to appear in Journal of Nonlinear and Convex Analysis, 2013

Properties of circular cone and spectral factorization associated with circular cone

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October 11, 2012

Abstract. Circular cone includes second-order cone as a special case when the rotation angle is 45 degree. This paper gives an insight on circular cone, in which we describe the tangent cone, normal cone, second order tangent cone, and second order regularity of circular cone. Moreover, we establish the spectral factorization associated with circular cone. These results are crucial to subsequent study regarding various analysis towards optimizations associated with circular cone.

Keywords. Circular cone, second order regular, spectral factorization.

1 Introduction

The circular cone [9] is a pointed closed convex cone having hyperspherical sections orthogonal to its axis of revolution about which the cone is invariant to rotation. Let its

[1]The author's work is supported by National Natural Science Foundation of China (11101248) and Shandong Province Natural Science Foundation (ZR2010AQ026).

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half-aperture angle be θ where $\theta \in (0, \frac{\pi}{2})$. Then, we denote the n-dimensional circular cone by $\mathcal{L}_\theta$ which is expressed as

$$\mathcal{L}_\theta := \left\{ x = (x_1, x_2)^T \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \cos\theta \|x\| \le x_1 \right\}. \tag{1}$$

Circular cone includes second-order cone (SOC), given by

$$\mathcal{K}^n := \{x = (x_1, x_2)^T \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\| \le x_1\}, \tag{2}$$

as a special case when the rotation angle is 45 degree. This can be verified by

$$\begin{aligned}
\mathcal{K}^n &= \left\{(x_1, x_2)^T \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\| \le x_1\right\} \\
&= \left\{(x_1, x_2)^T \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_1 \ge 0,\ 2\|x_2\|^2 \le 2x_1^2\right\} \\
&= \left\{(x_1, x_2)^T \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_1 \ge 0,\ 2x_1^2 + 2\|x_2\|^2 \le 4x_1^2\right\} \\
&= \left\{(x_1, x_2)^T \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \sqrt{2x_1^2 + 2\|x_2\|^2} \le 2x_1\right\} \\
&= \left\{(x_1, x_2)^T \in \mathbb{R} \times \mathbb{R}^{n-1} \;\middle|\; \frac{\sqrt{2}}{2}\sqrt{x_1^2 + \|x_2\|^2} \le x_1\right\} \\
&= \left\{(x_1, x_2)^T \in \mathbb{R} \times \mathbb{R}^{n-1} \;\middle|\; \frac{\sqrt{2}}{2}\|x\| \le x_1\right\} \\
&= \left\{(x_1, x_2)^T \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \cos\frac{\pi}{4}\|x\| \le x_1\right\}.
\end{aligned}$$

Considerable attentions have been devoted to the second-order cone $\mathcal{K}^n$ [5, 6, 7], a special case of self-dual cone. However, the study on the circular cone $\mathcal{L}_\theta$, a non-self-dual (or non-symmetric cone) is rather limited. In this paper, we show that there exists a close relationship between $\mathcal{K}^n$ and $\mathcal{L}_\theta$ by establishing an inequality regrading distance between $\mathcal{K}^n$ and $\mathcal{L}_\theta$. This nice property plays an essential role in our subsequence analysis and give us more information and insight on $\mathcal{L}_\theta$. In particular, we develop the formulae of tangent cone, normal cone, and second-order tangent cone of $\mathcal{L}_\theta$ in terms of $\mathcal{K}^n$ (the formula of the latter has been given by different scholars). Furthermore, we show that $\mathcal{L}_\theta$, as a non-self-dual and non-polytechnic cone, is also second-order regular. Note that we know the second-order cone and positive semi-definitive cone are both second order regular, but there are all symmetric. Thus this is an interesting case which indicates the second order regularity of a non-symmetric cone. Finally, we develop the spectral factorization of z in terms of $\mathcal{L}_\theta$ by studying the projection on $\mathcal{L}_\theta$ which will be useful in dealing with optimization associated circular cone.

