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Smooth and nonsmooth analyses of vector-valued functions associated with circular cones

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ABSTRACT

Let $\mathcal{L}_\theta$ be the circular cone in $\mathbb{R}^n$ which includes a second-order cone as a special case. For any function f from $\mathbb{R}$ to $\mathbb{R}$, one can define a corresponding vector-valued function $f^c(x)$ on $\mathbb{R}^n$ by applying f to the spectral values of the spectral decomposition of $x \in \mathbb{R}^n$ with respect to $\mathcal{L}_\theta$. We show that this vector-valued function inherits from f the properties of continuity, Lipschitz continuity, directional differentiability, Fréchet differentiability, continuous differentiability, as well as semismoothness. These results will play a crucial role in designing solution methods for optimization problem associated with the circular cone.

1. Introduction

The circular cone [1,2] is a pointed closed convex cone having hyperspherical sections orthogonal to its axis of revolution about which the cone is invariant to rotation. Let its half-aperture angle be θ with $\theta \in (0, \frac{\pi}{2})$. Then, the n-dimensional circular cone denoted by $\mathcal{L}_\theta$ can be expressed as

$$\begin{aligned} \mathcal{L}_\theta &:= \left\{x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \,|\, \|x\| \cos\theta \le x_1\right\} \\ &:= \left\{x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \,|\, \|x_2\| \cot\theta \le x_1\right\}. \end{aligned} \tag{1}$$

See Fig. 1.

When $\theta = 45^\circ$, the circular cone reduces to the well-known second-order cone (SOC, also called Lorentz cone) given by

$$\begin{aligned} \mathcal{K}^n &:= \left\{x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \,|\, \|x_2\| \le x_1\right\} \\ &:= \left\{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \,|\, \|x\| \cos 45^\circ \le x_1\right\}. \end{aligned}$$

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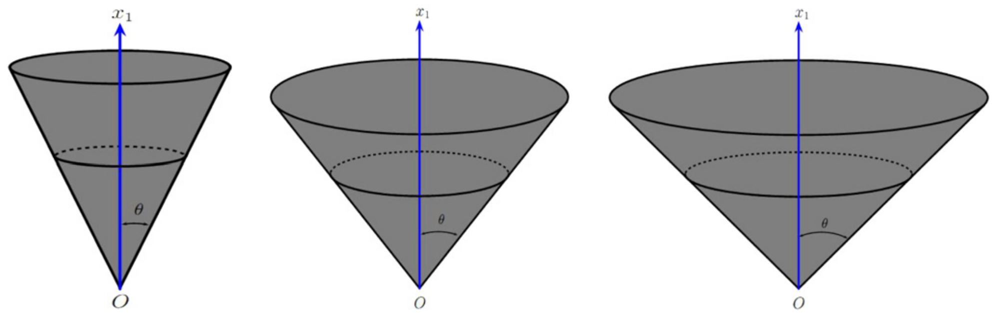

(a) $0 < \theta < 45°$. (b) $\theta = 45°$. (c) $45° < \theta < 90°$.

Fig. 1. The graphs of circular cones.

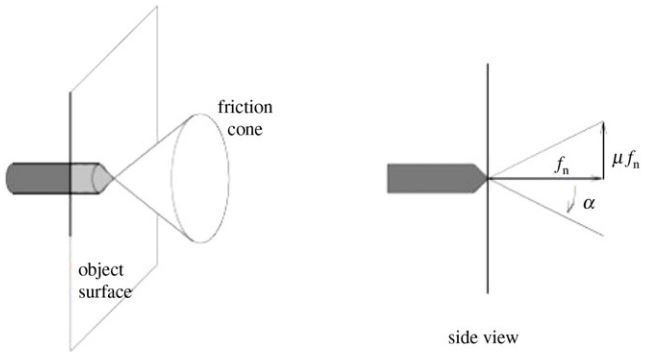

Fig. 2. The grasping force forms a circular cone where $\alpha = \tan^{-1}\mu < 45°$.

With respect to SOC, for any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we can decompose x as

$$x = \lambda_1(x)u_x^{(1)} + \lambda_2(x)u_x^{(2)}, \tag{2}$$

where $\lambda_1(x), \lambda_2(x)$ and $u_x^{(1)}, u_x^{(2)}$ are the *spectral values* and the associated *spectral vectors* of x with respect to $\mathcal{K}^n$, given by

$$\lambda_i(x) = x_1 + (-1)^i \|x_2\|,$$

$$u_x^{(i)} = \begin{cases} \frac{1}{2}\left(1, (-1)^i \frac{x_2}{\|x_2\|}\right), & \text{if } x_2 \neq 0, \\ \frac{1}{2}\left(1, (-1)^i w\right), & \text{if } x_2 = 0, \end{cases}$$

for $i = 1, 2$ with w being any vector in $\mathbb{R}^{n-1}$ satisfying $\|w\| = 1$. If $x_2 \neq 0$, the decomposition (2) is unique. With this spectral decomposition (2), for any function $f : \mathbb{R} \to \mathbb{R}$, the following vector-valued function associated with $\mathcal{K}^n$ ($n \geq 1$) is considered (see [3,4]):

$$f^{\text{soc}}(x) = f(\lambda_1)u^{(1)} + f(\lambda_2)u^{(2)} \quad \forall x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}. \tag{3}$$

If f is defined only on a subset of $\mathbb{R}$, then f^{soc} is defined on the corresponding subset of $\mathbb{R}^n$. The definition (3) is unambiguous whether $x_2 \neq 0$ or $x_2 = 0$. The above definition (3) is analogous to the one associated with the semidefinite cone $\mathcal{S}^n$; see [5,6]. It was shown [4] that the properties of continuity, strict continuity, Lipschitz continuity, directional differentiability, differentiability, continuous differentiability, and semismoothness are each inherited by f^{soc} from f. These results are useful in the design and analysis of smooth and nonsmooth methods for solving second-order cone programs (SOCP) and second-order cone complementarity problem (SOCCP); see [3,4,7,8] and references therein.

Recently, there have been found circular cone constraints involved in real engineering problems. For example, in the formulation of optimal grasping manipulation for multi-fingered robots, the grasping force of the i-th finger is subject to a contact friction constraint expressed as

$$|(u_{i1}, u_{i3})| \leq \mu u_{i1} \tag{4}$$

where μ is the friction coefficient; see Fig. 2.

Indeed, (4) is a circular cone constraint corresponding to $u_i = (u_{i1}, u_{i2}, u_{i3}) \in \mathcal{L}_\theta$ with $\theta = \tan^{-1}\mu < 45°$. Note that the circular cone $\mathcal{L}_\theta$ is a non-self-dual (or non-symmetric) cone and its related study is rather limited. Nonetheless, motivated by the real world application regarding circular cone, the structures and properties about $\mathcal{L}_\theta$ are investigated in [2]. In particular, the spectral factorization of z associated with the circular cone is characterized in [2, Theorem 3.1]. For convenience, we restate it as follows.

Theorem 1.1 (*[2, Theorem 3.1]*)**.** *For any $z = (z_1, z_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, one has*

$$z = \lambda_1(z) \cdot u_z^{(1)} + \lambda_2(z) \cdot u_z^{(2)} \tag{5}$$

where

$$\begin{cases} \lambda_1(z) = z_1 - \|z_2\| \cot\theta \\ \lambda_2(z) = z_1 + \|z_2\| \tan\theta \end{cases} \tag{6}$$

and

$$\begin{cases} u_z^{(1)} = \dfrac{1}{1+\cot^2\theta} \begin{bmatrix} 1 & 0 \\ 0 & \cot\theta \end{bmatrix} \begin{bmatrix} 1 \\ -w \end{bmatrix} = \begin{bmatrix} \sin^2\theta \\ -(\sin\theta\cos\theta)w \end{bmatrix} \\ u_z^{(2)} = \dfrac{1}{1+\tan^2\theta} \begin{bmatrix} 1 & 0 \\ 0 & \tan\theta \end{bmatrix} \begin{bmatrix} 1 \\ w \end{bmatrix} = \begin{bmatrix} \cos^2\theta \\ (\sin\theta\cos\theta)w \end{bmatrix} \end{cases} \tag{7}$$

with $w = \frac{z_2}{\|z_2\|}$ if $z_2 \neq 0$, and any vector in $\mathbb{R}^{n-1}$ satisfying $\|w\| = 1$ if $z_2 = 0$.

Analogous to (3), with the spectral factorization (5), for any function $f : \mathbb{R} \to \mathbb{R}$, we consider the following vector-valued function associated with $\mathcal{L}_\theta$ ($n \geq 1$):

$$f^c(z) = f(\lambda_1)u_z^{(1)} + f(\lambda_2)u_z^{(2)} \quad \forall z = (z_1, z_2) \in \mathbb{R} \times \mathbb{R}^{n-1}. \tag{8}$$

Can the properties of continuity, strict continuity, Lipschitz continuity, directional differentiability, differentiability, continuous differentiability, and semismoothness be each inherited by f^c from f? These are what we want to explore in this paper.

