\to x\\ v^*\to y^*}} \widehat{D}^*\Pi_{\mathcal{L}_\theta}(v)(v^*) \\ &= \operatorname*{Lim\,sup}_{\substack{v\to x\\ v^*\to y^*}} \left[\nabla\Pi_{\mathcal{L}_\theta}(v)v^*\right] = \left\{\nabla\Pi_{\mathcal{L}_\theta}(x)y^*\right\}. \end{aligned} \tag{3.22}$$

In case (b) we employ Theorem 3.5(b) along with (3.21), the construction of $A(x, y^*)$ in (3.19), definition (2.11), and representations (3.22) at the points of differentiability to get the relationships

$$\begin{aligned} D^*\Pi_{\mathcal{L}_\theta}(x)(y^*) &= \operatorname*{Lim\,sup}_{\substack{v\to x, v^*\to y^*\\ \lambda_1(v)\neq 0}} \widehat{D}^*\Pi_{\mathcal{L}_\theta}(v)(v^*) \bigcup \operatorname*{Lim\,sup}_{\substack{v\to x, v^*\to y^*\\ \lambda_1(v)=0}} \widehat{D}^*\Pi_{\mathcal{L}_\theta}(v)(v^*) \\ &= \operatorname*{Lim\,sup}_{\substack{v\to x, v^*\to y^*\\ v\in\mathcal{D}_{\Pi_{\mathcal{L}_\theta}}}} \left[\nabla\Pi_{\mathcal{L}_\theta}(v)v^*\right] \bigcup \operatorname*{Lim\,sup}_{\substack{v\to x, v^*\to y^*\\ \lambda_1(v)=0}} \left\{x^*\in\mathbb{R}^n \,\middle|\, v^*-x^*\in\mathbb{R}_+u_v^1,\ \langle x^*, u_v^1\rangle\geq 0\right\} \\ &= \operatorname*{Lim\,sup}_{\substack{v\to x, v^*\to y^*\\ v\in\mathcal{D}_{\Pi_{\mathcal{L}_\theta}}}} \left[\nabla\Pi_{\mathcal{L}_\theta}(v)v^*\right] \bigcup \operatorname*{Lim\,sup}_{\substack{v\to x, v^*\to y^*\\ \lambda_1(v)=0}} A(v, v^*) = \left[\partial_B(\Pi_{\mathcal{L}_\theta})(x)y^*\right] \bigcup A(x, y^*), \end{aligned}$$

where the set $\mathcal{D}_{\Pi_{\mathcal{L}_\theta}}$ unifies the points of differentiability of $\Pi_{\mathcal{L}_\theta}$, and where the last step is due to the outer semicontinuity of $A(v, v^*)$ at (x, y^*) meaning that

$$\operatorname*{Lim\,sup}_{\substack{v\to x, v^*\to y^*\\ \lambda_1(v)=0}} A(v, v^*) = A(x, y^*). \tag{3.23}$$

To verify (3.23), pick any sequences of $x_k \to x$, $v_k^* \to y^*$, and $x_k^* \in A(x_k, v_k^*)$ with $x_k^* \to x^*$ as $k \to \infty$ and then find by (3.19) such $\alpha_k \in \mathbb{R}_+$ that

$$v_k^* - x_k^* = \alpha_k u_{x_k}^1 \ \text{ and } \ \langle x_k^*, u_{x_k}^1\rangle \geq 0, \quad k \in \mathbb{N}. \tag{3.24}$$

This implies by (3.3) that $\alpha_k = (1+\operatorname{ctan}^2\theta)\langle v_k^* - x_k^*, u_{x_k}^1\rangle$ for all $k \in \mathbb{N}$. Since $u_{x_k}^1 \to u_x^1$ by $x_2 \neq 0$, $v_k^* \to y^*$, and $x_k^* \to x^*$, the sequence $\{\alpha_k\}$ also converges to some nonnegative number. Thus passing to the limit in (3.24) as $k \to \infty$ gives us $y^* - x^* \in \mathbb{R}_+u_x^1$ and $\langle x^*, u_x^1\rangle \geq 0$. This means that $x^* \in A(x, y^*)$ by (3.19) and so verifies the outer semicontinuity of A in (3.23).

To proceed in case (c), we employ Theorem 3.5(c) and get similarly to the above that

$$\begin{aligned}
D^*\Pi_{\mathcal{L}_\theta}(x)(y^*) &= \mathop{\mathrm{Lim\,sup}}_{\substack{v\to x, v^*\to y^*\\ \lambda_2(v)\neq 0}} \widehat{D}^*\Pi_{\mathcal{L}_\theta}(v)(v^*) \bigcup \mathop{\mathrm{Lim\,sup}}_{\substack{v\to x, v^*\to y^*\\ \lambda_2(v)=0}} \widehat{D}^*\Pi_{\mathcal{L}_\theta}(v)(v^*)\\
&= \mathop{\mathrm{Lim\,sup}}_{\substack{v\to x, v^*\to y^*\\ v\in\mathcal{D}_{\Pi_{\mathcal{L}_\theta}}}} \left[\nabla\Pi_{\mathcal{L}_\theta}(v)v^*\right] \bigcup \mathop{\mathrm{Lim\,sup}}_{\substack{v\to x, v^*\to y^*\\ \lambda_2(v)=0}} \left\{x^*\in\mathbb{R}^n \,\middle|\, x^*\in\mathbb{R}_+u_v^2,\ \langle v^*-x^*, u_v^2\rangle\geq 0\right\}\\
&= \mathop{\mathrm{Lim\,sup}}_{\substack{v\to x, v^*\to y^*\\ v\in\mathcal{D}_{\Pi_{\mathcal{L}_\theta}}}} \left[\nabla\Pi_{\mathcal{L}_\theta}(v)v^*\right] \bigcup \mathop{\mathrm{Lim\,sup}}_{\substack{v\to x, v^*\to y^*\\ \lambda_2(v)=0}} B(v,v^*) = \left[\partial_B(\Pi_{\mathcal{L}_\theta})(x)y^*\right]\bigcup B(x,y^*),
\end{aligned}$$

where the mapping $B(\cdot,\cdot)$ is defined in (3.20), and where the last step comes from the outer semicontinuity of this mapping at (x, y^*), which can proved similarly to (3.23).

It remains to consider case (d) with $x=0$. In this case we have by (3.21) and Theorem 3.5 that

$$\begin{aligned}
D^*\Pi_{\mathcal{L}_\theta}(0)(y^*) &= \mathop{\mathrm{Lim\,sup}}_{\substack{v\to 0, v^*\to y^*\\ \lambda_1(v)\lambda_2(v)\neq 0}} \widehat{D}^*\Pi_{\mathcal{L}_\theta}(v)(v^*) \bigcup \mathop{\mathrm{Lim\,sup}}_{\substack{v\to 0, v^*\to y^*\\ \lambda_1(v)=0,\lambda_2(v)>0}} \widehat{D}^*\Pi_{\mathcal{L}_\theta}(v)(v^*)\\
&\quad\bigcup \mathop{\mathrm{Lim\,sup}}_{\substack{v\to 0, v^*\to y^*\\ \lambda_1(v)<0,\lambda_2(v)=0}} \widehat{D}^*\Pi_{\mathcal{L}_\theta}(v)(v^*) \bigcup \mathop{\mathrm{Lim\,sup}}_{v^*\to y^*} \widehat{D}^*\Pi_{\mathcal{L}_\theta}(0)(v^*)\\
&= \mathop{\mathrm{Lim\,sup}}_{\substack{v\to 0, v^*\to y^*\\ v\in\mathcal{D}_{\Pi_{\mathcal{L}_\theta}}}} \left[\nabla\Pi_{\mathcal{L}_\theta}(v)v^*\right] \bigcup \mathop{\mathrm{Lim\,sup}}_{\substack{v\to 0, v^*\to y^*\\ \lambda_1(v)=0,\lambda_2(v)>0}} A(v,v^*)\\
&\quad\bigcup \mathop{\mathrm{Lim\,sup}}_{\substack{v\to 0, v^*\to y^*\\ \lambda_1(v)<0,\lambda_2(v)=0}} B(v,v^*) \bigcup \mathop{\mathrm{Lim\,sup}}_{v^*\to y^*} C(v^*),
\end{aligned}$$

where the set-valued mapping $C\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$C(v^*) := \left\{x^*\in\mathbb{R}^n \,\middle|\, x^*\in\mathcal{L}_\theta,\ v^*-x^*\in\mathcal{L}_{\frac{\pi}{2}-\theta}\right\}$$

is clearly outer semicontinuous at the reference point. To proceed further, we now claim that

$$\mathop{\mathrm{Lim\,sup}}_{\substack{v\to 0, v^*\to y^*\\ \lambda_1(v)=0,\lambda_2(v)>0}} A(v,v^*) = \bigcup_{\xi\in\Sigma_1}\left\{x^*\in\mathbb{R}^n \,\middle|\, y^*-x^*\in\mathbb{R}_+\xi,\ \langle x^*,\xi\rangle\geq 0\right\} \tag{3.25}$$

with the union in (3.25) taken over the set

$$\Sigma_1 := \left\{\frac{1}{1+\mathrm{ctan}^2\theta}\begin{bmatrix}1 & 0\\ 0 & \mathrm{ctan}\theta\end{bmatrix}\begin{pmatrix}1\\ w\end{pmatrix} \,\middle|\, w\in\mathbb{R}^{n-1},\ \|w\|=1\right\}.$$

The inclusion '$\subset$' in (3.25) easily follows from $u_v^1\in\Sigma_1$ and the closedness of Σ_1. To derive the converse inclusion '$\supset$', pick any x^* from the right-hand side of (3.25) and find $\alpha\geq 0$ and $w\in\mathbb{R}^{n-1}$ with $\|w\|=1$ satisfying the relationships

$$y^*-x^*=\alpha\xi \quad\text{and}\quad \langle x^*,\xi\rangle\geq 0, \quad\text{where}\quad \xi=\frac{1}{1+\mathrm{ctan}^2\theta}\begin{bmatrix}1 & 0\\ 0 & \mathrm{ctan}\theta\end{bmatrix}\begin{pmatrix}1\\ w\end{pmatrix}. \tag{3.26}$$

Letting $v(t) := t(\mathrm{ctan}\theta, -w)^T$ with $t>0$, we get from (3.2) that $\lambda_1(v(t))=0$, $\lambda_2(v(t))>0$, and $u^1_{v(t)}=\xi$. Hence (3.26) means that $x^*\in A(v(t), y^*)$ for all $t>0$, which ensures that x^* belongs to the right-hand side of (3.25). Similarly we arrive at the representation

$$\operatorname*{Lim\,sup}_{\substack{v\to 0, v^*\to y^*\\ \lambda_1(v)<0, \lambda_2(v)=0}} B(v, v^*) = \bigcup_{\eta\in\Sigma_2} \left\{x^* \in \mathbb{R}^n \,\middle|\, x^* \in \mathbb{R}_+\eta,\ \langle y^* - x^*, \eta\rangle \geq 0\right\},$$

where the union is taken over the set

$$\Sigma_2 := \left\{ \frac{1}{1+\tan^2\theta} \begin{bmatrix} 1 & 0 \\ 0 & \tan\theta \end{bmatrix} \begin{pmatrix} 1 \\ w \end{pmatrix} \,\middle|\, w \in \mathbb{R}^{n-1},\ \|w\| = 1 \right\}.$$

This follows from the proof of (3.25) with replacing $v(t)$ therein by $v(t) := t(-\tan\theta, w)^T$ as $t > 0$.