In fact, it is not hard to see that

$$\begin{aligned}
\mathcal{L}_\theta &= \left\{(x_1, x_2)^T \in \mathrm{I\!R} \times \mathrm{I\!R}^{n-1} \mid x_1 \ge 0,\ \|x\|^2 \cos^2\theta \le x_1^2\right\} \\
&= \left\{(x_1, x_2)^T \in \mathrm{I\!R} \times \mathrm{I\!R}^{n-1} \mid x_1 \ge 0,\ (x_1^2 + \|x_2\|^2)\cos^2\theta \le x_1^2\right\} \\
&= \left\{(x_1, x_2)^T \in \mathrm{I\!R} \times \mathrm{I\!R}^{n-1} \mid x_1 \ge 0,\ \|x_2\|^2 \le x_1^2 \tan^2\theta\right\} \\
&= \left\{(x_1, x_2)^T \in \mathrm{I\!R} \times \mathrm{I\!R}^{n-1} \mid \|x_2\| \le x_1 \tan\theta\right\},
\end{aligned}$$

which yields

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{L}_\theta \quad \Longleftrightarrow \quad \begin{bmatrix} \tan\theta x_1 \\ x_2 \end{bmatrix} \in \mathcal{K}^n \quad \Longleftrightarrow \quad \begin{bmatrix} \tan\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{K}^n. \tag{3}$$

For simplicity, let us denote

$$A := \begin{bmatrix} \tan\theta & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, the above expression (3) is equivalent to

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{L}_\theta \quad \Longleftrightarrow \quad A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{K}^n. \tag{4}$$

We point out that the matrix A is positive definite whose inverse matrix is

$$A^{-1} = \begin{bmatrix} \mathrm{ctan}\theta & 0 \\ 0 & 1 \end{bmatrix} \quad \text{where} \quad \mathrm{ctan}\theta := \frac{1}{\tan\theta}.$$

To close this section, we say a few words about the notations. For a convex cone $\mathcal{K}$, its *dual cone* is defined by

$$(\mathcal{K})^* = \{v \mid \langle v, x\rangle \ge 0,\ \forall x \in \mathcal{K}\},$$

while its *polar cone* is given by

$$(\mathcal{K})^\circ = \{v \mid \langle v, x\rangle \le 0,\ \forall x \in \mathcal{K}\}.$$

2 Insight on circular cone

In this section, we give an insight on circular cone in which we shall study some properties of $\mathcal{L}_\theta$, including characterizing its tangle cone, normal cone, second-order tangent cone, etc.. To this end, we first describe the relationship between $\mathcal{K}^n$ and $\mathcal{L}_\theta$.

Theorem 2.1. *Let $\mathcal{L}_\theta$ and $\mathcal{K}^n$ be defined as in (1) and (2), respectively. Then, we have*

(a) $\mathcal{L}_\theta = A^{-1}\mathcal{K}^n$ *and* $\mathcal{K}^n = A\mathcal{L}_\theta$.

(b) $A\mathcal{K}^n = \mathcal{L}_{\frac{\pi}{2}-\theta}$ *and* $\mathcal{L}_{\frac{\pi}{2}-\theta} = A^2\mathcal{L}_\theta$.

(c) $\mathcal{L}_\theta^* = \mathcal{L}_{\frac{\pi}{2}-\theta}$ *and* $(\mathcal{L}_\theta^*)^* = \mathcal{L}_\theta$.

Proof. (a) This follows from equivalence (4) because

$$\begin{array}{rcl} \mathcal{L}_\theta &=& \{x \mid x \in \mathcal{L}_\theta\} \\ &=& \{x \mid Ax \in \mathcal{K}^n\} \\ &=& \left\{x \mid x \in A^{-1}\mathcal{K}^n\right\} \\ &=& A^{-1}\mathcal{K}^n. \end{array}$$

(b) According to part(a), we have

$$\mathcal{L}_{\frac{\pi}{2}-\theta} = \begin{bmatrix} \text{ctan}(\frac{\pi}{2}-\theta) & 0 \\ 0 & 1 \end{bmatrix} \mathcal{K}^n = \begin{bmatrix} \tan\theta & 0 \\ 0 & 1 \end{bmatrix} \mathcal{K}^n = A\mathcal{K}^n = A(A\mathcal{L}_\theta) = A^2\mathcal{L}_\theta$$

which is the desired result.

(c) It is known that $\mathcal{K}^n$ is self-dual. Hence, we have

$$\begin{array}{rcl} \mathcal{K}^n &=& (\mathcal{K}^n)^* = \{v \mid \langle v, k\rangle \geq 0,\ \forall k \in \mathcal{K}^n\} \\ &=& \{v \mid \langle v, Az\rangle \geq 0,\ \forall z \in \mathcal{L}_\theta\} \\ &=& \{v \mid \langle Av, z\rangle \geq 0,\ \forall z \in \mathcal{L}_\theta\} \\ &=& \{v \mid Av \in \mathcal{L}_\theta^*\} \\ &=& A^{-1}\mathcal{L}_\theta^* \end{array}$$

which implies $\mathcal{L}_\theta^* = A\mathcal{K}^n = \mathcal{L}_{\frac{\pi}{2}-\theta}$ by part(b). The remaining part is true for all closed convex cone. $\square$