At last, we say a few words about notations. In what follows, for any differentiable (in the Fréchet sense) mapping $F : \mathbb{R}^n \to \mathbb{R}^m$, we denote its Jacobian (not transposed) at $x \in \mathbb{R}^n$ by $\nabla F(x) \in \mathbb{R}^{m\times n}$, i.e., $(F(x+u) - F(x) - \nabla F(x)u)/\|u\| \to 0$ as $u \to 0$. ":=" means "define". We write $z = O(\alpha)$ (respectively, $z = o(\alpha)$), with $\alpha \in \mathbb{R}$ and $z \in \mathbb{R}^n$, to mean $\|z\|/|\alpha|$ is uniformly bounded (respectively, tends to zero) as $\alpha \to 0$.

2. Preliminaries

In this section, we review some basic concepts regarding vector-valued functions. These contain continuity, (local) Lipschitz continuity, directional differentiability, differentiability, continuous differentiability, as well as semismoothness.

Suppose $F : \mathbb{R}^n \to \mathbb{R}^m$. Then, F is continuous at $x \in \mathbb{R}^n$ if $F(y) \to F(x)$ as $y \to x$, and F is continuous if F is continuous at every $x \in \mathbb{R}^n$. We say F is strictly continuous (also called "locally Lipschitz continuous") at $x \in \mathbb{R}^n$ if there exist scalars $\kappa > 0$ and $\delta > 0$ such that

$$\|F(y) - F(z)\| \leq \kappa\|y - z\| \quad \forall y, z \in \mathbb{R}^n \text{ with } \|y - x\| \leq \delta,\ \|z - x\| \leq \delta;$$

and F is strictly continuous if F is strictly continuous at every $x \in \mathbb{R}^n$. We say F is directionally differentiable at $x \in \mathbb{R}^n$ if

$$F'(x; h) := \lim_{t\to 0^+} \frac{F(x + th) - F(x)}{t} \quad \text{exists } \forall h \in \mathbb{R}^n;$$

and F is directionally differentiable if F is directionally differentiable at every $x \in \mathbb{R}^n$. F is differentiable (in the Fréchet sense) at $x \in \mathbb{R}^n$ if there exists a linear mapping $\nabla F(x) : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$F(x + h) - F(x) - \nabla F(x)h = o(\|h\|).$$

If F is differentiable at every $x \in \mathbb{R}^n$ and ∇F is continuous, then F is continuously differentiable. We notice that, in the above expression about strict continuity of F, if δ can be taken to be ∞, then F is called Lipschitz continuous with Lipschitz constant κ.

It is well-known that if F is strictly continuous, then F is almost everywhere differentiable by Rademacher's Theorem; see [9] and [10, Section 9J]. In this case, the generalized Jacobian $\partial F(x)$ of F at x (in the Clarke sense) can be defined as the convex hull of the generalized Jacobian $\partial_B F(x)$, where

$$\partial_B F(x) := \left\{ \lim_{x^j \to x} \nabla F(x^j) \mid F \text{ is differentiable at } x^j \in \mathbb{R}^n \right\}.$$

The notation ∂_B is adopted from [11]. In [10, Chapter 9], the case of $m = 1$ is considered and the notations "$\bar{\nabla}$" and "$\bar{\partial}$" are used instead of, respectively, "∂_B" and "∂". Assume $F : \mathbb{R}^n \to \mathbb{R}^m$ is strictly continuous, then F is said to be semismooth at x if F is directionally differentiable at x and, for any $V \in \partial F(x+h)$, we have

$$F(x+h) - F(x) - Vh = o(\|h\|).$$

Moreover, F is called ρ-order semismooth at x $(0 < \rho < \infty)$ if F is semismooth at x and, for any $V \in \partial F(x+h)$, we have

$$F(x+h) - F(x) - Vh = O(\|h\|^{1+\rho}).$$

The following lemma, proven by Sun and Sun [5, Theorem 3.6] using the definition of generalized Jacobian, enables one to study the semismooth property of f^c by examining only those points $x \in \mathbb{R}^n$ where f^c is differentiable and thus work only with the Jacobian of f^c, rather than the generalized Jacobian. It is a very useful working lemma for verifying semismoothness property in Section 4.

Lemma 2.1. *Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is strictly continuous and directionally differentiable in a neighborhood of $x \in \mathbb{R}^n$. Then, for any $0 < \rho < \infty$, the following two statements are equivalent.*

(a) *For any $v \in \partial F(x+h)$ and $h \to 0$,*

$$F(x+h) - F(x) - vh = o(\|h\|) \quad (\text{respectively, } O(\|h\|)^{1+\rho}).$$

(b) *For any $h \to 0$ such that F is differentiable at $x+h$,*

$$F(x+h) - F(x) - \nabla F(x+h)h = o(\|h\|) \quad (\text{respectively}, O(\|h\|)^{1+\rho}).$$

We say F is semismooth (respectively, ρ-order semismooth) if F is semismooth (respectively, ρ-order semismooth) at every $x \in \mathbb{R}^n$. We say F is *strongly semismooth* if it is 1-order semismooth. Convex functions and piecewise continuously differentiable functions are examples of semismooth functions. The composition of two (respectively, ρ-order) semismooth functions is also a (respectively, ρ-order) semismooth function. The property of semismoothness, as introduced by Mifflin [12] for functionals and scalar-valued functions and further extended by Qi and Sun [13] for vector-valued functions, is of particular interest due to the key role it plays in the superlinear convergence analysis of certain generalized Newton methods [11,13–16]. For extensive discussions of semismooth functions, see [12,13,17].

3. Properties of continuity and differentiability

In this section, we focus on the properties of continuity and differentiability between f and f^c. We need some technical lemmas which come from the simple structure of the circular cone and basic definitions before starting the proofs.

Lemma 3.1. *Let $\lambda_1 \le \lambda_2$ be the spectral values of $x \in \mathbb{R}^n$ and $m_1 \le m_2$ be the spectral values of $y \in \mathbb{R}^n$. Then, we have*

$$|\lambda_1 - m_1|^2 \sin^2\theta + |\lambda_2 - m_2|^2 \cos^2\theta = \|x-y\|^2, \tag{9}$$

and hence, $|\lambda_i - m_i| \le c\|x-y\|$, $\forall i = 1, 2$, where $c = \max\{\sec\theta,\ \csc\theta\}$.

Proof. The proof follows from a direct computation. □

Lemma 3.2. *Let $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$.*

(a) *If $x_2 \ne 0, y_2 \ne 0$, then we have*

$$\|u^{(i)} - v^{(i)}\| \le \frac{2\sin\cos\theta}{\|x_2\|}\|x-y\|, \quad i = 1, 2, \tag{10}$$

where $u^{(i)}$, $v^{(i)}$ are the unique spectral vectors of x and y, respectively.

(b) *If either $x_2 = 0$ or $y_2 = 0$, then we can choose $u^{(i)}$, $v^{(i)}$ such that the left hand side of inequality (10) is zero.*

Proof. (a) From the spectral factorization (5), we know that

$$u^{(1)} = \sin^2\theta\left(1, (-1)\cot\theta\frac{x_2}{\|x_2\|}\right), \qquad v^{(1)} = \sin^2\theta\left(1, (-1)\cot\theta\frac{y_2}{\|y_2\|}\right),$$

where $u^{(1)}, v^{(1)}$ are unique. This gives $u^{(1)} - v^{(1)} = \sin^2\theta \left(0, (-1)\cot\theta(\frac{x_2}{\|x_2\|} - \frac{y_2}{\|y_2\|})\right)$. Then,

$$\begin{aligned}
\|u^{(1)} - v^{(1)}\| &= \sin\theta\cos\theta \left\| \frac{x_2}{\|x_2\|} - \frac{y_2}{\|y_2\|} \right\| \\
&= \sin\theta\cos\theta \left\| \frac{x_2 - y_2}{\|x_2\|} + \frac{(\|y_2\| - \|x_2\|)y_2}{\|x_2\| \cdot \|y_2\|} \right\| \\
&\le \sin\theta\cos\theta \left(\frac{1}{\|x_2\|}\|x_2 - y_2\| + \frac{1}{\|x_2\|} \left| \|y_2\| - \|x_2\| \right| \right) \\
&\le \sin\theta\cos\theta \left(\frac{1}{\|x_2\|}\|x_2 - y_2\| + \frac{1}{\|x_2\|}\|x_2 - y_2\| \right) \\
&\le \frac{2\sin\theta\cos\theta}{\|x_2\|}\|x - y\|,
\end{aligned}$$

where the inequalities follow from the triangle inequality. Similar arguments apply for $\|u^{(2)} - v^{(2)}\|$.