To complete the proof of the theorem in case (d), let us finally verify that

$$\mathrm{bd}(\mathcal{L}_{\frac{\pi}{2}-\theta}) = \mathbb{R}_+\Sigma_1 \ \text{ and } \ \mathrm{bd}(\mathcal{L}_\theta) = \mathbb{R}_+\Sigma_2. \tag{3.27}$$

The inclusion $\mathrm{bd}(\mathcal{L}_{\frac{\pi}{2}-\theta}) \supset \mathbb{R}_+\Sigma_1$ is trivial. Conversely, take $x = (x_1, x_2) \in \mathrm{bd}(\mathcal{L}_{\frac{\pi}{2}-\theta})$, which means that $\mathrm{ctan}\theta x_1 = \|x_2\|$. If $x_1 = 0$, then $x = 0$ and therefore $x \in \mathbb{R}_+\Sigma_1$. If $x_1 \neq 0$, then $x_1 > 0$ and

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ \frac{x_2}{\tan\theta\|x_2\|} \end{pmatrix} = x_1\left(1+\mathrm{ctan}^2\theta\right)\frac{1}{1+\mathrm{ctan}^2\theta}\begin{bmatrix} 1 & 0 \\ 0 & \mathrm{ctan}\theta \end{bmatrix}\begin{pmatrix} 1 \\ \bar{x}_2 \end{pmatrix} \in \mathbb{R}_+\Sigma_1,$$

which justifies the first equality in (3.27). The second equality therein can be verified similarly. $\square$

It is worth mentioning that the sets Σ_1 and Σ_2, which play an important role in the proof of Theorem 3.6, agree in the symmetric case $\theta = 45°$ of the second-order cone, while $\Sigma_1 \neq \Sigma_2$ in the general case $\theta \neq 45°$ of the circular cone under consideration.

The last result of this section provides precise formulas for calculating the *regular derivative* (2.7) of the projection operator $\Pi_{\mathcal{L}_\theta}$ onto the circular cone.

Theorem 3.7 (calculating the regular derivative of the projection operator) *For any $x \in \mathbb{R}^n$ with decomposition (3.1) and any $w \in \mathbb{R}^n$, the regular derivative (2.7) of the projection operator (3.5) onto the circular cone is calculated as follows:*

(a) *If $\lambda_1(x)\lambda_2(x) \neq 0$, then $\widehat{D}\Pi_{\mathcal{L}_\theta}(x)(w) = \left\{\nabla\Pi_{\mathcal{L}_\theta}(x)w\right\}$.*

(b) *If $\lambda_1(x) = 0$ and $\lambda_2(x) > 0$, then*

$$\widehat{D}\Pi_{\mathcal{L}_\theta}(x)(w) = \begin{cases} \{w\} & \text{if } \ w \in (u_x^1)^\perp, \\ \emptyset & \text{otherwise.} \end{cases}$$

(c) *If $\lambda_1(x) < 0$ and $\lambda_2(x) = 0$, then*

$$\widehat{D}\Pi_{\mathcal{L}_\theta}(x)(w) = \begin{cases} \{0\} & \text{if } \ w \in (u_x^2)^\perp, \\ \emptyset & \text{otherwise.} \end{cases}$$

(d) *If $\lambda_1(x) = \lambda_2(x) = 0$, then*

$$\widehat{D}\Pi_{\mathcal{L}_\theta}(0)(w) = \begin{cases} \{0\} & \text{if } \ w = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof It follows from duality/polarity correspondences in variational analysis that

$$v \in \widehat{D}\Pi_{\mathcal{L}_\theta}(x)(w) \Longleftrightarrow \langle x^*, w\rangle \le \langle y^*, v\rangle \;\text{ when }\; x^* \in D^*\Pi_{\mathcal{L}_\theta}(x)(y^*); \tag{3.28}$$

see, e.g. [30, Proposition 8.37]. This allows us to proceed with calculating the regular derivative of $\Pi_{\mathcal{L}_\theta}$ by using the corresponding coderivative and B-subdifferential formulas obtained above.

Consider first case (a) of the theorem and pick any $v \in \widehat{D}\Pi_{\mathcal{L}_\theta}(x)(w)$. Then it follows from (3.28) and Theorem 3.6(a) that

$$\left\langle y^*, \nabla\Pi_{\mathcal{L}_\theta}(x)w\right\rangle = \left\langle \nabla\Pi_{\mathcal{L}_\theta}(x)y^*, w\right\rangle = \langle x^*, w\rangle \le \langle y^*, v\rangle \quad \text{for any } \; y^* \in \mathbb{R}^n.$$

This readily implies that $v = \nabla\Pi_{\mathcal{L}_\theta}(x)w$, i.e. $\widehat{D}\Pi_{\mathcal{L}_\theta}(x)(w) = \{\nabla\Pi_{\mathcal{L}_\theta}(x)w\}$.

In case (b) we take any pair $(w, v) \in \operatorname{gph}(\widehat{D}\Pi_{\mathcal{L}_\theta}(x))$ and get from Theorem 3.6(b) the inclusion $\partial_B(\Pi_{\mathcal{L}_\theta})(x)y^* \subset D^*\Pi_{\mathcal{L}_\theta}(x)(y^*)$, which readily ensures by Lemma 3.1(b) that $y^* \in D^*\Pi_{\mathcal{L}_\theta}(x)(y^*)$ and also that $[I - (1 + \operatorname{ctan}^2\theta)u_x^1(u_x^1)^T]y^* \in D^*\Pi_{\mathcal{L}_\theta}(x)(y^*)$. Thus it yields by (3.28) the

$$\langle y^*, w\rangle \le \langle y^*, v\rangle \;\text{ and }\; \left\langle y^* - (1 + \operatorname{ctan}^2\theta)u_x^1(u_x^1)^T y^*, w\right\rangle \le \langle y^*, v\rangle \quad \text{for all } y^* \in \mathbb{R}^n. \tag{3.29}$$

The first inequality in (3.29) tells us that $v = w$, while substituting it into the second one gives us $\langle u_x^1, w\rangle = 0$. Picking now $x^* \in A(x, y^*)$, we get by (3.19) that $y^* - x^* = \alpha u_x^1$ for some $\alpha \ge 0$. Hence

$$\langle x^*, w\rangle = \langle y^* - \alpha u_x^1, w\rangle = \langle y^*, w\rangle - \alpha\langle u_x^1, w\rangle = \langle y^*, v\rangle,$$

where the last step is due to $v = w$ and $\langle u_x^1, w\rangle = 0$ as shown above. Thus, we arrive at the representation of the regular derivative $\widehat{D}\Pi_{\mathcal{L}_\theta}(x)(w)$ claimed in (b).

To proceed case (c) of the theorem, pick $(w, v) \in \operatorname{gph} \widehat{D}\Pi_{\mathcal{L}_\theta}(x))$ and deduce from Theorem 3.6(c) and Lemma 3.1(c) that $0 \in D^*\Pi_{\mathcal{L}_\theta}(x)(y^*)$ and $(1 + \tan^2\theta)(u_x^2)^T y^* u_x^2 \in D^*\Pi_{\mathcal{L}_\theta}(x)(y^*)$. Hence we get from (3.28) that $\langle 0, w\rangle \le \langle y^*, v\rangle$ and $\langle (1 + \tan^2\theta)(u_x^2)^T y^* u_x^2, w\rangle \le \langle y^*, v\rangle$ whenever $y^* \in \mathbb{R}^n$; therefore $v = 0$ and $\langle u_x^2, w\rangle = 0$. Taking $x^* \in B(x, y^*)$ gives us by (3.20) that $x^* = \beta u_x^2$ for some $\beta \ge 0$. Then $\langle x^*, w\rangle = \langle \beta u_x^2, w\rangle = 0 = \langle y^*, v\rangle$, where the last step is due to $v = 0$ and $\langle u_x^2, w\rangle = 0$. This justifies the regular derivative formula in case (c).

Our final case is (d), where $x = 0$. Picking any $(w, v) \in \operatorname{gph}(\widehat{D}\Pi_{\mathcal{L}_\theta}(0))$, we get from Theorem 3.6(d) and Lemma 3.1(d) in this case that $y^* \in D^*\Pi_{\mathcal{L}_\theta}(x)(y^*)$ and $0 \in D^*\Pi_{\mathcal{L}_\theta}(x)(y^*)$ for all $y^* \in \mathbb{R}^n$. Plugging these into (3.28) yields $\langle y^*, w\rangle \le \langle y^*, v\rangle$ and $\langle 0, w\rangle \le \langle y^*, v\rangle$. Hence $v = w$ and $v = 0$, which gives us the formula claimed in (d) and thus completes the proof of the theorem. □

4. Generalized derivatives of the circular cone projection operator via orthogonal projections of spectral vectors

This section establishes useful relationships between the generalized derivatives of the projection operator onto the circular cone and the projection operator itself onto the orthogonal spaces to the *spectral vectors* (3.3) associated with $\mathcal{L}_\theta$; we call them for brevity the *orthogonal projections*. Besides being of their own 'qualitative' interest, they essentially simplify the numerical procedure for calculating the generalized derivatives while providing much shorter formulas that do not explicitly depend on the angle θ.

First we calculate orthogonal projections over a hyperplane based on convex optimization. The general result on the projection over $\{x \in \mathbb{R}^n | Mx = b\}$ with $M \in \mathbb{R}^{m\times n}$ being row-full rank and $b \in \mathbb{R}^m$ can be found in [36, Exercise 2D.10]. Here we provide the proof for completeness.

Lemma 4.1 (orthogonal projections) *Given a nonzero vector $\xi \in \mathbb{R}^n$, we have*

$$\Pi_{\xi^\perp}(y) = \left[I - \frac{1}{\|\xi\|^2}\xi\xi^T\right] y \quad \textit{for any} \;\; y \in \mathbb{R}^n.$$

Proof Note that the projection of y onto $\xi^\perp$ solves the following *convex quadratic* optimization problem:

$$\left\{\begin{array}{ll} \min & \frac{1}{2}\|y-x\|^2 \\ \text{s.t.} & x \in \left\{\xi^\perp\right\} \end{array}\right. \iff \left\{\begin{array}{ll} \min & \frac{1}{2}\|y-x\|^2 \\ \text{s.t.} & \langle \xi, x\rangle = 0. \end{array}\right. \tag{4.1}$$

Denote $\bar{x} := \Pi_{\xi^\perp}(y)$ and apply to it the classical *necessary and sufficient* condition for optimality of $\bar{x}$ in (4.1), which characterizes $\bar{x}$ as follows: there is a multiplier $\lambda \in \mathbb{R}$ such that

$$\bar{x} - y + \lambda\xi = 0 \;\text{ and }\; \langle \xi, \bar{x}\rangle = 0,$$

This can be equivalently rewritten as

$$0 = \langle y - \lambda\xi, \xi\rangle = \langle y, \xi\rangle - \lambda\|\xi\|^2, \quad \text{i.e. } \lambda = \frac{1}{\|\xi\|^2}\langle y, \xi\rangle.$$

It allows us to express the projection $\Pi_{\xi^\perp}(y)$ in the form of

$$\Pi_{\xi^\perp}(y) = \bar{x} = y - \lambda\xi = y - \frac{1}{\|\xi\|^2}\langle y, \xi\rangle\xi = y - \frac{1}{\|\xi\|^2}\xi\xi^T y = \left[I - \frac{1}{\|\xi\|^2}\xi\xi^T\right] y$$

and therefore completes the proof of the lemma. □

Having this in hand, we derive now alternative versions of the main results of Section 3 establishing relationships between the major generalized derivatives of $\Pi_{\mathcal{L}_\theta}$ and orthogonal projections associated with the spectral vectors u_x^1, u_x^2 of the circular cone. We start with the *B-subdifferential* (2.11) of $\Pi_{\mathcal{L}_\theta}$.