Theorem 2.2. *For any* $x, z \in \mathbb{R}^n$, *we have*

$$\|A\|^{-1}\ \text{dist}(Az, \mathcal{K}^n) \leq \text{dist}(z, \mathcal{L}_\theta) \leq \|A^{-1}\|\ \text{dist}(Az, \mathcal{K}^n) \tag{5}$$

and

$$\|A^{-1}\|^{-1}\ \text{dist}(A^{-1}x, \mathcal{L}_\theta) \leq \text{dist}(x, \mathcal{K}^n) \leq \|A\|\ \text{dist}(A^{-1}x, \mathcal{L}_\theta). \tag{6}$$

Proof. First, we observe the following:

$$\begin{array}{rl} & \text{dist}(x, \mathcal{K}^n) \\ = & \min\limits_{k\in\mathcal{K}^n} \|x-k\| = \min\limits_{k\in A\mathcal{L}_\theta} \|x-k\| = \min\limits_{z\in\mathcal{L}_\theta} \|x-Az\| = \min\limits_{z\in\mathcal{L}_\theta} \|A(A^{-1}x) - Az\| \\ = & \min\limits_{z\in\mathcal{L}_\theta} \|A(A^{-1}x - z)\| \leq \|A\| \min\limits_{z\in L_\theta} \|A^{-1}x - z\| = \|A\|\ \text{dist}\left(A^{-1}x, \mathcal{L}_\theta\right), \end{array} \tag{7}$$

$$
\begin{aligned}
& \operatorname{dist}(z, \mathcal{L}_\theta) \\
= \ & \min_{u \in \mathcal{L}_\theta} \|z - u\| = \min_{u \in A^{-1}\mathcal{K}^n} \|z - u\| = \min_{k \in \mathcal{K}^n} \|z - A^{-1}k\| = \min_{k \in \mathcal{K}^n} \|A^{-1}(Az) - A^{-1}k\| \\
= \ & \min_{k \in \mathcal{K}^n} \|A^{-1}(Az - k)\| \le \|A^{-1}\| \min_{k \in \mathcal{K}^n} \|Az - k\| = \|A^{-1}\| \operatorname{dist}(Az, \mathcal{K}^n). \qquad (8)
\end{aligned}
$$

These prove the second inequality in (5) and (6), respectively. Next, plugging $z = A^{-1}x$ and $x = Az$ in (7) and (8), respectively, yields the first inequality in (5) and (6), respectively. Thus, the proof is complete. $\square$

Theorem 2.2 indicates that the distances of arbitrary points to $\mathcal{K}^n$ and $\mathcal{L}_\theta$ are equivalent. This is an essential property for analyzing the tangent cone and normal cone of $\mathcal{L}_\theta$. Before we move on, we recall the definitions of tangent cone and normal cone. Given a subset $S \subset \mathbb{R}^n$ and $x \in S$, the contingent cone $T_S(x)$ and inner tangent cone $T_S^i(x)$ of S at x are defined respectively as

$$
T_S(x) := \{ d \in \mathbb{R}^n | \ \exists t_n \downarrow 0, \ \operatorname{dist}(x + t_n d, S) = o(t_n) \}
$$

and

$$
T_S^i(x) := \{ d \in \mathbb{R}^n | \ \operatorname{dist}(x + th, S) = o(t), \ t \ge 0) \}.
$$

In general, these two cones can be different. However, when S is convex, they are equal to each other and to the closure of the radial cone, see [4, page 45]. Hence for convex sets, we simply speak of tangent cone rather than contingent or inner tangent cones. Moreover, the Fréchet/regular normal cone (also known as the prenormal cone), written as $\widehat{N}_S(x)$, is defined as

$$
\widehat{N}_S(x) := \{ v \in \mathbb{R}^n | \ \langle v, z - x \rangle \le o(\|z - x\|), \ \text{for } z \in S \},
$$

and the Mordukhovich/limiting normal cone (or simply normal cone) is defined as

$$
N_S(x) := \limsup_{z \xrightarrow[S]{} x} \widehat{N}_S(z).
$$

When S is convex, $N_S(x) = \widehat{N}_S(x)$ and is the polar cone of $T_S(x)$, i.e.,

$$
N_S(x) := \{ v \in \mathbb{R}^n \mid \langle v, d \rangle \le 0, \ \forall d \in T_S(x) \}.
$$

Theorem 2.3. *For any $z \in \mathcal{L}_\theta$, we have*

(a) $T_{\mathcal{L}_\theta}(z) = A^{-1} T_{\mathcal{K}^n}(Az)$,

(b) $N_{\mathcal{L}_\theta}(z) = A N_{\mathcal{K}^n}(Az)$.