(b) We can choose the same spectral vectors for x and y from the spectral factorization (5) since either $x_2 = 0$ or $y_2 = 0$. Then, it is obvious. □

Lemma 3.3. *For any $w \neq 0 \in \mathbb{R}^n$, we have $\nabla_w \left(\frac{w}{\|w\|}\right) = \frac{1}{\|w\|}\left(I - \frac{ww^T}{\|w\|^2}\right)$.*

Proof. See [18, Lemma 3.3] or check it by direct computation. □

Now, we are ready to present our first main result about continuity between f and f^c.

Theorem 3.1. *For any $f : \mathbb{R} \to \mathbb{R}$, f^c is continuous at $x \in \mathbb{R}^n$ with spectral values λ_1, λ_2 if and only if f is continuous at λ_1, λ_2.*

Proof. "$\Leftarrow$" Suppose f is continuous at λ_1, λ_2. For any fixed $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y \to x$, let the spectral factorizations of x, y be $x = \lambda_1 u^{(1)} + \lambda_2 u^{(2)}$ and $y = m_1 v^{(1)} + m_2 v^{(2)}$, respectively. Then, we discuss two cases.

Case (i). If $x_2 \neq 0$, then we have

$$f^c(y) - f^c(x) = f(m_1)\left[v^{(1)} - u^{(1)}\right] + [f(m_1) - f(\lambda_1)]\,u^{(1)} + f(m_2)\left[v^{(2)} - u^{(2)}\right] + [f(m_2) - f(\lambda_2)]\,u^{(2)}. \tag{11}$$

Since f is continuous at λ_1, λ_2, and from Lemma 3.1, $|m_i - \lambda_i| \le c\ \|y - x\|$, we know that $f(m_i) \longrightarrow f(\lambda_i)$ as $y \to x$. In addition, by Lemma 3.2, we have $\|v^{(i)} - u^{(i)}\| \longrightarrow 0$ as $y \to x$. Thus, Eq. (11) yields $f^c(y) \longrightarrow f^c(x)$ as $y \to x$ because both $f(m_i)$ and $\|u^{(i)}\|$ are bounded. Hence, f^c is continuous at $x \in \mathbb{R}^n$.

Case (ii). If $x_2 = 0$, no matter y_2 is zero or not, we can arrange that x, y have the same spectral vectors. Thus, $f^c(y) - f^c(x) = [f(m_1) - f(\lambda_1)]\,u^{(1)} + [f(m_2) - f(\lambda_2)]\,u^{(2)}$. Then, f^c is continuous at $x \in \mathbb{R}^n$ by similar arguments.

"$\Rightarrow$" The proof for this direction is straightforward or refer to similar arguments for [4, Prop. 2]. □

Theorem 3.2. *For any $f : \mathbb{R} \to \mathbb{R}$, f^c is directionally differentiable at $x \in \mathbb{R}^n$ with spectral values λ_1, λ_2 if and only if f is directionally differentiable at λ_1, λ_2.*

Proof. "$\Leftarrow$" Suppose f is directionally differentiable at λ_1, λ_2. Fix any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, then we discuss two cases as below.

Case (i). If $x_2 \neq 0$, we have $f^c(x) = f(\lambda_1)u^{(1)} + f(\lambda_2)u^{(2)}$ where $\lambda_i = x_1 + (-1)^i(\tan\theta)^{(-1)^i}\|x_2\|$ and $u^{(i)} = (-1)^i \sin\theta\cos\theta\left((\tan\theta)^{(-1)^i}, \frac{x_2^T}{\|x_2\|}\right)$ for all $i = 1, 2$. From Lemma 3.3, we know that $u^{(i)}$ is Fréchet-differentiable with respect to x, with

$$\nabla_x u^{(i)} = \frac{(-1)^i \sin\theta\cos\theta}{\|x_2\|}\begin{bmatrix} 0 & 0 \\ 0 & I - \dfrac{x_2 x_2^T}{\|x_2\|^2} \end{bmatrix} \quad \forall i = 1, 2. \tag{12}$$

Also by the expression of λ_i, we know that λ_i is Fréchet-differentiable with respect to x, with

$$\nabla_x \lambda_i = \left(1, (-1)^i \tan^{(-1)^i}\theta \frac{x_2^T}{\|x_2\|}\right) \quad \forall i = 1, 2. \tag{13}$$

In general, we cannot apply the chain rule, when functions are only directionally differentiable. But, it works well for single-variable functions, that is, when single-variable functions are composed of a differentiable function. From the hypothesis, f

is directionally differentiable at λ_1, then it is easy to compute

$$\lim_{t\to 0^+}\frac{f(\lambda_1+t\times 1)-f(\lambda_1)}{t}=f'(\lambda_1;1),$$
$$\lim_{t\to 0^+}\frac{f(\lambda_1-t\times 1)-f(\lambda_1)}{t}=f'(\lambda_1;-1),$$
$$\lim_{t\to 0^+}\frac{f(\lambda_1+o(t))-f(\lambda_1)}{t}=0.$$

Note that the spectral value function $\lambda_1(x)=x_1-\cot\theta\|x_2\|$ is differentiable when $x_2\neq 0$, which yields

$$\lambda_1(x+th)=\lambda_1(x)+t\nabla_x\lambda_1 h+o(t).$$

Let $y:=\nabla_x\lambda_1 h+\frac{o(t)}{t}$. For the case of $\nabla_x\lambda_1 h<0$, we know $y<0$ as t is small. Thus,

$$\begin{aligned}\lim_{t\to 0^+}\frac{f(\lambda_1(x+th))-f(\lambda_1(x))}{t}&=\lim_{t\to 0^+}\frac{f(\lambda_1(x)+ty)-f(\lambda_1(x))}{t}\\&=\lim_{t\to 0^+}\frac{f(\lambda_1(x)-(-ty))-f(\lambda_1(x))}{-ty}(-y)=\lim_{-ty\to 0^+}\frac{f(\lambda_1(x)-(-ty))-f(\lambda_1(x))}{-ty}\lim_{t\to 0^+}(-y)\\&=f'(\lambda_1(x);-1)(-\nabla_x\lambda_1 h)=f'(\lambda_1(x);\nabla_x\lambda_1 h).\end{aligned}$$

Here the positively homogeneous property of directionally differentiable functions is used in the last equation. Similarly, for the other case of $\nabla_x\lambda_1 h\geq 0$, we have

$$\lim_{t\to 0^+}\frac{f(\lambda_1(x+th))-f(\lambda_1(x))}{t}=f'(\lambda_1(x);\nabla_x\lambda_1 h).$$

In summary, the composite function $f\circ\lambda_1(\cdot)$ is directionally differentiable at x. Now we can apply the chain rule and the product rule on $f^c(x)=f(\lambda_1)u^{(1)}+f(\lambda_2)u^{(2)}$. In other words,

$$\begin{aligned}(f^c)'(x;h)&=f(\lambda_1)\nabla_x u^{(1)}h+f'(\lambda_1;\nabla_x\lambda_1 h)u^{(1)}+f(\lambda_2)\nabla_x u^{(2)}h+f'(\lambda_2;\nabla_x\lambda_2 h)u^{(2)}\\&=(A_1,A_2)\in\mathbb{R}\times\mathbb{R}^{n-1},\end{aligned}$$

where

$$A_1=f'\left(\lambda_1;h_1-\cot\theta\frac{x_2^Th_2}{\|x_2\|}\right)\sin^2\theta+f'\left(\lambda_2;h_1+\tan\theta\frac{x_2^Th_2}{\|x_2\|}\right)\cos^2\theta \tag{14}$$

and

$$\begin{aligned}A_2=&\left[f'\left(\lambda_2;h_1+\tan\theta\frac{x_2^Th_2}{\|x_2\|}\right)-f'\left(\lambda_1;h_1-\cot\theta\frac{x_2^Th_2}{\|x_2\|}\right)\right]\sin\theta\cos\theta\frac{x_2}{\|x_2\|}\\&+\frac{f(\lambda_2)-f(\lambda_1)}{\lambda_2-\lambda_1}\left(I-\frac{x_2x_2^T}{\|x_2\|^2}\right)h_2,\end{aligned} \tag{15}$$

with $h=(h_1,h_2)\in\mathbb{R}\times\mathbb{R}^{n-1}$.