Proposition 4.2 (B-subdifferential via orthogonal projections) *Let $\lambda_i(x)$ and u_x^i, $i = 1, 2$, be the spectral values* (3.2) *and spectral vectors* (3.3)*, respectively, associated with the vector $x \in \mathbb{R}^n$ in the spectral decomposition* (3.1)*. Then we have the following relationships between the B-subdifferential of the projection operator $\Pi_{\mathcal{L}_\theta}$ and the orthogonal projections generated by u_x^i:*

(a) *If $\lambda_1(x)\lambda_2(x) \neq 0$, then*

$$\partial_B(\Pi_{\mathcal{L}_\theta})(x) = \left\{\begin{array}{ll} I & \text{if} \;\; \lambda_1(x) > 0 \text{ and } \lambda_2(x) > 0, \\ 0 & \text{if} \;\; \lambda_1(x) < 0 \text{ and } \lambda_2(x) < 0, \\ \frac{\lambda_2(x)}{\lambda_2(x)-\lambda_1(x)}\Pi_{(u_x^1)^\perp} + \left(1 - \frac{\lambda_2(x)}{\lambda_2(x)-\lambda_1(x)}\right)\left[I - \Pi_{(u_x^2)^\perp}\right] & \text{if} \;\; \lambda_1(x) < 0 \text{ and } \lambda_2(x) > 0. \end{array}\right.$$

(b) *If* $\lambda_1(x) = 0$ *and* $\lambda_2(x) > 0$, *then* $\partial_B(\Pi_{\mathcal{L}_\theta})(x) = \{I, \Pi_{(u_x^1)^\perp}\}$.
(c) *If* $\lambda_1(x) < 0$ *and* $\lambda_2(x) = 0$, *then* $\partial_B(\Pi_{\mathcal{L}_\theta})(x) = \{0, I - \Pi_{(u_x^2)^\perp}\}$.
(d) *If* $\lambda_1(x) = \lambda_2(x) = 0$, *then*

$$\partial_B(\Pi_{\mathcal{L}_\theta})(x) = \{0, I\} \bigcup \mathrm{co}\Big\{\Pi_{(u_z^1)^\perp}, I - \Pi_{(u_z^2)^\perp} \Big|\ \|z_2\| = 1\Big\}.$$

Proof Applying Lemma 4.1 to $\xi = u_x^1, u_x^2$ and using formulas (3.3) for the spectral vectors give us

$$\Pi_{(u_x^1)^\perp} = I + \frac{1}{1+\mathrm{ctan}^2\theta}\begin{bmatrix} -1 & \mathrm{ctan}\theta\bar{x}_2 \\ \mathrm{ctan}\theta\bar{x}_2 & -\mathrm{ctan}^2\theta\bar{x}_2\bar{x}_2^T \end{bmatrix}, \tag{4.2}$$

$$\Pi_{(u_x^2)^\perp} = I - \frac{1}{1+\tan^2\theta}\begin{bmatrix} 1 & \tan\theta\bar{x}_2 \\ \tan\theta\bar{x}_2 & \tan^2\theta\bar{x}_2\bar{x}_2^T \end{bmatrix}. \tag{4.3}$$

Then the result follows by comparison (4.2) and (4.3) with the corresponding calculations of Lemma 3.1 and taking into account the gradient expression (3.6) for $\Pi_{\mathcal{L}_\theta}$ at x in case (a). □

Next we establish relationships between the *directional derivative* of $\Pi_{\mathcal{L}_\theta}$ and the orthogonal projections generated by the spectral vectors.

Proposition 4.3 (directional derivative via orthogonal projections) *In the setting of Proposition 4.2 we have the following expressions for the directional derivative of* $\Pi_{\mathcal{L}_\theta}$*:*

(a) *If* $\lambda_1(x)\lambda_2(x) \neq 0$, *then*

$$\Pi'_{\mathcal{L}_\theta}(x;h) = \begin{cases} h & \text{if } \lambda_1(x) > 0 \text{ and } \lambda_2(x) > 0, \\ 0 & \text{if } \lambda_1(x) < 0 \text{ and } \lambda_2(x) < 0, \\ \frac{\lambda_2(x)}{\lambda_2(x)-\lambda_1(x)}\Pi_{(u_x^1)^\perp}(h) + \left(1 - \frac{\lambda_2(x)}{\lambda_2(x)-\lambda_1(x)}\right)\left[h - \Pi_{(u_x^2)^\perp}(h)\right] & \text{if } \lambda_1(x) < 0 \text{ and } \lambda_2(x) > 0. \end{cases}$$

(b) *If* $\lambda_1(x) = 0$ *and* $\lambda_2(x) > 0$, *then*

$$\Pi'_{\mathcal{L}_\theta}(x;h) = \begin{cases} h & \text{if } \langle u_x^1, h\rangle \geq 0, \\ \Pi_{(u_x^1)^\perp}(h) & \text{otherwise,} \end{cases}$$

(c) *If* $\lambda_1(x) < 0$ *and* $\lambda_2(x) = 0$, *then*

$$\Pi'_{\mathcal{L}_\theta}(x;h) = \begin{cases} 0 & \text{if } \langle u_x^2, h\rangle \leq 0, \\ h - \Pi_{(u_x^2)^\perp}(h) & \text{otherwise.} \end{cases}$$

Proof Similar to Proposition 4.2 with applying Lemma 3.2 instead of Lemma 3.1. □

Due to Proposition 3.4, the expression of graphical derivative $D\Pi_{\mathcal{L}_\theta}(x)$ via orthogonal projections can be obtain as well. To proceed now with coderivatives, we first consider the case of the *regular coderivative* (2.8).

Proposition 4.4 (regular coderivative via orthogonal projections) *In the setting of Proposition 4.2 we have the following expressions for the regular coderivative of* $\Pi_{\mathcal{L}_\theta}$ *at* x *for any* $y^* \in \mathbb{R}^n$*:*

(a) *If* $\lambda_1(x)\lambda_2(x) \neq 0$, *then*

$$\widehat{D}^*\Pi_{\mathcal{L}_\theta}(x)(y^*) = \begin{cases} y^* & \text{if } \lambda_1(x) > 0 \text{ and } \lambda_2(x) > 0, \\ 0 & \text{if } \lambda_1(x) < 0 \text{ and } \lambda_2(x) < 0, \\ \frac{\lambda_2(x)}{\lambda_2(x)-\lambda_1(x)}\Pi_{(u_x^1)^\perp}(y^*) + \left(1 - \frac{\lambda_2(x)}{\lambda_2(x)-\lambda_1(x)}\right)\left[y^* - \Pi_{(u_x^2)^\perp}(y^*)\right] & \text{if } \lambda_1(x) < 0 \text{ and } \lambda_2(x) > 0. \end{cases}$$

(b) *If* $\lambda_1(x) = 0$ *and* $\lambda_2(x) > 0$, *then*

$$\widehat{D}^*\Pi_{\mathcal{L}_\theta}(x)(y^*) = \begin{cases} \operatorname{co}\left\{y^*, \Pi_{(u_x^1)^\perp}(y^*)\right\} & \text{if } \langle y^*, u_x^1\rangle \geq 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

(c) *If* $\lambda_1(x) < 0$ *and* $\lambda_2(x) = 0$, *then*

$$\widehat{D}^*\Pi_{\mathcal{L}_\theta}(x)(y^*) = \begin{cases} \operatorname{co}\left\{0, y^* - \Pi_{(u_x^2)^\perp}(y^*)\right\} & \text{if } \langle y^*, u_x^2\rangle \geq 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

(d) *If* $\lambda_1(x) = \lambda_2(x) = 0$ *and thus* $x = 0$, *then* $\widehat{D}^*\Pi_{\mathcal{L}_\theta}(x)(y^*) = \mathcal{L}_\theta \bigcap \left\{y^* - \mathcal{L}_{\frac{\pi}{2}-\theta}\right\}$.

Proof We need to justify the following representation of the mapping A from (3.19):

$$A(x, y^*) = \begin{cases} \operatorname{co}\left\{y^*, \Pi_{(u_x^1)^\perp}(y^*)\right\} & \text{if } \langle y^*, u_x^1\rangle \geq 0, \\ \emptyset & \text{otherwise.} \end{cases} \tag{4.4}$$

To proceed, take $x^* \in A(x, y^*)$ and get by (3.19) that $x^* = y^* - \alpha u_x^1$ with $\alpha \geq 0$ and $\langle x^*, u_x^1\rangle \geq 0$. Thus

$$0 \leq \langle x^*, u_x^1\rangle = \langle y^*, u_x^1\rangle - \alpha\|u_x^1\|^2 = \langle y^*, u_x^1\rangle - \frac{1}{1+\operatorname{ctan}^2\theta}\alpha,$$

which yields $\alpha \leq (1+\operatorname{ctan}^2\theta)\langle y^*, u_x^1\rangle$. If $\langle y^*, u_x^1\rangle < 0$, we have $A(x, y^*) = \emptyset$ while $\langle y^*, u_x^1\rangle \geq 0$ gives us $0 \leq \alpha \leq (1+\operatorname{ctan}^2\theta)\langle y^*, u_x^1\rangle$. The case of $\alpha = 0$ corresponds to

$$y^* = x^* \in A(x, y^*)$$

while the other one $\alpha = (1+\operatorname{ctan}^2\theta)\langle y^*, u_x^1\rangle$ corresponds to

$$x^* = y^* - (1+\operatorname{ctan}^2\theta)\langle y^*, u_x^1\rangle u_x^1 = y^* - (1+\operatorname{ctan}^2\theta)u_x^1(u_x^1)^T y^* = \Pi_{(u_x^1)^\perp}(y^*).$$

In this way we arrive at the inclusion

$$A(x, y^*) \subset \operatorname{co}\left\{y^*, \Pi_{(u_x^1)^\perp}(y^*)\right\} \quad \text{as} \quad \langle y^*, u_x^1\rangle \geq 0.$$

To verify the converse inclusion in (4.4), it suffices to show, by the convexity of $A(x, y^*)$, that y^* and $\Pi_{(u_x^1)^\perp}(y^*)$ are both in $A(x, y^*)$. Since $\langle y^*, u_x^1\rangle \geq 0$, we clearly have $y^* \in A(x, y^*)$. Observe further that

$$y^* = \Pi_{(u_x^1)^\perp}(y^*) + \Pi_{\operatorname{span}\{u_x^1\}}(y^*),$$

and so $y^* - \Pi_{(u_x^1)^\perp}(y^*) = \beta u_x^1$ for some $\beta \in \mathbb{R}$. Thus

$$\beta\|u_x^1\|^2 = \langle y^* - \Pi_{(u_x^1)^\perp}(y^*), u_x^1\rangle = \langle y^*, u_x^1\rangle \geq 0,$$

i.e. $\beta \geq 0$. This ensures together with $\langle \Pi_{(u_x^1)^\perp}(y^*), u_x^1 \rangle = 0$ that $\Pi_{(u_x^1)^\perp}(y^*) \in A(x, y^*)$, which therefore verifies representation (4.4). The following representation

$$B(x, y^*) = \begin{cases} \operatorname{co}\left\{0, y^* - \Pi_{(u_x^2)^\perp}(y^*)\right\} & \text{if} \quad \langle y^*, u_x^2 \rangle \geq 0, \\ \emptyset & \text{otherwise} \end{cases} \tag{4.5}$$

can be derived similarly. We complete the proof by putting Theorem 3.5, (4.4), and (4.5) together. □

We present the main result of this section establishing expressions of the *coderivative* (2.6) of the projection operator $\Pi_{\mathcal{L}_\theta}$ via the orthogonal projections generated by the spectral vectors (3.3).