Proof. (a) Let us first show that $T_{\mathcal{L}_\theta}(z) \subseteq A^{-1}T_{\mathcal{K}^n}(Az)$. Choose $d \in T_{\mathcal{L}_\theta}(z)$. Then, by definition of tangent cone, we have

$$\mathrm{dist}(z + td, \mathcal{L}_\theta) = o(t). \tag{9}$$

Plugging $x = A(z + td)$ into (6) yields

$$\|A^{-1}\|^{-1}\ \mathrm{dist}(z + td, \mathcal{L}_\theta) \le \mathrm{dist}\left(A(z + td), \mathcal{K}^n\right) \le \|A\|\ \mathrm{dist}(z + td, \mathcal{L}_\theta).$$

This together with (9) implies $\mathrm{dist}(Az + tAd, \mathcal{K}^n) = o(t)$. Thus, $Ad \in T_{\mathcal{K}^n}(Az)$, which says $d \in A^{-1}T_{\mathcal{K}^n}(Az)$.

Conversely, let $d \in A^{-1}T_{\mathcal{K}^n}(Az)$. Since $Ad \in T_{\mathcal{K}^n}(Az)$, from definition of tangent cone, we know

$$\mathrm{dist}(Az + tAd, \mathcal{K}^n) = o(t). \tag{10}$$

Replacing z in (5) by $z + td$ gives

$$\|A\|^{-1}\ \mathrm{dist}(Az + tAd, \mathcal{K}^n) \le \mathrm{dist}(z + td, \mathcal{L}_\theta) \le \|A^{-1}\|\ \mathrm{dist}(Az + tAd, \mathcal{K}^n).$$

This together with (10) implies $\mathrm{dist}(z + td, \mathcal{L}_\theta) = o(t)$, which says $d \in T_{\mathcal{L}_\theta}(z)$.

(b) The desired result follows from

$$\begin{aligned}
N_{\mathcal{L}_\theta}(z) &= \{v \in \mathbb{R}^n \mid \langle v, d\rangle \le 0,\ \forall d \in T_{\mathcal{L}_\theta}(z)\} \\
&= \left\{v \in \mathbb{R}^n \mid \langle v, A^{-1}w\rangle \le 0,\ \forall w \in T_{\mathcal{K}^n}(Az)\right\} \\
&= \left\{v \in \mathbb{R}^n \mid \langle A^{-1}v, w\rangle \le 0,\ \forall w \in T_{\mathcal{K}^n}(Az)\right\} \\
&= \left\{v \in \mathbb{R}^n \mid A^{-1}v \in N_{\mathcal{K}^n}(Az)\right\} \\
&= AN_{\mathcal{K}^n}(Az).
\end{aligned}$$

□

Theorem 2.3 tells us that the explicit formula of tangent cone $T_{\mathcal{L}_\theta}(z)$ can be established by $T_{\mathcal{K}^n}(Az)$, which has been given in [3].

It is well known that in the study of second order analysis for optimization problems, we need the following inner and outer second order tangent sets to describe the possible curvature of the feasible region. Below, we state their official definitions.

Definition 2.1. [4, Definition 3.28] *The set limits*

$$T_S^{i,2}(x, d) := \left\{ w \in \mathbb{R}^n \;\middle|\; \mathrm{dist}\left(x + td + \frac{1}{2}t^2 w, S\right) = o(t^2),\ t \ge 0 \right\}$$

and

$$T_S^2(x,d) = \left\{ w \in \mathbb{R}^n \;\middle|\; \exists\ t_n \downarrow 0 \text{ such that dist}\left(x + t_n d + \frac{1}{2}t_n^2 w, S \right) = o(t_n^2) \right\}$$

are called the inner and outer second order tangent sets, respectively, to the set S *at* x *in the direction* d.

Definition 2.2. [4, Definition 3.32] *We say that the set* S *is second order directionally differentiable at a point* $x \in S$ *in a direction* $d \in T_S(x)$*, if* $T_S^i(x) = T_S(x)$ *and* $T_S^{i,2}(x,d) = T_S^2(x,d)$*. We simply say that* S *is second order directionally differentiable at a point* $x \in S$ *if it is second order directionally differentiable in all directions* $d \in T_S(x)$.