Now, applying Eqs. (12) and (13) and using the fact that $\lambda_2-\lambda_1=\frac{\|x_2\|}{\sin\theta\cos\theta}$ in the A_2 term, we see that $(f^c)'(x;h)$ can be rewritten in a more compact form as below:

$$(f^c)'(x;h)=f'\left(\lambda_1;h_1-\cot\theta\frac{x_2^Th_2}{\|x_2\|}\right)u^{(1)}+f'\left(\lambda_2;h_1+\tan\theta\frac{x_2^Th_2}{\|x_2\|}\right)u^{(2)}+\frac{f(\lambda_2)-f(\lambda_1)}{\lambda_2-\lambda_1}\left(I-\frac{x_2x_2^T}{\|x_2\|^2}\right)h_2. \tag{16}$$

Case (ii). If $x_2=0$, we compute the directional derivative $(f^c)'(x;h)$ at x for any direction h by definition. Let $h=(h_1,h_2)\in\mathbb{R}\times\mathbb{R}^{n-1}$. We have two subcases. First, consider the subcase of $h_2\neq 0$. From the spectral factorization, we can choose $u^{(1)}=\left(\sin^2\theta,-\sin\theta\cos\theta\frac{h_2}{\|h_2\|}\right)$ and $u^{(2)}=\left(\cos^2\theta,\sin\theta\cos\theta\frac{h_2}{\|h_2\|}\right)$ such that

$$\begin{cases}f^c(x+th)=f(\lambda+\triangle\lambda_1)u^{(1)}+f(\lambda+\triangle\lambda_2)u^{(2)}\\f^c(x)=f(\lambda)u^{(1)}+f(\lambda)u^{(2)}\end{cases}$$

where $\lambda=x_1$ and $\triangle\lambda_i=t\left(h_1+(-1)^i\tan^{(-1)^i}\theta\|h_2\|\right)$ for all $i=1,2$. Thus, we obtain

$$f^c(x+th)-f^c(x)=[f(\lambda+\triangle\lambda_1)-f(\lambda)]\,u^{(1)}+[f(\lambda+\triangle\lambda_2)-f(\lambda)]\,u^{(2)}.$$

Using the following facts

$$\lim_{t\to 0^+}\frac{f(\lambda+\triangle\lambda_1)-f(\lambda)}{t}=\lim_{t\to 0^+}\frac{f(\lambda+t(h_1-\cot\theta\|h_2\|))-f(\lambda)}{t}=f'(\lambda;h_1-\cot\theta\|h_2\|)$$
$$\lim_{t\to 0^+}\frac{f(\lambda+\triangle\lambda_2)-f(\lambda)}{t}=\lim_{t\to 0^+}\frac{f(\lambda+t(h_1+\tan\theta\|h_2\|))-f(\lambda)}{t}=f'(\lambda;h_1+\tan\theta\|h_2\|)$$

yields

$$\begin{aligned}\lim_{t\to 0^+}\frac{f^c(x+th)-f^c(x)}{t}&=\lim_{t\to 0^+}\frac{f(\lambda+\triangle\lambda_1)-f(\lambda)}{t}u^{(1)}+\lim_{t\to 0^+}\frac{f(\lambda+\triangle\lambda_2)-f(\lambda)}{t}u^{(2)}\\&=f'(\lambda;h_1-\cot\theta\|h_2\|)u^{(1)}+f'(\lambda;h_1+\tan\theta\|h_2\|)u^{(2)}\end{aligned}\tag{17}$$

which says $(f^c)'(x;h)$ exists.

Second, for the subcase of $h_2=0$, the same arguments apply except $h_2/\|h_2\|$ is replaced by any $w\in\mathbb{R}^{n-1}$ with $\|w\|=1$, i.e., choosing $u^{(1)}=\left(\sin^2\theta,-\sin\theta\cos\theta w\right)$ and $u^{(2)}=\left(\cos^2\theta,\sin\theta\cos\theta w\right)$. Analogously, we obtain

$$\lim_{t\to 0^+}\frac{f^c(x+th)-f^c(x)}{t}=f'(\lambda;h_1)u^{(1)}+f'(\lambda;h_1)u^{(2)}\tag{18}$$

which implies $(f^c)'(x;h)$ exists in the form of (18). From all the above, it shows that f^c is directionally differentiable at x when $x_2=0$ and its directional derivative $(f^c)'(x;h)$ is either in the form of (17) or (18).

"⇒" Suppose f^c is directionally differentiable at $x\in\mathbb{R}^n$ with spectral values λ_1,λ_2, we will prove that f is directionally differentiable at λ_1,λ_2. For $\lambda_1\in\mathbb{R}$ and any direction $d_1\in\mathbb{R}$, let $h:=d_1u^{(1)}+0u^{(2)}$ where $x=\lambda_1u^{(1)}+\lambda_2u^{(2)}$. Then, $x+th=(\lambda_1+td_1)u^{(1)}+\lambda_2u^{(2)}$ and

$$\frac{f^c(x+th)-f^c(x)}{t}=\frac{f(\lambda_1+td_1)-f(\lambda_1)}{t}u^{(1)}.$$

Since f^c is directionally differentiable at x, the above equation implies

$$f'(\lambda_1;d_1)=\lim_{t\to 0^+}\frac{f(\lambda_1+td_1)-f(\lambda_1)}{t}\quad\text{exists.}$$

This means f is directionally differentiable at λ_1. Similarly, f is also directionally differentiable at λ_2. □

Theorem 3.3. *For any $f:\mathbb{R}\to\mathbb{R}$, f^c is differentiable at $x=(x_1,x_2)\in\mathbb{R}\times\mathbb{R}^{n-1}$ with spectral values λ_1,λ_2 if and only if f is differentiable at λ_1,λ_2. Moreover, for given $h=(h_1,h_2)\in\mathbb{R}\times\mathbb{R}^{n-1}$, we have*

$$\nabla f^c(x)h=\begin{bmatrix} b & \dfrac{cx_2^T}{\|x_2\|}\\ \dfrac{cx_2}{\|x_2\|} & aI+(\bar b-a)\dfrac{x_2x_2^T}{\|x_2\|^2}\end{bmatrix}\begin{bmatrix}h_1\\h_2\end{bmatrix},\quad \text{when } x_2\neq 0,$$

where

$$\begin{aligned}a&=\frac{f(\lambda_2)-f(\lambda_1)}{\lambda_2-\lambda_1},\\ b&=f'(\lambda_1)\sin^2\theta+f'(\lambda_2)\cos^2\theta,\\ \bar b&=f'(\lambda_1)\cos^2\theta+f'(\lambda_2)\sin^2\theta,\\ c&=[f'(\lambda_2)-f'(\lambda_1)]\sin\theta\cos\theta.\end{aligned}$$

When $x_2=0$, $\nabla f^c(x)=f'(\lambda)I$ with $\lambda=x_1$.

Proof. "⇐" The proof of this direction is identical to the proof shown as in Theorem 3.2, in which only "directionally differentiable" needs to be replaced by "differentiable". Since f is differentiable at λ_1 and λ_2, we have that $f'(\lambda_1;\cdot)$ and $f'(\lambda_2;\cdot)$ are linear, which means $f'(\lambda_i;a+b)=f'(\lambda_i)a+f'(\lambda_i)b$. This together with Eqs. (14) and (15) yield

$$\begin{aligned}A_1&=f'\left(\lambda_1;h_1-\cot\theta\frac{x_2^Th_2}{\|x_2\|}\right)\sin^2\theta+f'\left(\lambda_2;h_1+\tan\theta\frac{x_2^Th_2}{\|x_2\|}\right)\cos^2\theta\\&=f'(\lambda_1)h_1\sin^2\theta-f'(\lambda_1)\cot\theta\frac{x_2^Th_2}{\|x_2\|}\sin^2\theta+f'(\lambda_2)h_1\cos^2\theta+f'(\lambda_2)\tan\theta\frac{x_2^Th_2}{\|x_2\|}\cos^2\theta\\&=\left[f'(\lambda_1)\sin^2\theta+f'(\lambda_2)\cos^2\theta\right]h_1+\left[f'(\lambda_2)-f'(\lambda_1)\right]\sin\theta\cos\theta\frac{x_2^T}{\|x_2\|}h_2\end{aligned}$$

and

$$\begin{aligned}
A_2 &= \left[f'\left(\lambda_2; h_1 + \tan\theta \frac{x_2^T h_2}{\|x_2\|}\right) - f'\left(\lambda_1; h_1 - \cot\theta \frac{x_2^T h_2}{\|x_2\|}\right)\right] \sin\theta \cos\theta \frac{x_2}{\|x_2\|} + \frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1}\left(I - \frac{x_2 x_2^T}{\|x_2\|^2}\right) h_2 \\
&= \left[f'(\lambda_2)h_1 - f'(\lambda_1)h_1 + f'(\lambda_2)\tan\theta \frac{x_2^T h_2}{\|x_2\|} + f'(\lambda_1)\cot\theta \frac{x_2^T h_2}{\|x_2\|}\right] \sin\theta\cos\theta \frac{x_2}{\|x_2\|} \\
&\quad + \frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1}\left(I - \frac{x_2 x_2^T}{\|x_2\|^2}\right) h_2 \\
&= \left[f'(\lambda_2) - f'(\lambda_1)\right] \sin\theta\cos\theta \frac{x_2}{\|x_2\|} h_1 + \left[f'(\lambda_2)\sin^2\theta + f'(\lambda_1)\cos^2\theta\right] \frac{x_2 x_2^T}{\|x_2\|^2} h_2 \\
&\quad + \frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1}\left(I - \frac{x_2 x_2^T}{\|x_2\|^2}\right) h_2. \qquad (19)
\end{aligned}$$