THEOREM 4.5 (coderivative via orthogonal projections) *In the setting of Proposition 4.2 we have the following expressions for the coderivative of $\Pi_{\mathcal{L}_\theta}$ at x for any $y^* \in \mathbb{R}^n$:*

(a) *If $\lambda_1(x)\lambda_2(x) \neq 0$, then*

$$D^*\Pi_{\mathcal{L}_\theta}(x)(y^*) = \begin{cases} y^* & \text{if} \quad \lambda_1(x) > 0 \text{ and } \lambda_2(x) > 0, \\ 0 & \text{if} \quad \lambda_1(x) < 0 \text{ and } \lambda_2(x) < 0, \\ \frac{\lambda_2(x)}{\lambda_2(x)-\lambda_1(x)} \Pi_{(u_x^1)^\perp}(y^*) + \left(1 - \frac{\lambda_2(x)}{\lambda_2(x)-\lambda_1(x)}\right)\left[y^* - \Pi_{(u_x^2)^\perp}(y^*)\right] & \text{if} \quad \lambda_1(x) < 0 \text{ and } \lambda_2(x) > 0. \end{cases}$$

(b) *If $\lambda_1(x) = 0$ and $\lambda_2(x) > 0$, then*

$$D^*\Pi_{\mathcal{L}_\theta}(x)(y^*) = \begin{cases} \operatorname{co}\left\{y^*, \Pi_{(u_x^1)^\perp}(y^*)\right\} & \text{if} \quad \langle y^*, u_x^1 \rangle \geq 0, \\ \left\{y^*, \Pi_{(u_x^1)^\perp}(y^*)\right\} & \text{otherwise.} \end{cases}$$

(c) *If $\lambda_1(x) < 0$ and $\lambda_2(x) = 0$, then*

$$D^*\Pi_{\mathcal{L}_\theta}(x)(y^*) = \begin{cases} \operatorname{co}\left\{0, y^* - \Pi_{(u_x^2)^\perp}(y^*)\right\} & \text{if} \quad \langle y^*, u_x^2 \rangle \geq 0, \\ \left\{0, y^* - \Pi_{(u_x^2)^\perp}(y^*)\right\} & \text{otherwise.} \end{cases}$$

(d) *If $\lambda_1(x) = \lambda_2(x) = 0$ and thus $x = 0$, then*

$$\begin{aligned} D^*\Pi_{\mathcal{L}_\theta}(0)(y^*) &= \left\{0, y^*\right\} \bigcup \left[\bigcup_{\|z_2\|=1} \operatorname{co}\left\{\Pi_{(u_z^1)^\perp}(y^*), y^* - \Pi_{(u_z^2)^\perp}(y^*)\right\} \right] \\ &\bigcup \left[\bigcup_{\substack{\xi \in \operatorname{bd}(\mathcal{L}_{\frac{\pi}{2}-\theta})/\{0\} \\ \langle y^*, \xi \rangle \geq 0}} \operatorname{co}\left\{y^*, \Pi_{\xi^\perp}(y^*)\right\} \right] \\ &\bigcup \left[\bigcup_{\substack{\eta \in \operatorname{bd}(\mathcal{L}_\theta)/\{0\} \\ \langle \eta, y^* \rangle \geq 0}} \operatorname{co}\left\{0, y^* - \Pi_{\eta^\perp}(y^*)\right\} \right] \bigcup \left[\left(\mathcal{L}_\theta \bigcap \left\{y^* - \mathcal{L}_{\frac{\pi}{2}-\theta}\right\} \right) \right]. \end{aligned}$$

Proof Follows the proof of Proposition 4.4 with employing Theorem 3.6 and Proposition 4.2. □

Proceeding similarly with the usage of Theorem 3.7, we arrive at the following result.

THEOREM 4.6 (regular derivative via orthogonal projections) *In the setting of Proposition 4.2 we have the following expressions for the regular derivative of* $\Pi_{\mathcal{L}_\theta}$ *at* x *for any* $w \in \mathbb{R}^n$*:*

(a) *If* $\lambda_1(x)\lambda_2(x) \neq 0$*, then* $\widehat{D}\Pi_{\mathcal{L}_\theta}(x)(w) = \left\{\nabla\Pi_{\mathcal{L}_\theta}(x)w\right\}$.
(b) *If* $\lambda_1(x) = 0$ *and* $\lambda_2(x) > 0$*, then*

$$\widehat{D}\Pi_{\mathcal{L}_\theta}(x)(w) = \begin{cases} \{\Pi_{(u_x^1)^\perp}(w)\} & \text{if } w \in (u_x^1)^\perp, \\ \emptyset & \text{otherwise.} \end{cases}$$

(c) *If* $\lambda_1(x) < 0$ *and* $\lambda_2(x) = 0$*, then*

$$\widehat{D}\Pi_{\mathcal{L}_\theta}(x)(w) = \begin{cases} \left\{\left[I - \Pi_{(u_x^2)^\perp}\right](w)\right\} & \text{if } w \in (u_x^2)^\perp, \\ \emptyset & \text{otherwise.} \end{cases}$$

(d) *If* $\lambda_1(x) = \lambda_2(x) = 0$*, then*

$$\widehat{D}\Pi_{\mathcal{L}_\theta}(0)(w) = \begin{cases} \{\Pi_{(u_x^1)^\perp}(0)\} \text{ or } \left\{\left[I - \Pi_{(u_x^2)^\perp}\right](0)\right\} & \text{if } w = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

We conclude this section with two remarks clarifying our approach and possible applications.

Remark 4.7 (on reduction to the second-order cone) Recalling relationships (1.6) between the circular and second-order cones, a natural question arises about the possibility to calculate generalized derivatives of the circular cone $\mathcal{L}_\theta$ by reducing it to the second-order cone $\mathcal{K}^n$. In particular, would it be possible to derive the results on calculating the coderivatives of $\Pi_{\mathcal{L}_\theta}$ in Theorems 3.5 and 3.6 from the known results of [35] for $\Pi_{\mathcal{K}^n}$? Our discussions below show that such an approach meets *principal difficulties* and does not seem to be implemented. That is why we give the detailed proof and arguments on some main results given above, although some analysis techniques are inspired by [35]. Indeed, for any closed and convex set E, the projection and normal cone have the following well-known formulas

$$\Pi_E = (I + N_E)^{-1} \text{ and } N_E = \Pi_E^{-1} - I. \tag{4.6}$$

Applying to (1.6) the calculus rules from [29, Corollary 1.15] and [30, Exercise 6.7] gives us

$$N_{\mathcal{L}_\theta}(x) = AN_{\mathcal{K}^n}(Ax) \text{ and } T_{\mathcal{L}_\theta}(x) = A^{-1}T_{\mathcal{K}^n}(Ax). \tag{4.7}$$

Unifying (4.6) and (4.7) yields the projection representation for $\mathcal{L}_\theta$ via the *inverse* projection for $\mathcal{K}^n$ by

$$\Pi_{\mathcal{L}_\theta} = (I + AN_{\mathcal{K}^n} \circ A)^{-1} = \left(I + A(\Pi_{\mathcal{K}^n}^{-1} - I) \circ A\right)^{-1}. \tag{4.8}$$

The presence of the inverse operator in (4.8) does not make it possible to represent the regular and limiting coderivatives of $\Pi_{\mathcal{L}_\theta}$ via the corresponding constructions for $\Pi_{\mathcal{K}^n}$ by employing the known calculus rules. Let us demonstrate it for the case of the coderivative D^* having in mind that the same arguments work for the case of $\widehat{D}^*$. To proceed, we get from (4.8) the equivalences

$$\begin{aligned}(x,y)\in \operatorname{gph}\Pi_{\mathcal{L}_\theta} &\iff x-y\in AN_{\mathcal{K}^n}(Ay)\iff A^{-1}x-A^{-1}y\in N_{\mathcal{K}^n}(Ay)\\ &\iff Ay=\Pi_{\mathcal{K}^n}(A^{-1}x-A^{-1}y+Ay)\\ &\iff (A^{-1}x-A^{-1}y+Ay,Ay)\in \operatorname{gph}\Pi_{\mathcal{K}^n}\\ &\iff \begin{bmatrix}A^{-1} & A-A^{-1}\\ 0 & A\end{bmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\in \operatorname{gph}\Pi_{\mathcal{K}^n}.\end{aligned}\tag{4.9}$$

Consider further the extended matrix

$$\Gamma:=\begin{bmatrix}A^{-1} & A-A^{-1}\\ 0 & A\end{bmatrix}\ \text{ with }\ \Gamma^{-1}=\begin{bmatrix}A & -A+A^{-1}\\ 0 & A^{-1}\end{bmatrix}$$

and observe from (4.9) that $\operatorname{gph}\Pi_{\mathcal{L}_\theta}=\Gamma^{-1}(\operatorname{gph}\Pi_{\mathcal{K}^n})$. This gives us the representation

$$N_{\operatorname{gph}\Pi_{\mathcal{L}_\theta}}\big(x,\Pi_{\mathcal{L}_\theta}(x)\big)=\Gamma N_{\operatorname{gph}\Pi_{\mathcal{K}^n}}\left(\Gamma^T\begin{pmatrix}x\\ \Pi_{\mathcal{L}_\theta}(x)\end{pmatrix}\right).$$