Theorem 2.4. *Let* $z \in \mathcal{L}_\theta$ *and* $d \in T_{\mathcal{L}_\theta}(z)$*. Then,*

$$T_{\mathcal{L}_\theta}^{i,2}(z,d) = T_{\mathcal{L}_\theta}^2(z,d) = A^{-1}T_{\mathcal{K}^n}^2(Az, Ad).$$

Proof. The first equality is due to the second order directionally differentiable of $\mathcal{K}^n$ as shown in [10, Proposition 3.1] and the second equality can be proved by the same arguments as in Theorem 2.3. □

Definition 2.3. [4, Definition 3.85] *We say that a subset* $S \subset \mathbb{R}^n$ *is second order regular at* x *if it satisfies*

(i) $T_S^2(x,d) = T_S^{i,2}(x,d)$ *for all* $d \in T_S(x)$*;*

(ii) *for any* $d \in T_S(x)$ *and for any sequence* $x + t_n d + \frac{1}{2}t_n^2 r_n \in S$ *such that* $t_n r_n \to 0$*, the following condition holds:*

$$\lim_{n\to\infty} \text{dist}\left(r_n, T_S^2(x,d)\right) = 0.$$

Theorem 2.5. *The circular cone* $\mathcal{L}_\theta$ *is second order regular.*

Proof. Let $z \in \mathcal{L}_\theta$ and $d \in T_{\mathcal{L}_\theta}(z)$. According to Theorem 2.4, it suffices to show that for any sequence $z + t_n d + \frac{1}{2}t_n^2 r_n \in \mathcal{L}_\theta$ with $t_n r_n \to 0$, there holds

$$\lim_{n\to\infty} \text{dist}\left(r_n, T_{\mathcal{L}_\theta}^2(z,d)\right) = 0. \tag{11}$$

We will complete the proof by using the relationship between $\mathcal{L}_\theta$ and $\mathcal{K}^n$. Since $z + t_n d + \frac{1}{2}t_n^2 r_n \in \mathcal{L}_\theta$, we know $Az + t_n Ad + \frac{1}{2}t_n^2 Ar_n \in \mathcal{K}$ by Theorem 2.1(a). Note that $t_n Ar_n \to 0$

because $\|t_n A r_n\| \leq \|A\| \cdot \|t_n r_n\|$. In addition, $\mathcal{K}^n$ is second order regular (see [10] for detailed proof), from Definition 2.3, we have

$$\lim_{n\to\infty} \operatorname{dist}\left(Ar_n, T^2_{\mathcal{K}^n}(Az, Ad)\right) = 0. \tag{12}$$

On the other hand, we observe that

$$\begin{aligned} \operatorname{dist}\left(r_n, T^2_{\mathcal{L}_\theta}(z,d)\right) &= \operatorname{dist}\left(r_n, A^{-1}T^2_{\mathcal{K}^n}(Az, Ad)\right) \\ &= \operatorname{dist}\left(A^{-1}(Ar_n), A^{-1}T^2_{\mathcal{K}^n}(Az, Ad)\right) \\ &\leq \|A^{-1}\| \operatorname{dist}\left(Ar_n, T^2_{\mathcal{K}^n}(Az, Ad)\right). \end{aligned}$$

This together with (12) implies the validity of (11). $\Box$

3 Spectral factorization associated with circular cone

In this section, we will develop the spectral factorization associated with circular cone which is the basis of further investigations for optimization associated with circular cone. To this end, we start with studying the projection on $\mathcal{L}_\theta$, i.e.,

$$\Pi_{\mathcal{L}_\theta}(z) := \arg\min_{x\in\mathcal{L}_\theta} \|z - x\| = \{x \in \mathcal{L}_\theta \mid \|z - x\| \leq \|z - u\|,\ \forall u \in \mathcal{L}_\theta\}.$$

It should be mentioned that the projection cannot be obtained by using the relationship between $\mathcal{L}_\theta$ and $\mathcal{K}^n$ because

$$\|A^{-1}x\| \leq \|A^{-1}y\| \quad \not\Rightarrow \quad \|x\| \leq \|y\| \quad \text{whenever} \quad \theta \neq \pi/4.$$

For example, let $x = (8, 1)$, $y = (4, 2)$, and $\theta = \cot^{-1}(1/8)$. Then,

$$\|A^{-1}x\| = \sqrt{2} < \sqrt{17}/2 = \|A^{-1}y\|, \quad \text{but} \quad \|x\| = \sqrt{65} > \sqrt{20} = \|y\|.$$