Thus, for $x_2 \neq 0$, we have

$$\nabla f^c(x)h = \begin{bmatrix} b & \dfrac{c x_2^T}{\|x_2\|} \\ \dfrac{c x_2}{\|x_2\|} & aI + (\bar{b} - a)\dfrac{x_2 x_2^T}{\|x_2\|^2} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \qquad (20)$$

with

$$\begin{aligned}
a &= \frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1}, \\
b &= f'(\lambda_1)\sin^2\theta + f'(\lambda_2)\cos^2\theta, \\
\bar{b} &= f'(\lambda_1)\cos^2\theta + f'(\lambda_2)\sin^2\theta, \qquad (21) \\
c &= \left[f'(\lambda_2) - f'(\lambda_1)\right]\sin\theta\cos\theta.
\end{aligned}$$

From Eq. (16), $\nabla f^c(x)h$ can also be recast in a more compact form:

$$\nabla f^c(x)h = f'(\lambda_1)\left(h_1 - \cot\theta \frac{x_2^T h_2}{\|x_2\|}\right)u^{(1)} + f'(\lambda_2)\left(h_1 + \tan\theta \frac{x_2^T h_2}{\|x_2\|}\right)u^{(2)} + \frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1}\left(I - \frac{x_2 x_2^T}{\|x_2\|^2}\right)h_2. \qquad (22)$$

For the case of $x_2 = 0$, with linearity of $f'(\lambda; \cdot)$ and Eqs. (17) and (18), we have

$$\nabla f^c(x) = f'(\lambda)I, \qquad (23)$$

where $\lambda = \lambda_1 = \lambda_2 = x_1$.

"$\Rightarrow$" Let f^c be Fréchet-differentiable at $x \in \mathbb{R}^n$ with spectral eigenvalues λ_1, λ_2, we will show that f is Fréchet-differentiable at λ_1, λ_2. Suppose not, then f is not Fréchet-differentiable at λ_i for some $i \in \{1, 2\}$. Thus, either the right- and left-directional derivatives of f at λ_i is unequal or one of them does not exist. In either case, this implies that there exist two sequences of non-zero scalars t^ν and τ^ν, $\nu = 1, 2, \ldots,$ converging to zero such that the limits

$$\lim_{\nu\to\infty} \frac{f(\lambda_i + t^\nu) - f(\lambda_i)}{t^\nu}, \qquad \lim_{\nu\to\infty} \frac{f(\lambda_i + \tau^\nu) - f(\lambda_i)}{\tau^\nu}$$

either are unequal or one of them does not exist. Now for any $x = \lambda_1 u^{(1)} + \lambda_2 u^{(2)}$, let $h := 1 \cdot u^{(1)} + 0 \cdot u^{(2)} = u^{(1)}$. Then, we know $x + th = (\lambda_1 + t)u^{(1)} + \lambda_2 u^{(2)}$ and $f^c(x + th) = f(\lambda_1 + t)u^{(1)} + f(\lambda_2)u^{(2)}$, which give

$$\begin{aligned}
\lim_{\nu\to\infty} \frac{f^c(x + t^\nu h) - f^c(x)}{t^\nu} &= \lim_{\nu\to\infty} \frac{f(\lambda_1 + t^\nu) - f(\lambda_1)}{t^\nu} u^{(1)} \\
\lim_{\nu\to\infty} \frac{f^c(x + \tau^\nu h) - f^c(x)}{\tau^\nu} &= \lim_{\nu\to\infty} \frac{f(\lambda_1 + \tau^\nu) - f(\lambda_1)}{\tau^\nu} u^{(1)}.
\end{aligned}$$

It follows that these two limits either are unequal or one of them does not exist. This implies that f^c is not Fréchet-differentiable at x, which is a contradiction. □

Theorem 3.4. *For any $f : \mathbb{R} \to \mathbb{R}$, f^c is continuously differentiable (smooth) at $x \in \mathbb{R}^n$ with spectral values λ_1, λ_2 if and only if f is continuously differentiable (smooth) at λ_1, λ_2.*

Proof. "$\Leftarrow$" Suppose f is continuously differentiable at $x \in \mathbb{R}^n$. From Eq. (20), it can be seen that ∇f^c is continuous at every x with $x_2 \neq 0$. It remains to show that ∇f^c is continuous at every x with $x_2 = 0$. Fix any $x = (x_1, 0) \in \mathbb{R}^n$, which says $\lambda_1 = \lambda_2 = x_1$. Let $y^\nu = (y_1^\nu, y_2^\nu) \in \mathbb{R} \times \mathbb{R}^{n-1}$ be any sequence converging to x. For those $y_2^\nu = 0$, applying Eq. (23) gives $\nabla f^c(y^\nu) = f'(\lambda(y^\nu))I$. Suppose $y_2^\nu \neq 0$, from Eq. (21), we have

$$\lim_{y^\nu \to x, y_2^\nu \neq 0} a = \lim_{y^\nu \to x, y_2^\nu \neq 0} \frac{f(\lambda_2(y^\nu)) - f(\lambda_1(y^\nu))}{\lambda_2(y^\nu) - \lambda_1(y^\nu)} = f'(x_1),$$

$$\lim_{y^\nu \to x, y_2^\nu \neq 0} b = \lim_{y^\nu \to x, y_2^\nu \neq 0} \left[f'(\lambda_1(y^\nu)) \sin^2\theta + f'(\lambda_2(y^\nu)) \cos^2\theta\right] = f'(x_1),$$

$$\lim_{y^\nu \to x, y_2^\nu \neq 0} c \frac{y_2^\nu}{\|y_2^\nu\|} = \lim_{y^\nu \to x, y_2^\nu \neq 0} \sin\theta\cos\theta \left[f'(\lambda_2(y^\nu)) - f'(\lambda_1(y^\nu)) \right] \frac{y_2^\nu}{\|y_2^\nu\|} = 0,$$

$$\lim_{y^\nu \to x, y_2^\nu \neq 0} (\bar{b} - a) \frac{y_2^\nu y_2^{\nu T}}{\|y_2^\nu\|^2} = \lim_{y^\nu \to x, y_2^\nu \neq 0} \left\{\left[f'(\lambda_1(y^\nu)) \cos^2\theta + f'(\lambda_2(y^\nu)) \sin^2\theta\right] - \frac{f(\lambda_2(y^\nu)) - f(\lambda_1(y^\nu))}{\lambda_2(y^\nu) - \lambda_1(y^\nu)}\right\} \frac{y_2^\nu y_2^{\nu T}}{\|y_2^\nu\|^2} = 0.$$

Using the facts that both $\frac{y_2^\nu}{\|y_2^\nu\|}$ and $\frac{y_2^\nu y_2^{\nu T}}{\|y_2^\nu\|^2}$ are bounded by 1 and then taking the limit in (20) as $y \to x$ yield $\lim_{y \to x} \nabla f^c(y) = f'(x_1)I = \nabla f^c(x)$. This says ∇f^c is continuous at every $x \in \mathbb{R}^n$.

"$\Rightarrow$" The proof for this direction is similar to the one for [4, Prop. 5], so we omit it. □

Next, we move to the property of (locally) Lipschitz continuity. To this end, we need the following result, which is from [10, Theorem 9.67].

Lemma 3.4 *([10, Theorem 9.67]). Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is strictly continuous. Then, there exist continuously differentiable functions $f^\nu : \mathbb{R}^n \to \mathbb{R}$, $\nu = 1, 2, \ldots$, converging uniformly to f on any compact set C in $\mathbb{R}^n$ and satisfying*

$$\|\nabla f^\nu(x)\| \leq \sup_{y \in C} \operatorname{Lip} f(y) \quad \forall x \in C, \ \nu = 1, 2, 3, \ldots$$

where $\operatorname{Lip} f(x) := \limsup_{y,z \to x, y \neq z} \frac{\|f(y) - f(z)\|}{\|y - z\|}$.