Using this and the coderivative definition (2.6), we have the following transformations:

$$\begin{aligned}&y^*\in D^*\Pi_{\mathcal{L}_\theta}(x)(x^*)\iff(y^*,-x^*)\in N_{\operatorname{gph}\Pi_{\mathcal{L}_\theta}}\big(x,\Pi_{\mathcal{L}_\theta}(x)\big)\\ &\quad=\Gamma N_{\operatorname{gph}\Pi_{\mathcal{K}^n}}\left(\Gamma\begin{pmatrix}x\\ \Pi_{\mathcal{L}_\theta}(x)\end{pmatrix}\right)\\ &\quad\iff\Gamma^{-1}(y^*,-x^*)\in N_{\operatorname{gph}\Pi_{\mathcal{K}^n}}\left(\Gamma\begin{pmatrix}x\\ \Pi_{\mathcal{L}_\theta}(x)\end{pmatrix}\right)\\ &\quad\iff(Ay^*+Ax^*-A^{-1}x^*,-A^{-1}x^*)\in N_{\operatorname{gph}\Pi_{\mathcal{K}^n}}\\ &\qquad\times\big(A^{-1}x+A\Pi_{\mathcal{L}_\theta}(x)-A^{-1}\Pi_{\mathcal{L}_\theta}(x),A\Pi_{\mathcal{L}_\theta}(x)\big)\\ &\quad\iff(A^{-1}x^*,Ay^*+Ax^*-A^{-1}x^*)\in \operatorname{gph}D^*\Pi_{\mathcal{K}^n}\\ &\qquad\times\big(A\Pi_{\mathcal{L}_\theta}(x)+A^{-1}\Pi_{(\mathcal{L}_\theta)^\circ}(x),A\Pi_{\mathcal{L}_\theta}(x)\big)\\ &\quad\iff\begin{bmatrix}A-A^{-1} & A\end{bmatrix}\begin{pmatrix}x^*\\ y^*\end{pmatrix}\in D^*\Pi_{\mathcal{K}^n}\big(A\Pi_{\mathcal{L}_\theta}(x)+A^{-1}\Pi_{(\mathcal{L}_\theta)^\circ}(x)\big)(A^{-1}x^*).\end{aligned}\tag{4.10}$$

Since the operator $\begin{bmatrix}A-A^{-1} & A\end{bmatrix}$ in (4.10) is *not invertible*, the obtained relationship does not allow us to easily find (x^*,y^*) and so to reduce calculating the coderivative of $\Pi_{\mathcal{L}_\theta}$ to that of $\Pi_{\mathcal{K}^n}$.

Remark 4.8 (applications of generalized derivatives) It has been well recognized in variational analysis and optimization that the generalized derivatives considered in this paper are very instrumental for characterizing fundamental properties of solutions maps to constraint and variational systems related to Lipschitzian stability, metric regularity, openness, calmness, etc., as well as for a variety of applications to optimization, equilibrium, and control problems. Among numerous publications on these topics, we refer the reader to the books [29,30] and to the recent paper [8] devoted to general problems of conic programming; see also the bibliographies therein. It is crucial to emphasize that the results established in these directions are the most effective when the aforementioned generalized derivative can be calculated entirely in terms of the initial problem data. This is done in this paper for the projection operator onto the circular cone. Detailed applications of the obtained results to particular issues of variational analysis and optimization will be done in our future research; see also the next section.

5. Full and tilt stability in circular cone programming

The final section of the paper provides applications of the generalized derivatives to complete characterizations of the fundamental notions of *tilt* and *full stability* of locally optimal solutions for the case of mathematical programs with circular cone constrains, or problems of circular come programming. It has been known from the general optimization theory that the most effective characterizations of these notions are given via the *second-order subdifferential* of the indicator functions to the corresponding constraint mapping, which in our case is the circular cone $\mathcal{L}_\theta$. Since the second-order subdifferential (2.10) is the coderivative of the first-order subdifferential mapping for the indicator function $\delta_{\mathcal{L}_\theta}$ of $\mathcal{L}_\theta$, the calculation of this construction plays a crucial role in the desired stability characterizations. The coderivative calculation results for the projection operator $\Pi_{\mathcal{L}_\theta}$ obtained in Sections 3 and 4 could be very instrumental to proceed in this direction due to the relationships in (4.6) between the normal cone to a convex set and the projection onto it. Implementing it in detail requires careful and lengthy considerations, similarly to what has done in the recent paper [8] in the case of the second-order cone.

In what follows we are able to significantly simplify this work in the case of the circular cone by taking into account its relationship (1.6) with the second-order cone and applying the results of [8] together with calculus rules of generalized differentiation. First we present three technical lemmas of their own interest, which are needed for our subsequent analysis.

Lemma 5.1 (interior and boundary relationships) *The following relationships hold between the interiors and boundaries of the circular and second-order cones:*

(a) $\operatorname{int}\mathcal{L}_\theta = A^{-1}\operatorname{int}\mathcal{K}^n$ *and* $\operatorname{bd}\mathcal{L}_\theta = A^{-1}\operatorname{bd}\mathcal{K}^n$*;*
(b) $\operatorname{int}\mathcal{K}^n = A\operatorname{int}\mathcal{L}_\theta$ *and* $\operatorname{bd}\mathcal{K}^n = A\operatorname{bd}\mathcal{L}_\theta$.

Proof Note that A is *nonsingular*. It is easy to verify that

$$\begin{aligned} \operatorname{int}\mathcal{L}_\theta &= \left\{x \in \mathbb{R}^n \,\middle|\, x_1\tan\theta > \|x_2\|\right\}, \quad &\operatorname{int}\mathcal{K}^n &= \left\{x \in \mathbb{R}^n \,\middle|\, x_1 > \|x_2\|\right\},\\ \operatorname{bd}\mathcal{L}_\theta &= \left\{x \in \mathbb{R}^n \,\middle|\, x_1\tan\theta = \|x_2\|\right\}, \quad &\operatorname{bd}\mathcal{K}^n &= \left\{x \in \mathbb{R}^n \,\middle|\, x_1 = \|x_2\|\right\}. \end{aligned}$$

which directly leads us to the claimed relationships. □

Lemma 5.2 (normal and tangent cones to the circular cone) *For $x \in \mathcal{L}_\theta$ we have the following formulas for the normal and tangent cones to $\mathcal{L}_\theta$:*

$$\text{(a)}\quad N_{\mathcal{L}_\theta}(x) = \begin{cases} -\mathcal{L}_{\frac{\pi}{2}-\theta} & \text{if } x = 0,\\ \{0\} & \text{if } x \in \operatorname{int}\mathcal{L}_\theta,\\ \mathbb{R}_+(-x_1\tan^2\theta, x_2) & \text{if } x \in \operatorname{bd}\mathcal{L}_\theta/\{0\}. \end{cases}$$

(b) $T_{\mathcal{L}_\theta}(x) = \begin{cases} \mathcal{L}_\theta & \text{if } \ x = 0, \\ \mathbb{R}^n & \text{if } \ x \in \operatorname{int} \mathcal{L}_\theta, \\ \left\{(d_1, d_2) \in \mathbb{R}^n \,\middle|\, d_2^T x_2 - \tan^2\theta x_1 d_1 \le 0\right\} & \text{if } \ x \in \operatorname{bd} \mathcal{L}_\theta/\{0\}. \end{cases}$

(c) $\operatorname{span}\left\{N_{\mathcal{L}_\theta}(x)\right\} = \begin{cases} \mathbb{R}^n & \text{if } \ x = 0, \\ \{0\} & \text{if } \ x \in \operatorname{int} \mathcal{L}_\theta, \\ \mathbb{R}(-x_1 \tan^2\theta, x_2) & \text{if } \ x \in \operatorname{bd} \mathcal{L}_\theta/\{0\}. \end{cases}$

Proof We combine (4.7) with calculations of $N_{\mathcal{K}^n}$ and its span obtained in [6, Lemma 25] and [25, Lemma 3.5 and (3.16)], respectively, to get the expressions

$$\begin{aligned} \operatorname{span}\{N_{\mathcal{L}_\theta}(x\} &= A \operatorname{span}\left\{N_{\mathcal{K}^n}(Ax)\right\} \\ &= A \begin{cases} \mathbb{R}^n & \text{if } \ Ax = 0, \\ \{0\} & \text{if } \ Ax \in \operatorname{int} \mathcal{K}^n, \\ \mathbb{R}(-x_1 \tan\theta, x_2) & \text{if } \ Ax \in \operatorname{bd} \mathcal{K}^n/\{0\} \end{cases} \\ &= \begin{cases} \mathbb{R}^n & \text{if } \ x = 0, \\ \{0\} & \text{if } \ x \in \operatorname{int} \mathcal{L}_\theta, \\ \mathbb{R}(-x_1 \tan^2\theta, x_2) & \text{if } \ x \in \operatorname{bd} \mathcal{L}_\theta/\{0\}, \end{cases} \end{aligned}$$

which thus complete the proof of the lemma. □

LEMMA 5.3 (relationships between the first-order and second-order subdifferentials of $\delta_{\mathcal{L}_\theta}$ and $\delta_{\mathcal{K}^n}$) *For any $x \in \mathcal{L}_\theta$ the following relationships hold:*

(a) $\partial\delta_{\mathcal{L}_\theta}(x) = A\partial\delta_{\mathcal{K}^n}(Ax)$;
(b) $\partial^2\delta_{\mathcal{L}_\theta}(x, w)(u) = A\partial^2\delta_{\mathcal{K}^n}(Ax, A^{-1}w)(Au)$ *whenever* $w \in \partial\delta_{\mathcal{L}_\theta}(x)$ *and* $u \in \mathbb{R}^n$.

Proof To verify (a), observe from (1.6) that $\delta_{\mathcal{L}_\theta}(x) = \delta_{\mathcal{K}^n}(Ax)$, which yields $\partial\delta_{\mathcal{L}_\theta}(x) = A\partial\delta_{\mathcal{K}^n}(Ax)$. This implies therefore the graph relationship

$$\operatorname{gph}\big(\partial\delta_{\mathcal{L}_\theta}\big) = \begin{bmatrix} A^{-1} & 0 \\ 0 & A \end{bmatrix} \operatorname{gph}\big(\partial\delta_{\mathcal{K}^n}\big).$$

To prove (b), we take the latter into account, pick $(x, w) \in \operatorname{gph}(\partial\delta_{\mathcal{L}_\theta})$, and apply the calculus rule from [29, Theorem 1.17] to obtain the equality

$$N_{\operatorname{gph}(\partial\delta_{\mathcal{L}_\theta})}(x, w) = \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} N_{\operatorname{gph}(\partial\delta_{\mathcal{K}^n})}(Ax, A^{-1}w).$$

This gives us by definition (2.6) that

$$\begin{aligned} (u, v) \in \operatorname{gph}(D^*\partial\delta_{\mathcal{L}_\theta})(x, w) &\iff (v, -u) \in N_{\operatorname{gph}(\partial\delta_{\mathcal{L}_\theta})}(x, w) \\ &\iff (A^{-1}v, -Au) \in N_{\operatorname{gph}(\partial\delta_{\mathcal{K}^n})}(Ax, A^{-1}w) \\ &\iff (Au, A^{-1}v) \in \operatorname{gph}(D^*\delta_{\mathcal{K}^n})(Ax, A^{-1}w), \end{aligned}$$

which readily ensures the equivalences

$$\begin{aligned} v \in (D^*\partial\delta_{\mathcal{L}_\theta})(x, w)(u) &\iff A^{-1}v \in (D^*\delta_{\mathcal{K}^n})(Ax, A^{-1}w)(Au) \\ &\iff v \in A(D^*\delta_{\mathcal{K}^n})(Ax, A^{-1}w)(Au). \end{aligned}$$

By construction (2.10) we have therefore

$$\begin{aligned}\partial^2\delta_{\mathcal{L}_\theta}(x,w)(u) &= D^*(\partial\delta_{\mathcal{L}_\theta})(x,w)(u) = A(D^*\delta_{\mathcal{K}^n})(Ax,A^{-1}w)(Au)\\ &= A\partial^2\delta_{\mathcal{K}^n}(Ax,A^{-1}w)(Au), \end{aligned}\tag{5.1}$$

which justifies the second-order relationship claimed in (b). $\square$

Now we are ready to obtain major calculations for the second-order subdifferential of $\delta_{\mathcal{L}_\theta}$ that are crucial for deriving the main results of this section.