Therefore, we seek another way to characterize the projection. First, we note that for any closed convex cone Ω

$$\Pi_{-\Omega}(x) = -\Pi_\Omega(-x).$$

In fact, letting $a = \Pi_{-\Omega}(x)$ yields

$$\|(-x) - (-a)\| = \|x - a\| \leq \|x - (-y)\| = \|(-x) - y\| \quad \forall y \in \Omega,$$

where the inequality comes from the fact that $a = \Pi_{-\Omega}(x)$ by definition of projection. This means that $-a = \Pi_\Omega(-x)$. Besides, it is well known that any vector $z \in \mathbb{R}^n$ can be written as

$$z = \Pi_\Omega(z) + \Pi_{\Omega^\circ}(z).$$

Hence,

$$\begin{aligned} z &= \Pi_{\mathcal{L}_\theta}(z) + \Pi_{\mathcal{L}_\theta^\circ}(z) = \Pi_{\mathcal{L}_\theta}(z) + \Pi_{-\mathcal{L}_\theta^*}(z) \\ &= \Pi_{\mathcal{L}_\theta}(z) - \Pi_{\mathcal{L}_\theta^*}(-z) = \Pi_{\mathcal{L}_\theta}(z) - \Pi_{\mathcal{L}_{\frac{\pi}{2}-\theta}}(-z). \end{aligned} \tag{13}$$

Due to the special structure of $\mathcal{L}_\theta$, the explicit formula of projection is given below.

$$\Pi_{\mathcal{L}_\theta}(z) = \begin{cases} z, & \text{if } z \in \mathcal{L}_\theta \\ 0, & \text{if } z \in -\mathcal{L}_\theta^* \\ u, & \text{otherwise,} \end{cases} \tag{14}$$

where

$$u = \begin{bmatrix} \dfrac{z_1 + \|z_2\| \tan\theta}{1+\tan^2\theta} \\ \left(\dfrac{z_1 + \|z_2\| \tan\theta}{1+\tan^2\theta} \tan\theta \right) \dfrac{z_2}{\|z_2\|} \end{bmatrix}.$$

In fact, formula (14) can be found in several places, for example, [8], [1, page 508] or [2, Theorem 3.3.6]. For completeness we provide the detailed argument on (14), nonetheless, by a different approach from that in [2, Theorem 3.3.6], which leads us to establish the spectral factorization associated with $\mathcal{L}_\theta$.

The first two cases in (14) follow from (13) directly. Now, consider the third case. Note that it corresponds to $z_1 \tan\theta < \|z_2\|$ and $-z_1 \text{ctan}\theta < \|z_2\|$. Hence we must have $z_2 \neq 0$, because otherwise, we would have $z_1 < 0$ and $z_1 > 0$, which is impossible. Let us calculate the projection in the third case by solving the Karush-Kuhn-Tucker conditions for the following convex programming problems

$$\begin{array}{ll} \min & \dfrac{1}{2}\|x - z\|^2 \\ \text{s.t.} & x \in \mathcal{L}_\theta \end{array}$$

which is equivalent to

$$\begin{array}{ll} \min & \dfrac{1}{2}\|x - z\|^2 \\ \text{s.t.} & \|x_2\| - x_1 \tan\theta \leq 0. \end{array}$$

The KKT point of the above convex programming is to find $x \in \mathcal{L}_\theta$ and $\lambda \geq 0$ such that

$$\begin{bmatrix} x_1 - z_1 \\ x_2 - z_2 \end{bmatrix} + \lambda \left\{ \begin{bmatrix} 0 \\ \dfrac{x_2}{\|x_2\|} \end{bmatrix} - \tan\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} = 0,$$

which is equivalent to solving

$$\begin{cases} x_1 = z_1 + \lambda \tan\theta, \\ x_2 = \dfrac{1}{1 + (\lambda/\|x_2\|)}\, z_2. \end{cases} \tag{15}$$