Theorem 3.5. *For any $f : \mathbb{R} \to \mathbb{R}$, the following results hold:*

(a) *f^c is strictly continuous at $x \in \mathbb{R}^n$ with spectral values λ_1, λ_2 if and only if f is strictly continuous at λ_1, λ_2.*
(b) *f^c is Lipschitz continuous (with respect to $\|\cdot\|$) with constant κ if and only if f is Lipschitz continuous with constant κ.*

Proof. (a) "$\Leftarrow$" Fix any $x \in \mathbb{R}^n$ with spectral values λ_1 and λ_2 given by (6). Suppose f is strictly continuous at λ_1 and λ_2. Then, there exist $\kappa_i > 0$ and $\delta_i > 0$ for $i = 1, 2$ such that

$$|f(b) - f(a)| \leq \kappa_i |b - a|, \quad \forall a, b \in [\lambda_i - \delta_i, \lambda_i + \delta_i] \ i = 1, 2.$$

Let $\bar{\delta} := \min\{\delta_1, \delta_2\}$ and $C := [\lambda_1 - \bar{\delta}_1, \lambda_1 + \bar{\delta}] \cup [\lambda_2 - \bar{\delta}, \lambda_2 + \bar{\delta}]$. Define a real-valued function $\bar{f} : \mathbb{R} \to \mathbb{R}$ as

$$\bar{f}(a) = \begin{cases} f(a) & \text{if } a \in C, \\ (1-t)f(\lambda_1 + \bar{\delta}) & \text{if } \lambda_1 + \bar{\delta} < \lambda_2 - \bar{\delta} \text{ and, for some } t \in (0,1), \\ +tf(\lambda_2 - \bar{\delta}) & a = (1-t)(\lambda_1 + \bar{\delta}) + t(\lambda_2 - \bar{\delta}), \\ f(\lambda_1 - \bar{\delta}) & \text{if } a < \lambda_1 - \bar{\delta}, \\ f(\lambda_2 + \bar{\delta}) & \text{if } a > \lambda_2 + \bar{\delta}. \end{cases}$$

From the above, we know that $\bar{f}$ is Lipschitz continuous, which means there exists a scalar $\kappa > 0$ such that $\operatorname{Lip} \bar{f}(a) \leq \kappa$ for all $a \in \mathbb{R}$. Since C is compact, by Lemma 3.4, there exist continuously differentiable functions $f^\nu : \mathbb{R} \to \mathbb{R}$, $\nu = 1, 2, \ldots$, converging uniformly to $\bar{f}$ and satisfying

$$\left|(f^\nu)'(a)\right| \leq \kappa, \quad \forall a \in C, \ \forall \nu.$$

On the other hand, from Lemma 3.1, there exists a δ such that C contains all spectral values of $w \in B(x, \delta)$. Moreover, for any $w \in B(x, \delta)$ with spectral factorization $w = \mu_1 u^{(1)} + \mu_2 u^{(2)}$, by direct computation, we have

$$\left\|(f^\nu)^c(w) - f^c(w)\right\|^2 = \sin^2\theta |f^\nu(\mu_1) - f(\mu_2)|^2 + \cos^2\theta |f^\nu(\mu_2) - f(\mu_2)|^2.$$

This together with f^ν converging uniformly to f on C implies that $(f^\nu)^c$ converges uniformly to f^c on $B(x, \delta)$.

Next, we explain that $\|\nabla(f^\nu)^c(w)\|$ is uniformly bounded. Indeed, for $w_2 = 0$, from Eq. (23) we have $\|\nabla(f^\nu)^c(w)\| = |(f^\nu)'(w_1)| \leq \kappa$. For general $w_2 \neq 0$, it is not hard to check $\|\nabla(f^\nu)^c(w)\| \leq M$ for some uniform bound $M \geq \kappa$ on the set C by using Eq. (22).

Fix any $y, z \in B(x, \delta)$. Since $(f^\nu)^c$ converges uniformly to f^c, for any $\epsilon > 0$ there exists an integer ν_0 such that for all $\nu \geq \nu_0$ we have

$$\|(f^\nu)^c(w) - f^c(w)\| \leq \epsilon\|y - z\| \quad \forall w \in B(x, \delta).$$

Note that f^ν is continuously differentiable, Theorem 3.4 implies $(f^\nu)^c$ is also continuously differentiable. Then, by the fact that $\|\nabla(f^\nu)^c(w)\|$ is uniform bounded by M and the Mean Value Theorem for continuously differentiable functions, we obtain

$$\begin{aligned}\left\|f^c(y) - f^c(z)\right\| &= \left\|f^c(y) - (f^\nu)^c(y) + (f^\nu)^c(y) - (f^\nu)^c(z) + (f^\nu)^c(z) - f^c(z)\right\| \\ &\leq \left\|f^c(y) - (f^\nu)^c(y)\right\| + \left\|(f^\nu)^c(y) - (f^\nu)^c(z)\right\| + \left\|(f^\nu)^c(z) - f^c(z)\right\| \\ &\leq 2\epsilon\|y - z\| + \left\|\int_0^1 \nabla(f^\nu)^c(z + t(y - z))(y - z)dt\right\| \\ &\leq (M + 2\epsilon)\|y - z\|.\end{aligned}$$

This shows that f^c is strictly continuous at x.

"⇒" Suppose that f^c is strictly continuous at x with eigenvalues λ_1 and λ_2 and spectral vectors $u^{(1)}$ and $u^{(2)}$. This means there exist δ and M such that for $y, z \in B(x, \delta)$, we have

$$\left\|f^c(y) - f^c(z)\right\| \leq M\|y - z\|.$$

For any $i \in \{1, 2\}$ and any $a, b \in [\lambda_i - \delta, \lambda_i + \delta]$, denote

$$y := x + (a - \lambda_i)u^{(i)}, \qquad z := x + (b - \lambda_i)u^{(i)}.$$

Then, $\|y - x\| = |a - \lambda_i|\|u^{(i)}\| \leq \delta$ and $\|z - x\| = |b - \lambda_i|\|u^{(i)}\| \leq \delta$. Thus,

$$|f(b) - f(a)| \cdot \left\|u^{(i)}\right\| = \left\|f^c(y) - f^c(z)\right\| \leq M\|y - z\|$$

which says that f is strictly continuous at λ_1 and λ_2 because $\|u^{(1)}\| = \sin\theta$ and $\left\|u^{(2)}\right\| = \cos\theta$.

(b) This is the immediate consequence of part (a). □

4. Semismoothness property

This section is devoted to presenting a semismooth property between f and f^c. As mentioned earlier, Lemma 2.1 will be employed frequently in our analysis.

Theorem 4.1. *For any $f : \mathbb{R} \to \mathbb{R}$, f^c is semismooth at $x \in \mathbb{R}^n$ with spectral values λ_1, λ_2 if and only if f is semismooth at λ_1, λ_2.*

Proof. "⇒" Suppose f^c is semismooth, then f^c is strictly continuous and directionally differentiable. By Theorem 3.2 and Theorem 3.5, f is strictly continuous and directionally differentiable. Now, for any $\alpha \in \mathbb{R}$ and any $\eta \in \mathbb{R}$ such that f is differentiable at $\alpha + \eta$, Theorem 3.2 yields that f^c is differentiable at $x + h$, where $x := (\alpha, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $h := (\eta, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Hence, we can choose the same spectral vectors for $x + h = (\alpha + \eta, 0)$ and $x = (\alpha, 0)$ such that

$$\begin{cases} f^c(x + h) = f(\alpha + \eta)u^{(1)} + f(\alpha + \eta)u^{(2)}, \\ f^c(x) = f(\alpha)u^{(1)} + f(\alpha)u^{(2)}. \end{cases}$$

Since f^c is semismooth, by Lemma 2.1, we know

$$f^c(x + h) - f^c(x) - \nabla f^c(x + h)h = o(\|h\|). \tag{24}$$

On the other hand, Eq. (23) yields $\nabla f^c(x + h)h = f'(\alpha + \eta)Ih = \big(f'(\alpha + \eta)\eta, 0\big)$. Plugging this into Eq. (24) yields $f(\alpha + \eta) - f(\alpha) - f'(\alpha + \eta)\eta = o(|\eta|)$. Thus, by Lemma 2.1 again, it follows that f is semismooth at α. Since α is arbitrary, f is semismooth.