THEOREM 5.4 (major characteristics of $\partial^2\delta_{\mathcal{L}_\theta}$) *For any $x \in \mathcal{L}_\theta$ and $w \in N_{\mathcal{L}_\theta}(x)$ we have the following calculation formulas for the domain*

$$\begin{aligned}&\operatorname{dom}\partial^2\delta_{\mathcal{L}_\theta}(x,w)\\ &= \left\{ u\in\mathbb{R}^n \;\middle|\; \begin{array}{ll} u=0 & \text{if } -w\in\operatorname{int}\mathcal{L}_{\frac{\pi}{2}-\theta},\\ u\in\mathbb{R}(-w_1\mathrm{ctan}^2\theta, w_2) & \text{if } -w\in\operatorname{bd}\mathcal{L}_{\frac{\pi}{2}-\theta}/\{0\},\ x=0,\\ \langle u,w\rangle=0 & \text{if } -w\in\operatorname{bd}\mathcal{L}_{\frac{\pi}{2}-\theta}/\{0\},\ x\in\operatorname{bd}\mathcal{L}_\theta/\{0\} \end{array}\right\}\end{aligned}\tag{5.2}$$

and the value of the second-order subdifferential of $\delta_{\mathcal{L}_\theta}(x,w)$ at zero given by

$$\partial^2\delta_{\mathcal{L}_\theta}(x,w)(0) = \begin{cases}\mathbb{R}^n & \text{if } x=0,\\ \{0\} & \text{if } x\in\operatorname{int}\mathcal{L}_\theta,\\ \mathbb{R}(-x_1\tan^2\theta, x_2) & \text{if } x\in\operatorname{bd}\mathcal{L}_\theta/\{0\}.\end{cases}\tag{5.3}$$

Proof Employing Lemmas 5.1–5.3 as well as [25, Theorem 3.6 and Lemma 4.6] yields

$$\begin{aligned}\operatorname{dom}\partial^2\delta_{\mathcal{L}_\theta}(x,w) &= A^{-1}\operatorname{dom}\partial^2\delta_{\mathcal{K}^n}(Ax,A^{-1}w)\\ &= A^{-1}\left\{ u\in\mathbb{R}^n \;\middle|\; \begin{array}{ll} u=0 & \text{if } -A^{-1}w\in\operatorname{int}\mathcal{K}^n,\\ \mathbb{R}(-w_1\mathrm{ctan}\theta, w_2) & \text{if } -A^{-1}w\in\operatorname{bd}\mathcal{K}^n/\{0\},\ Ax=0,\\ \langle u,A^{-1}w\rangle=0 & \text{if } Ax,-A^{-1}w\in\operatorname{bd}\mathcal{K}^n/\{0\}, \end{array}\right\}\\ &= A^{-1}\left\{ u\in\mathbb{R}^n \;\middle|\; \begin{array}{ll} u=0 & \text{if } -A^{-2}w\in\operatorname{int}\mathcal{L}_\theta,\\ \mathbb{R}(-w_1\mathrm{ctan}\theta, w_2) & \text{if } -A^{-2}w\in\operatorname{bd}\mathcal{L}_\theta/\{0\},\ x=0,\\ \langle u,A^{-1}w\rangle=0 & \text{if } x,-A^{-2}w\in\operatorname{bd}\mathcal{L}_\theta/\{0\}, \end{array}\right\}\\ &= \left\{ u\in\mathbb{R}^n \;\middle|\; \begin{array}{ll} u=0 & \text{if } -A^{-2}w\in\operatorname{int}\mathcal{L}_\theta,\\ \mathbb{R}(-w_1\mathrm{ctan}^2\theta, w_2) & \text{if } -A^{-2}w\in\operatorname{bd}\mathcal{L}_\theta/\{0\},\ x=0,\\ \langle u,w\rangle=0 & \text{if } x,-A^{-2}w\in\operatorname{bd}\mathcal{L}_\theta/\{0\} \end{array}\right\}\\ &= \left\{ u\in\mathbb{R}^n \;\middle|\; \begin{array}{ll} u=0 & \text{if } -w\in\operatorname{int}\mathcal{L}_{\frac{\pi}{2}-\theta},\\ \mathbb{R}(-w_1\mathrm{ctan}^2\theta, w_2) & \text{if } -w\in\operatorname{bd}\mathcal{L}_{\frac{\pi}{2}-\theta}/\{0\},\ x=0,\\ \langle u,w\rangle=0 & \text{if } -w\in\operatorname{bd}\mathcal{L}_{\frac{\pi}{2}-\theta}/\{0\},\ x\in\operatorname{bd}\mathcal{L}_\theta/\{0\} \end{array}\right\},\end{aligned}$$

where we use the fact from [2] ensuring that $A^2\mathcal{L}_\theta = \mathcal{L}_{\frac{\pi}{2}-\theta}$. This justifies (5.2).

To verify (5.3), observe from (4.7) and (5.1) that

$$\begin{aligned}\partial^2\delta_{\mathcal{L}_\theta}(x,w)(0) &= AD^*\delta_{\mathcal{K}^n}(Ax,A^{-1}w)(0) = A\operatorname{span}\big\{N_{\mathcal{K}^n}(Ax)\big\}\\ &= \operatorname{span}\big\{AN_{\mathcal{K}^n}(Ax)\big\} = \operatorname{span}\big\{N_{\mathcal{L}_\theta}(x)\big\},\end{aligned}\tag{5.4}$$

where the second equality comes from [25, Theorem 3.6]. Then (5.3) follows from Lemma 5.2(c), which therefore completes the proof of the theorem. $\square$

After all these preparations from generalized differentiation, now we are able to proceed with the main topic of this section concerning *second-order characterizations* of the fundamental notions of *till and full stability* in problems of circular programming formulated as follows:

$$\text{minimize} \;\; f(x) \;\; \text{subject to} \;\; g(x) \in \mathcal{L}_\theta, \tag{5.5}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are assumed to be twice continuously differentiable at the reference points. Note that we consider the case of only one circular cone constraint in (5.5) just for simplicity. It is possible to carry out the case of *products* of circular cones without any difficulties.

Along with (5.5), consider its perturbed two-parametric version $\mathcal{P}(w, v)$ given by

$$\text{minimize} \;\; \varphi(x, w) - \langle v, x\rangle \;\; \text{over} \;\; x \in \mathbb{R}^n \;\; \text{with} \;\; \varphi(x, w) := f(x, w) + \delta_{\mathcal{L}_\theta}\big(g(x, w)\big), \tag{5.6}$$

where the vector $w \in \mathbb{R}^d$ signifies *basic* perturbations with $f(x, 0) = f(x)$ and $g(x, 0) = g(x)$, while the vector $v \in \mathbb{R}^n$ stands for *tilt* perturbations. For a locally optimal solution $\bar{x}$ to (5.5), fix a positive number γ and consider the (local) *optimal value function*

$$m_\gamma(w, v) := \inf \big\{\varphi(x, w) - \langle v, x\rangle \big| \; \|x - \bar{x}\| \le \gamma\big\}, \quad (w, v) \in \mathbb{R}^d \times \mathbb{R}^n,$$

and the corresponding *optimal solution map*

$$M_\gamma(w, v) := \operatorname{argmin} \big\{\varphi(x, w) - \langle v, x\rangle \big| \; \|x - \bar{x}\| \le \gamma\big\}, \quad (w, v) \in \mathbb{R}^d \times \mathbb{R}^n.$$

Following the scheme of [15] for general optimization problems, we define the notion of *full stability* of locally optimal solutions to (5.6) and its *tilt stability* predecessor.[16]

Definition 5.5 (full and tilt stability) A point $\bar{x}$ is a FULLY STABLE locally optimal solution to $\mathcal{P}(\bar{w}, \bar{v})$ if there exist a number $\gamma > 0$, the neighbourhoods W of $\bar{w}$ and V of $\bar{v}$ such that the mapping $(w, v) \mapsto M_\gamma(w, v)$ is single-valued and Lipschitz continuous with $M_\gamma(\bar{w}, \bar{v}) = \bar{x}$ and furthermore the function $(w, v) \mapsto m_\gamma(w, v)$ is Lipschitz continuous on $W \times V$. If φ in (5.6) does not depend on w, then $\bar{x}$ is called TILT- STABILE local minimizer to $\mathcal{P}(\bar{v})$.

It is easy to see that in the case of tilt stability the value function $m_\gamma(v)$ is automatically Lipschitz continuous around $\bar{v}$. Although the notions of tilt and full stability have drawn much attention in the literature for various classes of optimization problems (see the references and discussions in Section 1), we are not familiar with any work in this direction for problems of circular cone programming.

In what follows we derive complete characterizations of these stability notions in the perturbed setting of $\mathcal{P}(w, v)$ from (5.6). Observe to this end that we can confine ourselves to characterizing full stability, which readily imply the corresponding results for tilt stability when the parameter w is absent in (5.6).

Considering further this perturbed setting of circular cone programming, recall that the partial *Robinson constraint qualification* (RCQ) holds at $(\bar{x}, \bar{w})$ if

$$0 \in \operatorname{int}\Big\{g(\bar{x}, \bar{w}) + \nabla_x g(\bar{x}, \bar{w})\mathbb{R}^n - \mathcal{L}_\theta\Big\}. \tag{5.7}$$

Our first characterization of full stability is obtained under RCQ via the so-called *partial strong metric regularity* (PSMR) of the subgradient mapping $\partial_x \varphi : \mathbb{R}^n \times \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ from (5.6) at $(\bar{x}, \bar{w}, \bar{v})$ meaning that the partially inverse mapping

$$S_\varphi(w, v) := \left\{ x \in \mathbb{R}^n \,\middle|\, v \in \partial_x \varphi(x, w) \right\}$$

admits a Lipschitzian single-valued localization around this point.