Thus,

$$\|z_2\| = (1 + (\lambda/\|x_2\|))\,\|x_2\| = \|x_2\| + \lambda = \|x_2\| + \frac{x_1 - z_1}{\tan\theta} = x_1 \tan\theta + \frac{x_1 - z_1}{\tan\theta},$$

where the third equality is due to (15) and the last equality comes from the fact that $\|x_2\| = x_1 \tan\theta$ since the projection point of $z \notin \mathcal{L}_\theta$ must lie in the boundary of L_θ. Then, we have

$$x_1 = \frac{z_1 + \|z_2\| \tan\theta}{1 + \tan^2\theta}.$$

Substituting this into the first equation in (15) yields

$$\lambda = \frac{\|z_2\| - z_1 \tan\theta}{1 + \tan^2\theta}.$$

Therefore, according to the second equation in (15), we obtain

$$x_2 = \left(\frac{z_1 + \|z_2\| \tan\theta}{1 + \tan^2\theta} \tan\theta \right) \frac{z_2}{\|z_2\|}$$

which says

$$\Pi_{\mathcal{L}_\theta}(z) = \begin{bmatrix} \dfrac{z_1 + \|z_2\| \tan\theta}{1 + \tan^2\theta} \\ \left(\dfrac{z_1 + \|z_2\| \tan\theta}{1 + \tan^2\theta} \tan\theta \right) \dfrac{z_2}{\|z_2\|} \end{bmatrix} \tag{16}$$

under this subcase.

From (13), we see that $\Pi_{\mathcal{L}_\theta^\circ}(z) = -\Pi_{\mathcal{L}_{\frac{\pi}{2}-\theta}}(-z)$ which implies

$$\begin{aligned} \Pi_{\mathcal{L}_\theta^\circ}(z) &= -\begin{bmatrix} \dfrac{-z_1 + \|z_2\| \text{ctan}\theta}{1 + \text{ctan}^2\theta} \\ \left(\dfrac{-z_1 + \|z_2\| \text{ctan}\theta}{1 + \text{ctan}^2\theta} \text{ctan}\theta \right) \dfrac{-z_2}{\|z_2\|} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{z_1 - \|z_2\| \text{ctan}\theta}{1 + \text{ctan}^2\theta} \\ \left(\dfrac{z_1 - \|z_2\| \text{ctan}\theta}{1 + \text{ctan}^2\theta} \text{ctan}\theta \right) \dfrac{-z_2}{\|z_2\|} \end{bmatrix}. \end{aligned} \tag{17}$$

According to the above arguments, we obtain the following result, which is called the *spectral factorization* for z associated with circular cone.

Theorem 3.1. *For any $z \in \mathbb{R}^n$, one has*

$$z = \lambda_1(z) \cdot u_z^{(1)} + \lambda_2(z) \cdot u_z^{(2)} \tag{18}$$

where

$$\begin{aligned}\lambda_1(z) &= z_1 - \|z_2\|\mathrm{ctan}\theta \\ \lambda_2(z) &= z_1 + \|z_2\|\tan\theta\end{aligned}$$

and

$$\begin{aligned}u_z^{(1)} &= \frac{1}{1+\mathrm{ctan}^2\theta}\begin{bmatrix}1 & 0\\ 0 & \mathrm{ctan}\theta\end{bmatrix}\begin{bmatrix}1\\ -w\end{bmatrix} \\ u_z^{(2)} &= \frac{1}{1+\tan^2\theta}\begin{bmatrix}1 & 0\\ 0 & \tan\theta\end{bmatrix}\begin{bmatrix}1\\ w\end{bmatrix}\end{aligned}$$

with $w = \dfrac{z_2}{\|z_2\|}$ *if* $z_2 \neq 0$*, and any vector in* $\mathbb{R}^{n-1}$ *satisfying* $\|w\| = 1$ *if* $z_2 = 0$*.*

Proof. The case of $z_2 = 0$ is clear by simply calculating (18). The case of $z_2 \neq 0$ follows from (13), (16), and (17) because

$$\begin{aligned} z &= \Pi_{\mathcal{L}_\theta}(z) + \Pi_{\mathcal{L}_\theta^\circ}(z) \\ &= \begin{bmatrix} \dfrac{z_1 + \|z_2\|\tan\theta}{1+\tan^2\theta} \\ \left(\dfrac{z_1 + \|z_2\|\tan\theta}{1+\tan^2\theta}\tan\theta\right)\dfrac{z_2}{\|z_2\|}\end{bmatrix} + \begin{bmatrix} \dfrac{z_1 - \|z_2\|\mathrm{ctan}\theta}{1+\mathrm{ctan}^2\theta} \\ \left(\dfrac{z_1 - \|z_2\|\mathrm{ctan}\theta}{1+\mathrm{ctan}^2\theta}\mathrm{ctan}\theta\right)\dfrac{-z_2}{\|z_2\|}\end{bmatrix} \\ &= \frac{z_1 + \|z_2\|\tan\theta}{1+\tan^2\theta}\begin{bmatrix}1 & 0\\ 0 & \tan\theta\end{bmatrix}\begin{bmatrix}1\\ \dfrac{z_2}{\|z_2\|}\end{bmatrix} + \frac{z_1 - \|z_2\|\mathrm{ctan}\theta}{1+\mathrm{ctan}^2\theta}\begin{bmatrix}1 & 0\\ 0 & \mathrm{ctan}\theta\end{bmatrix}\begin{bmatrix}1\\ -\dfrac{z_2}{\|z_2\|}\end{bmatrix}.\end{aligned}$$

□

With Theorem 3.1, we could derive another expression for the projection shown as below.