"⇐" Suppose f is semismooth, then f is strictly continuous and directionally differentiable. By Theorem 3.2 and Theorem 3.5, f^c is strictly continuous and directionally differentiable. For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^n$ and $h = (h_1, h_2) \in \mathbb{R} \times \mathbb{R}^n$ such that f^c is differentiable at $x + h$, we will verify that

$$f^c(x + h) - f^c(x) - \nabla f^c(x + h)h = o(\|h\|).$$

Case (i). If $x_2 \neq 0$, let λ_i be the spectral values of x and $u^{(i)}$ be the associated spectral vectors. We denote $x + h$ by z for convenience, i.e., $z := x + h$ and let m_i be the spectral values of z with the associated spectral vectors $v^{(i)}$. Hence, we have

$$\begin{cases} f^c(x) = f(\lambda_1)u^{(1)} + f(\lambda_2)u^{(2)}, \\ f^c(x + h) = f(m_1)v^{(1)} + f(m_2)v^{(2)}. \end{cases}$$

Suppose now f^c is differentiable at z. From (20), we know

$$\nabla f^c(x+h) = \begin{bmatrix} b & \dfrac{cz_2^T}{\|z_2\|} \\ \dfrac{cz_2}{\|z_2\|} & aI + (\bar{b} - a)\dfrac{z_2 z_2^T}{\|z_2\|^2} \end{bmatrix},$$

where

$$\begin{aligned} a &= \frac{f(m_2) - f(m_1)}{m_2 - m_1}, \\ b &= f'(m_1)\sin^2\theta + f'(m_2)\cos^2\theta, \\ \bar{b} &= f'(m_1)\cos^2\theta + f'(m_2)\sin^2\theta, \\ c &= \left[f'(m_2) - f'(m_1)\right]\sin\theta\cos\theta. \end{aligned}$$

With this, we can write out $f^c(x+h) - f^c(x) - \nabla f^c(x+h)h := (\Xi_1, \Xi_2)$ where $\Xi_1 \in \mathbb{R}$ and $\Xi_2 \in \mathbb{R}^{n-1}$. Since the expansion is very long, for simplicity, we denote Ξ_1 to be the first component and Ξ_2 to be the second component of the expansion. We will show that Ξ_1 and Ξ_2 are both $o(\|h\|)$. First, we compute the first component Ξ_1:

$$\begin{aligned} \Xi_1 &= \sin^2\theta\left\{f(m_1) - f(\lambda_1) - f'(m_1)\left(h_1 - \cot\theta\frac{z_2^T h_2}{\|z_2\|}\right)\right\} + \cos^2\theta\left\{f(m_2) - f(\lambda_2) - f'(m_2)\left(h_1 + \tan\theta\frac{z_2^T h_2}{\|z_2\|}\right)\right\} \\ &= \sin^2\theta\left\{f(m_1) - f(\lambda_1) - f'(m_1)\left(h_1 - \cot\theta(\|z_2\| - \|x_2\|)\right) + o(\|h\|)\right\} \\ &\quad + \cos^2\theta\left\{f(m_2) - f(\lambda_2) - f'(m_2)\left(h_1 + \tan\theta(\|z_2\| - \|x_2\|)\right) + o(\|h\|)\right\} \\ &= o\left(h_1 - (\|z_2\| - \|x_2\|)\right) + o(\|h\|) + o\left(h_1 + (\|z_2\| - \|x_2\|)\right) + o(\|h\|). \end{aligned}$$

In the above expression of Ξ_1, the third equality is obtained by the following:

$$\begin{aligned} \frac{z_2^T h_2}{\|z_2\|} &= \frac{z_2^T(z_2 - x_2)}{\|z_2\|} = \|z_2\| - \frac{\|z_2\|\|x_2\|}{\|z_2\|}\cos\alpha \\ &= \|z_2\| - \|x_2\|\left(1 + O(\alpha^2)\right) = \|z_2\| - \|x_2\|\left(1 + O(\|h\|^2)\right) \\ &= \|z_2\| - \|x_2\|\left(1 + o(\|h\|)\right) \end{aligned}$$

where α is the angle between x_2 and z_2 and note that $z_2 - x_2 = h_2$ gives $O(\alpha^2) = O(\|h\|^2)$. In addition, the last equality in the expression of Ξ_1 holds because f is semismooth and

$$m_i - \lambda_i = h_1 + (-1)^i(\tan\theta)^{(-1)^i}(\|z_2\| - \|x_2\|).$$

On the other hand, due to

$$\left|h_1 + (-1)^i(\tan\theta)^{(-1)^i}(\|z_2\| - \|x_2\|)\right| \le |h_1| + M\|z_2 - x_2\| \le M(|h_1| + \|h_2\|)$$

where $M = \max\{\tan\theta, \cot\theta\} \ge 1$, we observe that when $\|h\| \to 0$,

$$\begin{aligned} &|h_1| + (-1)^i(\tan\theta)^{(-1)^i}(\|z_2\| - \|x_2\|) \to 0 \\ &|h_1| + (-1)^i(\tan\theta)^{(-1)^i}(\|z_2\| - \|x_2\|) = O(\|h\|). \end{aligned}$$

Thus, we obtain $o\left(h_1 + (-1)^i(\tan\theta)^{(-1)^i}(\|z_2\| - \|x_2\|)\right) = o(\|h\|)$, which implies that the first component Ξ_1 is $o(\|h\|)$.

Now we consider the second component Ξ_2:

$$\begin{aligned} \Xi_2 &= -\sin\theta\cos\theta\left\{f(m_1)\frac{z_2}{\|z_2\|} - f(m_2)\frac{z_2}{\|z_2\|} - f(\lambda_1)\frac{x_2}{\|x_2\|} + f(\lambda_2)\frac{x_2}{\|x_2\|}\right\} \\ &\quad - \sin\theta\cos\theta\left(f'(m_2) - f'(m_1)\right)\frac{z_2 h_1}{\|z_2\|} - \frac{f(m_2) - f(m_1)}{m_2 - m_1}h_2 \\ &\quad - \left\{f'(m_2)\sin^2\theta - f'(m_1)\cos^2\theta - \frac{f(m_2) - f(m_1)}{m_2 - m_1}\right\}\frac{z_2 z_2^T h_2}{\|z_2\|^2} \\ &= -\sin\theta\cos\theta\left\{f(m_1)\frac{z_2}{\|z_2\|} - f(\lambda_1)\frac{x_2}{\|x_2\|} - f'(m_1)\frac{z_2 h_1}{\|z_2\|}\right. \\ &\quad \left. - \frac{f(m_1)}{\sin\theta\cos\theta(m_2 - m_1)}h_2 + \left(f'(m_1)\cot\theta + \frac{f(m_1)}{\sin\theta\cos\theta(m_2 - m_1)}\right)\frac{z_2 z_2^T h_2}{\|z_2\|^2}\right\} \end{aligned}$$

$$
\begin{aligned}
&+\sin\theta\cos\theta\left\{f(m_2)\frac{z_2}{\|z_2\|}-f(\lambda_2)\frac{x_2}{\|x_2\|}-f'(m_2)\frac{z_2h_1}{\|z_2\|}\right.\\
&\left.-\frac{f(m_2)}{\sin\theta\cos\theta(m_2-m_1)}h_2-\left(f'(m_2)\tan\theta+\frac{f(m_2)}{\sin\theta\cos\theta(m_2-m_1)}\right)\frac{z_2z_2^Th_2}{\|z_2\|^2}\right\}\\
&:=\Xi_2^{(1)}+\Xi_2^{(2)}
\end{aligned}
$$

where $\Xi_2^{(1)}$ denotes the first half part while $\Xi_2^{(2)}$ denotes the second half part. We will show that both $\Xi_2^{(1)}$ and $\Xi_2^{(2)}$ are $o(\|h\|)$. For symmetry, it is enough to show that $\Xi_2^{(1)}$ is $o(\|h\|)$. From the observations that $(m_2-m_1)\sin\theta\cos\theta=\|z_2\|$ we have the following:

$$
\begin{aligned}
\Xi_2^{(1)}&=-\sin\theta\cos\theta\left\{f(m_1)\frac{z_2}{\|z_2\|}-f(\lambda_1)\frac{x_2}{\|x_2\|}-f'(m_1)\frac{z_2h_1}{\|z_2\|}\right.\\
&\quad\left.-\frac{f(m_1)}{\sin\theta\cos\theta(m_2-m_1)}h_2+\left(f'(m_1)\cot\theta+\frac{f(m_1)}{\sin\theta\cos\theta(m_2-m_1)}\right)\frac{z_2z_2^Th_2}{\|z_2\|^2}\right\}\\
&=-\sin\theta\cos\theta\left\{\left[f(m_1)-f(\lambda_1)-f'(m_1)\left(h_1-\cot\theta\frac{z_2^Th_2}{\|z_2\|}\right)\right]\frac{z_2}{\|z_2\|}\right.\\
&\quad\left.+\left(f(m_1)-f(\lambda_1)\right)\left(\frac{-h_2}{\|z_2\|}+\frac{z_2z_2^Th_2}{\|z_2\|^3}\right)+f(\lambda_1)\left(\frac{z_2}{\|z_2\|}-\frac{x_2}{\|x_2\|}-\frac{h_2}{\|z_2\|}+\frac{z_2z_2^Th_2}{\|z_2\|^3}\right)\right\}.
\end{aligned}
$$