THEOREM 5.6 (full stability via PSMR) *Let $\bar{x}$ be a local minimizer of* (5.6) *under the validity of RCQ* (5.7)*. Then $\bar{x}$ is fully stable in problem $\mathcal{P}(\bar{w}, \bar{v})$ with $\bar{v} \in \partial_x \varphi(\bar{x}, \bar{w})$ if and only if the partial subgradient mapping $\partial_x \varphi$ is PSMR at $(\bar{x}, \bar{w}, \bar{v})$.*

Proof Employing relationship (1.6) between the circular and second-order cones, rewrite (5.5) as

$$\text{minimize} \quad f(x) \quad \text{subject to} \quad Ag(x) \in \mathcal{K}^n \tag{5.8}$$

and do the same with the perturbed version (5.6), where the corresponding function φ is represented by

$$\varphi(x, w) = f(x, w) + \delta_{\mathcal{K}^n}\big(Ag(x, w)\big), \quad (x, w) \in \mathbb{R}^n \times \mathbb{R}^d. \tag{5.9}$$

It is easy to observe the relationships

$$\begin{aligned} 0 \in \operatorname{int}\left\{g(\bar{x}, \bar{w}) + \nabla_x g(\bar{x}, \bar{w})\mathbb{R}^n - \mathcal{L}_\theta\right\} &\Longleftrightarrow 0 \in \operatorname{int}\left\{Ag(\bar{x}, \bar{w}) + A\nabla_x g(\bar{x}, \bar{w})\mathbb{R}^n - A\mathcal{L}_\theta\right\} \\ &\Longleftrightarrow 0 \in \operatorname{int}\left\{Ag(\bar{x}, \bar{w}) + A\nabla_x g(\bar{x}, \bar{w})\mathbb{R}^n - \mathcal{K}^n\right\}, \end{aligned} \tag{5.10}$$

which show that RCQ (5.7) for the circular cone program under consideration is equivalent to the corresponding version of RCQ for the perturbed version of the second-order cone program (5.8). Furthermore, we have for φ in (5.6) under RCQ (5.7) that

$$\begin{aligned} \partial_x \varphi(x, w) &= \nabla_x f(x, w) + \nabla_x g(x, w)^* N_{\mathcal{L}_\theta}\big(g(x, w)\big) \\ &= \nabla_x f(x, w) + \nabla_x g(x, w)^* A N_{\mathcal{K}^n}\big(Ag(x, w)\big) \\ &= \nabla_x f(x, w) + \big(A\nabla_x g(x, w)\big)^* N_{\mathcal{K}^n}\big(Ag(x, w)\big), \end{aligned}$$

which is the partial subdifferential of φ in (5.9) under (5.10). This deduces that the PSMR property of the subgradient mapping $\partial_x \varphi$ in the perturbed version of the circular cone program (5.5) agrees with the one for its second-order cone counterpart (5.8). Applying now the result of [25, Theorem 4.2] on characterizing full stability via PSMR in second-order cone programming justifies the equivalence claimed in this theorem. □

To derive further characterizations of full stability expressed entirely via the circular cone program data, we need more qualification conditions formulated in the following definition.

Definition 5.7 (partial second-order qualification and nondegeneracy) Let $(\bar{x}, \bar{w}) \in \mathbb{R}^n \times \mathbb{R}^d$ be such that $g(\bar{x}, \bar{w}) \in \mathcal{L}_\theta$ in the framework of (5.6). We say that:

(a) The partial SECOND-ORDER CONSTRAINT QUALIFICATION (SOCQ) holds at $(\bar{x}, \bar{w})$ if

$$\partial^2 \delta_{\mathcal{L}_\theta}\big(g(\bar{x}, \bar{w}), \bar{y}\big)(0) \cap \ker \nabla_x g(\bar{x}, \bar{w})^* = \{0\} \text{ for any } \bar{y} \in N_{\mathcal{L}_\theta}\big(g(\bar{x}, \bar{w})\big).$$

(b) The pair $(\bar{x}, \bar{w})$ is NONDEGENERATE for (5.6) if

$$\nabla_x g(\bar{x}, \bar{w})\mathbb{R}^n + \operatorname{lin}\big\{T_{\mathcal{L}_\theta}\big(g(\bar{x}, \bar{w})\big)\big\} = \mathbb{R}^n, \tag{5.11}$$

where $\operatorname{lin}\{T_{\mathcal{L}_\theta}(g(\bar{x}, \bar{w}))\}$ is the largest linear subspace contained in $T_{\mathcal{L}_\theta}(g(\bar{x}, \bar{w}))$.

The first condition in Definition 5.7 is a specification for (5.6) the qualification condition employed in [26] for developing chain rules of second-order generalized differentiation, while the nondegeneracy condition (5.11) is taken from [7], where it was considered for general problems of conic programming. Note they both conditions in Definition 5.7 imply the partial RCQ (5.7). It occurs that they are *equivalent* in the general framework of circular cone programming.

PROPOSITION 5.8 (equivalent descriptions) *The partial second-order qualification and nondegeneracy conditions from Definition 5.7 are equivalent at any point $(\bar{x}, \bar{w})$ feasible to (5.6).*

Proof It is easy to deduce from the normal-tangent duality for convex sets that the nondegeneracy condition (5.11) can be rewritten in the form

$$\operatorname{span}\big\{N_{\mathcal{L}_\theta}\big(g(\bar{x}, \bar{w})\big)\big\} \cap \ker \nabla_x g(\bar{x}, \bar{w})^* = \{0\}.$$

Then the claimed equivalence follows from the span expression (5.4) in the proof of Theorem 5.4. Observe furthermore that Theorem 5.4 allows us to represent the equivalent qualification conditions of Definition 5.7 *explicitly* in terms of the initial data of the circular cone $\mathcal{L}_\theta$. □

Next we formulate two second-order conditions, each of which completely characterizes full stability of local minimizers for circular cone programs under the validity of the equivalent nondegeneracy and SOCQ properties of Definition 5.7. Given $(\bar{x}, \bar{w}) \in \mathbb{R}^n \times \mathbb{R}^d$ and $\bar{v} \in \partial_x \varphi(\bar{x}, \bar{w})$, consider the *Lagrange function* for $\mathcal{P}(\bar{w}, \bar{v})$ defined by

$$L^{\mathcal{L}_\theta}(x, w, \lambda) := f(x, w) - \langle \lambda, g(x, w)\rangle \tag{5.12}$$

and form the corresponding *Karush–Kuhn–Tucker* (KKT) system

$$0 \in \nabla_x L^{\mathcal{L}_\theta}(\bar{x}, \bar{w}, \bar{\lambda}) - \bar{v}, \quad -\bar{\lambda} \in N_{\mathcal{L}_\theta}\big(g(\bar{x}, \bar{w})\big), \tag{5.13}$$

which admits the unique Lagrange multiplier $\bar{\lambda} \in \mathbb{R}^m$ under the validity of the nondegeneracy/SOCQ property at $(\bar{x}, \bar{w})$. Define also the associated *critical cone* at $(\bar{x}, \bar{w})$ by

$$\mathcal{C}^{\mathcal{L}_\theta}(\bar{x}, \bar{w}) := \big\{u \in \mathbb{R}^n \big|\ \nabla_x f(\bar{x}, \bar{w})u \le 0,\ \nabla_x g\big(\bar{x}, \bar{w}\big) \in T_{\mathcal{L}_\theta}\big(g(\bar{x}, \bar{w})\big)\big\}. \tag{5.14}$$

Definition 5.9 (second-order growth and strong sufficient optimality condition) Let $(\bar{x}, \bar{w}) \in \mathbb{R}^n \times \mathbb{R}^d$ be such that $g(\bar{x}, \bar{w}) \in \mathcal{L}_\theta$, and let $\bar{v} \in \partial_x \varphi(\bar{x}, \bar{w})$ in the framework of (5.6). We say that:

(a) The UNIFORM SECOND-ORDER GROWTH CONDITION (USOGC) holds at $(\bar{x}, \bar{w}, \bar{v})$ if there exist $\eta > 0$ and neighbourhoods U of $\bar{x}$, W of $\bar{w}$, and V of $\bar{v}$ such that for any $(w, v) \in W \times V$ and any $x_{wv} \in U$ satisfying $v \in \partial_x \varphi(x_{wv}, w)$ in (5.6) we have

$$\begin{aligned} f(x, w) \ \geq \ & f(x_{wv}, w) + \langle v, x - x_{wv} \rangle \\ & + \eta \|x - x_{wv}\|^2 \quad \text{whenever } \ x \in U \ \text{ with } \ g(x, w) \in \mathcal{L}_\theta. \end{aligned} \tag{5.15}$$

(b) The STRONG SECOND-ORDER SUFFICIENT OPTIMALITY CONDITION (SSOSC) holds at $(\bar{x}, \bar{w})$ if

$$\left\langle u, \nabla^2_{xx} L^{\mathcal{L}_\theta}(\bar{x}, \bar{w}, \bar{\lambda}) u \right\rangle + \left\langle \mathcal{H}^{\mathcal{L}_\theta}(\bar{x}, \bar{w}, \bar{\lambda}) u, u \right\rangle > 0 \ \text{ for all } \ u \in \operatorname{span}\left\{ \mathcal{C}^{\mathcal{L}_\theta}(\bar{x}, \bar{w}) \right\} / \{0\},$$

where $\bar{\lambda} \in \mathbb{R}^m$ is a unique solution of the KKT system (5.13), and where

$$\mathcal{H}^{\mathcal{L}_\theta}(\bar{x}, \bar{w}, \bar{\lambda}) := \begin{cases} -\dfrac{(\bar{\lambda})_1}{\left(g(\bar{x}, \bar{w})\right)_1} \nabla_x g(\bar{x}, \bar{w})^* \begin{bmatrix} 1 & 0^T \\ 0 & -\text{ctan}^2 \theta I \end{bmatrix} \nabla_x g(\bar{x}, \bar{w}) & \text{if } \ g(\bar{x}, \bar{w}) \in \operatorname{bd} \mathcal{L}_\theta / \{0\} \\ 0 & \text{otherwise.} \end{cases}$$

Now we are ready to establish the main result of this section providing complete second-order characterizations of full stability of locally optimal solutions to circular cone programs that are expressed entirely in terms of their initial data.

THEOREM 5.10 (second-order characterizations of full stability of locally optimal solutions to circular cone programs) *Let $\bar{x}$ be a feasible solution to the parameterized problem $\mathcal{P}(\bar{w}, \bar{v})$ from (5.6) with some basic parameter $\bar{w} \in \mathbb{R}^d$ and tilt parameter $\bar{v}$ satisfying*

$$\bar{v} \in \nabla_x f(\bar{x}, \bar{w}) + \nabla_x g(\bar{x}, \bar{w})^* N_{\mathcal{L}_\theta}\big(g(\bar{x}, \bar{w})\big). \tag{5.16}$$

Suppose that the equivalent partial SOCQ and nondegeneracy properties from Definition 5.7 hold at $(\bar{x}, \bar{w})$. Then each of the uniform second-order growth condition at $(\bar{x}, \bar{w}, \bar{v})$ and the strong second-order sufficient optimality condition at $(\bar{x}, \bar{w})$ formulated in Definition 5.9 is necessary and sufficient for full stability of $\bar{x}$ in the perturbed problem $\mathcal{P}(\bar{w}, \bar{v})$.