Theorem 3.2. *For any* $z \in \mathbb{R}^n$*, we have*

$$\Pi_{\mathcal{L}_\theta}(z) = \big(\lambda_1(z)\big)_+ \cdot u_z^{(1)} + \big(\lambda_2(z)\big)_+ \cdot u_z^{(2)}, \tag{19}$$

where $(a)_+ := \max\{0, a\}$*,* $\lambda_i(z)$ *and* u_z^i *for* $i = 1, 2$ *are given as in Theorem 3.1.*

Proof. The proof is divided into two cases, according to whether $z_2 = 0$ or $z_2 \neq 0$.

Case 1: $z_2 = 0$. If $z_1 \geq 0$, then $z_1 \tan\theta \geq 0 = \|z_2\|$ and $\lambda_i(z) = z_1 \geq 0$. Hence $z \in \mathcal{L}_\theta$ and both sides of (19) are z by (14) and (18). If $z_1 < 0$, then $-z_1\mathrm{ctan}\theta \geq 0 = \|z_2\|$ and $\lambda_i(z) = z_1 < 0$ for $i = 1, 2$. Hence, $z \in -\mathcal{L}_{\frac{\pi}{2}-\theta} = -\mathcal{L}_\theta^*$ and both sides of (19) are 0 by (14).

Case 2: $z_2 \neq 0$. If $z \in \mathcal{L}_\theta$, then $z_1 \tan\theta \geq \|z_2\|$ which implies $z_1 \geq 0$. Therefore, $\lambda_i(z) \geq 0$ for $i = 1, 2$ which gives $\Pi_{\mathcal{L}_\theta}(z) = z = \lambda_1(z)u_z^1 + \lambda_2(z)u_z^2$ by (14) and (18). If $z \in -\mathcal{L}_\theta^*$, then

$-z \in \mathcal{L}_{\frac{\pi}{2}-\theta}$, i.e., $-z_1 \text{ctan}\theta \geq \|z_2\|$, which says $z_1 \leq 0$. Hence, $\lambda_1(z) = z_1 - \|z_2\|\text{ctan}\theta \leq 0$ and $\lambda_2(z) = z_1 + \|z_2\| \tan\theta \leq 0$. This indicates that the right-hand side of (19) is zero and it coincides $\Pi_{\mathcal{L}_\theta}(z) = 0$ by (14) under this case. Other cases correspond to $z_1 \tan\theta < \|z_2\|$ and $-z_1\text{ctan}\theta < \|z_2\|$, i.e., $\lambda_1(z) = z_1 - \|z_2\|\text{ctan}\theta < 0$ and $\lambda_2(z) = z_1 + \|z_2\| \tan\theta > 0$. Simplifying the right-hand side of (19) with this, we see that (14) is also satisfied under this case. Thus, all the above shows the validity of (19). $\square$

In particular, when $\theta = \pi/4$, expressions (18) and (19) takes, respectively, the form of

$$z = (z_1 - \|z_2\|)\frac{1}{2}\begin{bmatrix} 1 \\ -w \end{bmatrix} + (z_1 + \|z_2\|)\frac{1}{2}\begin{bmatrix} 1 \\ w \end{bmatrix}$$

and

$$\Pi_{\mathcal{L}_\theta}(z) = (z_1 - \|z_2\|)_+\frac{1}{2}\begin{bmatrix} 1 \\ -w \end{bmatrix} + (z_1 + \|z_2\|)_+\frac{1}{2}\begin{bmatrix} 1 \\ w \end{bmatrix}$$

where $w = \dfrac{z_2}{\|z_2\|}$ if $z_2 \neq 0$, and any vector in $\mathbb{R}^{n-1}$ satisfying $\|w\| = 1$ if $z_2 = 0$. These are exactly the well-known spectral factorization and projection associated with $\mathcal{K}^n$.

We believe that the spectral factorization given in Theorem 3.1 is very important for developing theory and algorithm for optimization associated with $\mathcal{L}_\theta$ like the role played by the spectral factorization associated with $\mathcal{K}^n$ in second-order cone optimization. We leave it for our future research topic.

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