Following the same arguments as for the first component Ξ_1, it can be seen that

$$
f(m_1)-f(\lambda_1)-f'(m_1)\left(h_1-\cot\theta\frac{z_2^Th_2}{\|z_2\|}\right)=o(\|h\|).
$$

Since $m_1-\lambda_1=h_1-\cot\theta(\|z_2\|-\|x_2\|)=O(\|h\|)$ and f is strictly continuous, it follows that $f(m_1)-f(\lambda_1)=O(\|h\|)$. In addition, $-h_2/\|z_2\|+z_2z_2^Th_2/\|z_2\|^3=O(\|h\|)$. Hence,

$$
(f(m_1)-f(\lambda_1))\left(\frac{-h_2}{\|z_2\|}+\frac{z_2z_2^Th_2}{\|z_2\|^3}\right)=O(\|h\|^2)=o(\|h\|).
$$

Therefore, it remains to prove that the last part of $\Xi_2^{(1)}$ is $o(\|h\|)$. Indeed, with $z_2=x_2+h_2$, we have

$$
\frac{z_2}{\|z_2\|}-\frac{x_2}{\|x_2\|}-\frac{h_2}{\|z_2\|}+\frac{z_2z_2^Th_2}{\|z_2\|^3}=x_2\left(\frac{1}{\|z_2\|}-\frac{1}{\|x_2\|}+\frac{z_2^Th_2}{\|z_2\|^3}\right)+O(\|h\|^2).
$$

Let $\theta(z_2):=-1/\|z_2\|$, then we compute $\nabla\theta(z_2)=-\frac{-1}{\|z_2\|^2}\frac{z_2}{\|z_2\|}=\frac{z_2}{\|z_2\|^3}$ which implies

$$
\frac{1}{\|z_2\|}-\frac{1}{\|x_2\|}+\frac{z_2^Th_2}{\|z_2\|^3}=\theta(x_2)-\theta(z_2)-\nabla\theta(z_2)(x_2-z_2)=O(\|h\|^2),
$$

where the last equality is from first Taylor approximation. Thus, we obtain

$$
f(\lambda_1)\left(\frac{z_2}{\|z_2\|}-\frac{x_2}{\|x_2\|}-\frac{h_2}{\|z_2\|}+\frac{z_2z_2^Th_2}{\|z_2\|^3}\right)=o(\|h\|).
$$

From all the above, we therefore verified that (24) is satisfied, which says f^c is semismooth under case (i).

Case (ii). If $x_2=0$, we need to discuss two subcases. First subcase, consider $h_2\neq 0$. Then, $x=(x_1,0)$ and $x+h=(x_1+h_1,h_2)$. We can choose $u^{(1)}=\left(\sin^2\theta,-\sin\theta\cos\theta\frac{h_2}{\|h_2\|}\right)$ and $u^{(2)}=\left(\sin^2\theta,\sin\theta\cos\theta\frac{h_2}{\|h_2\|}\right)$ such that $x=\lambda u^{(1)}+\lambda u^{(2)}$ and $x+h=m_1u^{(1)}+m_2u^{(2)}$ with $\lambda=x_1$ and $m_i=x_1+h_1+(-1)^i(\tan\theta)^{(-1)^i}\|h_2\|$, $i=1,2$. Hence,

$$
\begin{cases}
f^c(x)=f(x_1)u^{(1)}+f(x_1)u^{(2)},\\
f^c(x+h)=f(m_1)u^{(1)}+f(m_2)u^{(2)}.
\end{cases}
$$

Beside, from Eq. (22), we know

$$
\begin{aligned}
\nabla f^c(x+h)h&=f'(m_1)\left(h_1-\cot\theta\frac{h_2^Th_2}{\|h_2\|}\right)u^{(1)}+f'(m_2)\left(h_1+\tan\theta\frac{h_2^Th_2}{\|h_2\|}\right)u^{(2)}\\
&\quad+\frac{f(m_2)-f(m_1)}{m_2-m_1}\left(I-\frac{h_2h_2^T}{\|h_2\|^2}\right)h_2.
\end{aligned}
$$

Note that $h_2^T h_2 = \|h_2\|^2$, hence $\left(I - \frac{h_2 h_2^T}{\|h_2\|^2}\right) h_2 = 0$. Therefore, we have

$$\nabla f^c(x+h)h = f'(m_1)(h_1 - \cot\theta\|h_2\|)u^{(1)} + f'(m_2)(h_1 + \tan\theta\|h_2\|)u^{(2)}.$$

Combining all of these, we obtain

$$f^{soc}(x+h) - f^{soc}(x) - \nabla f^{soc}(x+h)h = \left\{f(m_1) - f(x_1) - f'(m_1)(h_1 - \cot\theta\|h_2\|)\right\} u^{(1)} + \left\{f(m_2) - f(x_1) - f'(m_2)(h_1 + \tan\theta\|h_2\|)\right\} u^{(2)}.$$

Since f is semismooth at x_1 and $m_i - x_1 = h_1 + (-1)^i(\tan\theta)^{(-1)^i}\|h_2\|$, we have $f(m_i) - f(x_1) - f'(m_i)(h_1 + (-1)^i (\tan\theta)^{(-1)^i}\|h_2\|) = o(\|h\|)$. With $u^{(1)}$ and $u^{(2)}$ being uniformly bounded, the above expression implies that (24) is satisfied. Hence, f^c is semismooth under this subcase.

Second, for the subcase of $h_2 = 0$, we know $x = (x_1, 0)$ and $x + h = (x_1 + h_1, 0)$. We can choose $u^{(1)} = (\sin^2\theta, -\sin\theta\cos\theta w)$ and $u^{(2)} = \left(\cos^2\theta, \sin\theta\cos\theta w\right)$ with $\|w\| = 1$ such that $x = \lambda u^{(1)} + \lambda u^{(2)}$ and $x + h = mu^{(1)} + mu^{(2)}$, where $\lambda = x_1$ and $m = x_1 + h_1$. Hence,

$$\begin{cases} f^c(x) &= f(x_1)u^{(1)} + f(x_1)u^{(2)} \\ f^c(x+h) &= f(x_1+h_1)u^{(1)} + f(x_1+h_1)u^{(2)}. \end{cases}$$

In addition, (23) says $\nabla f^c(x+h) = f'(x_1+h_1)I$, and hence $\nabla f^c(x+h)h = (f'(x_1+h_1)h_1, 0)$. Combining all of these, we obtain

$$\begin{aligned} f^c(x+h) - f^c(x) - \nabla f^{soc}(x+h)h &= \left\{f(x_1+h_1)u^{(1)} + f(x_1+h_1)u^{(2)}\right\} \\ &\quad - \left\{f(x_1)u^{(1)} + f(x_1)u^{(2)}\right\} - \left(f'(x_1+h_1)h_1, 0\right) \\ &= \left(f(x_1+h_1) - f(x_1) - f'(x_1+h_1)h_1, 0\right) = (o(|h_1|), 0) \end{aligned}$$

where the third equality holds since f is semismooth and by Lemma 2.1. When h goes to zero, it implies h_1 goes to zero, so the above expression implies that (24) is satisfied which says f^c is semismooth in this subcase. From all the above, we proved that if f is semismooth then f^c is semismooth. □

5. Conclusion

In this paper, we have proved the following results of vector-valued functions associated with the circular cone, which are useful for designing and analyzing smooth and nonsmooth methods for solving circular cone problems.

(a) f^c is continuous at $x \in \mathbb{R}^n$ with spectral values λ_1, λ_2 if and only if f is continuous at λ_1, λ_2.
(b) f^c is directionally differentiable at $x \in \mathbb{R}^n$ with spectral values λ_1, λ_2 if and only if f is directionally differentiable at λ_1, λ_2.
(c) f^c is differentiable at $x \in \mathbb{R}^n$ with spectral values λ_1, λ_2 if and only if f is differentiable at λ_1, λ_2.
(d) f^c is continuously differentiable at $x \in \mathbb{R}^n$ with spectral values λ_1, λ_2 if and only if f is continuously differentiable at λ_1, λ_2.
(e) f^c is strictly continuous at $x \in \mathbb{R}^n$ with spectral values λ_1, λ_2 if and only if f is strictly continuous at λ_1, λ_2.
(f) f^c is Lipschitz continuous (with respect to $\|\cdot\|$) with constant κ if and only if f is Lipschitz continuous with constant κ.
(g) f^c is semismooth at an $x \in \mathbb{R}^n$ with spectral values λ_1, λ_2 if and only if f is semismooth λ_1, λ_2.

Our proofs involve more algebraic computations in general. Nonetheless, our proofs come from the straightforward, intuitive thinking and basic definitions as well as the simple structure of the circular cone. We believe that the intuitive way we presented here would be helpful for analysis of other merit functions used for solving circular cone problems. That is one of our future research interests.

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