Proof Note first that condition (5.16) is equivalent to the aforementioned stationary condition $\bar{v} \in \partial_x \varphi(\bar{x}, \bar{w})$ due to the elementary subdifferential sum and chain rules for the function φ from (5.6); see [29,30]. Rewriting the circular cone program (5.5) in the equivalent second-order cone programming form (5.8), we may proceed similarly to the proof of Theorem 5.6. The SOC counterparts of the second-order characterizations of full stability claimed in the theorem are obtained in [25, Theorem 4.4 and Theorem 4.8]. We need now to verify that the assumptions made and the second-order characterizations formulated in this theorem reduce to those given in the corresponding results of [25].

Let us start with the SOCQ from Definition 5.7(a). We have the equivalences

$$\begin{aligned} & \partial^2 \delta_{\mathcal{L}_\theta}\big(g(\bar{x}, \bar{w}), \bar{y}\big)(0) \cap \ker \nabla_x g(\bar{x}, \bar{w})^* = \{0\} \ \text{ with } \ \bar{y} \in N_{\mathcal{L}_\theta}\big(g(\bar{x}, \bar{w})\big) \\ \Longleftrightarrow \ & A \partial^2 \delta_{\mathcal{K}^n}\big(A g(\bar{x}, \bar{w}), A^{-1} \bar{y}\big)(0) \cap \ker \nabla_x g(\bar{x}, \bar{w})^* = \{0\} \ \text{ with } \ A^{-1} \bar{y} \in N_{\mathcal{K}^n}\big(A^{-1} g(\bar{x}, \bar{w})\big) \\ \Longleftrightarrow \ & \partial^2 \delta_{\mathcal{K}^n}\big(A g(\bar{x}, \bar{w}), A^{-1} \bar{y}\big)(0) \cap A^{-1} \ker \nabla_x g(\bar{x}, \bar{w})^* = \{0\} \ \text{ with } \ A^{-1} \bar{y} \in N_{\mathcal{K}^n}\big(A^{-1} g(\bar{x}, \bar{w})\big) \\ \Longleftrightarrow \ & \partial^2 \delta_{\mathcal{K}^n}\big(A g(\bar{x}, \bar{w}), A^{-1} \bar{y}\big)(0) \cap \ker \big(A \nabla_x g(\bar{x}, \bar{w})\big)^* = \{0\} \ \text{ with } \ A^{-1} \bar{y} \in N_{\mathcal{K}^n}\big(A^{-1} g(\bar{x}, \bar{w})\big), \end{aligned}$$

which show that the assumed SOCQ for the circular cone program is equivalent to the one in [25, Theorem 4.4 and Theorem 4.8] regarding the perturbed second-order cone program

(5.8). Since it is obviously to observe that USOGC (5.15) for the circular cone program agrees with that in [25, Theorem 4.4] for the second-order cone program (5.8), it remains to verify such a correspondence for the SSOSC in Definition 5.9(b) and that in [25, Theorem 4.8].

To proceed, observe that the Lagrange function (5.12) is represented as and

$$L^{\mathcal{L}_\theta}(x,w,\lambda)=f(x,w)-\langle\lambda,g(x,w)\rangle=f(x,w)-\left\langle A^{-1}\lambda,Ag(x,w)\right\rangle=L^{\mathcal{K}^n}\left(x,w,A^{-1}\lambda\right)$$

while the KKT system (5.13) can be written as

$$0\in\nabla_x L^{\mathcal{K}^n}\left(\bar{x},\bar{w},A^{-1}\bar{\lambda}\right)-\bar{v}\ \text{ and }\ -A^{-1}\bar{\lambda}\in N_{\mathcal{K}^n}\left(Ag(\bar{x},\bar{w})\right).$$

This means that $A^{-1}\bar{\lambda}$ is the Lagrange multiplier for the perturbed second-order cone program associated with (5.8). Furthermore, for the critical cone (5.14) we have

$$\mathcal{C}^{\mathcal{L}_\theta}(\bar{x},\bar{w})=\left\{u\in\mathbb{R}^n\,\middle|\,\nabla_x f(\bar{x},\bar{w})u\le 0,\ A\nabla_x g(\bar{x},\bar{w})\in T_{\mathcal{K}^n}\left(Ag(\bar{x},\bar{w})\right)\right\}=\mathcal{C}^{\mathcal{K}^n}(\bar{x},\bar{w}),$$

which ensures in turn the following representation of the function $\mathcal{H}^{\mathcal{L}_\theta}$ in Definition 5.9:

$$\begin{aligned}&\mathcal{H}^{\mathcal{L}_\theta}(\bar{x},\bar{w},\bar{\lambda})\\&=\begin{cases}-\dfrac{(A^{-1}\bar{\lambda})_1}{\left(Ag(\bar{x},\bar{w})\right)_1}\nabla_x g(\bar{x},\bar{w})^*A\begin{bmatrix}1&0^T\\0&-I\end{bmatrix}A\nabla_x g(\bar{x},\bar{w}) & \text{if } Ag(\bar{x},\bar{w})\in\operatorname{bd}\mathcal{K}^n/\{0\},\\ 0 & \text{otherwise}\end{cases}\\&=\mathcal{H}^{\mathcal{K}^n}\left(\bar{x},\bar{w},A^{-1}\bar{\lambda}\right).\end{aligned}$$

It gives us the corresponding SSOSC for the second-order cone program (5.8) used in [25, Theorem 4.8] and thus completes the proof of this theorem. □

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References

[1] Dattorro J. Convex optimization and euclidean distance geometry. Palo Alto (CA): Meboo; 2005.
[2] Zhou J-C, Chen J-S. Properties of circular cone and spectral factorization associated with circular cone. J. Nonlinear Convex Anal. 2013;14:807–816.
[3] Chen J-S. The convex and monotone functions associated with second-order cone. Optimization. 2006;55:363–385.
[4] Chen J-S, Chen X, Tseng P. Analysis of nonsmooth vector-valued functions associated with second-order cone. Math. Program. 2004;101:95–117.
[5] Ko C-H, Chen J-S. Optimal grasping manipulation for multifingered robots using semismooth Newton method. Math. Probl. Eng. 2013;2013:DOI:10.1155/2013/681710.
[6] Bonnans JF, Ramírez HC. Perturbation analysis of second-order cone programming problems. Math. Program. 2005;104:205–227.

[7] Bonnans JF, Shapiro A. Perturbation analysis of optimization problems. New York (NY): Springer; 2000.
[8] Mordukhovich BS, Outrata JV, Ramírez H. Second-order variational analysis in conic programming with applications to optimality conditions and stability. 2013. Available from: http://www.optimization-online.org/DB_HTML/2013/01/3723.html
[9] Matsukawa Y, Yoshise A. A primal barrier function Phase I algorithm for nonsymmetric conic optimization problems. Japan J. Indust. Appl. Math. 2012;29:499–517.
[10] Nesterov Y. Towards nonsymmetric conic optimization. CORE Discussion Paper # 2006/28, 2006.
[11] Skajaa A, Jorgensen JB, Hansen PC. On implementing a homogeneous interior-point algorithm for nonsymmetric conic optimization, IMM-Technical Report, No. 2011-02, 2011.
[12] Skajaa A, Ye Y-Y. A homogeneous interior-point algorithm for nonsymmetric convex conic optimization, preprint, 2012.
[13] Zhou J-C, Chen J-S. The vector-valued functions associated with circular cones. Abstract Appl. Anal. 2014;2014:DOI:10.1155/2014/603542.
[14] Zhou J-C, Chen J-S, Hung H-F. Circular cone convexity and some inequalities associated with circular cones. J. Inequal. Appl. 2013;2013:571, 17pp.
[15] Levy AB, Poliquin RA, Rockafellar RT. Stability of locally optimal solutions. SIAM J. Optim. 2000;10:580–604.
[16] Poliquin RA, Rockafellar RT. Tilt stability of a local minimum. SIAM J. Optim. 1998;8:287–299.
[17] Drusvyatskiy D, Lewis AS. Tilt stability, uniform quadratic growth, and strong metric regularity of the subdifferential. SIAM J. Optim. 2013;23:256–267.
[18] Drusvyatskiy D, Mordukhovich BS, Nghia TTA. Second-order growth, tilt stability, and metric regularity of the subdifferential. J. Convex Anal. 2014;21.
[19] Lewis AS, Zhang S. Partial smoothness, tilt stability, and generalized Hessians. SIAM J. Optim. 2013;23:74–94.
[20] Mordukhovich BS, Nghia TTA. Second-order variational analysis and characterizations of tilt-stable optimal solutions in infinite-dimensional spaces. Nonlinear Anal. 2013;86:159–180.
[21] Mordukhovich BS, Nghia TTA. Second-order characterizations of tilt stability with applications to nonlinear programming. Math. Program. Forthcoming. Available from: http://borismordukhovich.com/publications.php
[22] Mordukhovich BS, Nghia TTA. Full Lipschitzian and Hölderian stability in optimization with applications to mathematical programming and optimal control. SIAM J. Optim. Forthcoming. Available from: http://borismordukhovich.com/publications.php
[23] Mordukhovich BS, Nghia TTA, Rockafellar RT. Full stability in finite-dimensional optimization. Math. Oper. Res. Forthcoming. Available from: http://borismordukhovich.com/publications.php
[24] Mordukhovich BS, Outrata JV. Tilt stability in nonlinear programming under Mangasarian-Fromovitz constraint qualification. Kybernetiika. 2013;49:446–464.
[25] Mordukhovich BS, Outrata JV, Sarabi ME. Full stability of locally optimal solutions in second-order cone programming. SIAM J. Optim. Forthcoming. Available from: http://borismordukhovich.com/publications.php
[26] Mordukhovich BS, Rockafellar RT. Second-order subdifferential calculus with applications to tilt stability in optimization. SIAM J. Optim. 2012;22:953–986.
[27] Mordukhovich BS, Rockafellar RT, Sarabi ME. Characterizations of full stability in constrained optimization. SIAM J. Optim. 2013;23:1810–1849.
[28] Mordukhovich BS. Sensitivity analysis in nonsmooth optimization. In: Field DA, Komkov V, editors. Theoretical aspects of industrial design. Vol. 58, SIAM Proceedings in Applied Mathematics. Philadelphia, PA: SIAM; 1992. p. 32–46.
[29] Mordukhovich BS. Variational analysis and generalized differentiation: I, II. Berlin: Springer; 2006.
[30] Rockafellar RT, RJ-B Wets. Variational analysis. Berlin: Springer; 1998.

[31] Facchinei F, Pang J-S. Finite dimensional variational inequalities and complementarity problems. New York (NY): Springer; 2003.
[32] Qi L. Convergence analysis of some algorithms for solving nonsmooth equations. Math. Oper. Res. 1993;18:227–244.
[33] Hayashi S, Yamashita N, Fukushima M. A combined smoothing and regularization method for monotone second-order cone complementarity problems. SIAM J. Optim. 2005;15:593–615.
[34] Potra F, Qi L, Sun D. Secant methods for semismooth equations. Numer. Math. 1998;80:305–324.
[35] Outrata JV, Sun D. On the coderivative of the projection operator onto the second-order cone. Set-Valued Anal. 2008;16:999–1014.
[36] Dontchev AL, Rockafellar RT. Implicit functions and solution mappings: a view from variational analysis. New York (NY): Springer; 2009.