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ORIGINAL PAPER

# The same growth of FB and NR symmetric cone complementarity functions

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**Abstract** We establish that the Fischer–Burmeister (FB) complementarity function and the natural residual (NR) complementarity function associated with the symmetric cone have the same growth, in terms of the classification of Euclidean Jordan algebras. This, on the one hand, provides an affirmative answer to the second open question proposed by Tseng (J Optim Theory Appl 89:17–37, 1996) for the matrix-valued FB and NR complementarity functions, and on the other hand, extends the third important inequality of Lemma 3.1 in the aforementioned paper to the setting of Euclidean Jordan algebras. It is worthwhile to point out that the proof is surprisingly simple.

**Keywords** Symmetric cone · FB and NR complementarity functions · Growth

## 1 Introduction

Let $\mathbb{A} = (\mathbb{V}, \circ, \langle\cdot,\cdot\rangle_{\mathbb{V}})$ be a Euclidean Jordan algebra (see Sect. 2 for details). Let $\mathcal{K}$ be the set of all squares in $\mathbb{V}$. Given the continuously differentiable mappings $F, G : \mathbb{V} \to \mathbb{V}$, we consider the symmetric cone complementarity problem (SCCP): *to*

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*find a vector* $\zeta \in \mathbb{V}$ *such that*

$$F(\zeta) \in \mathcal{K}, \quad G(\zeta) \in \mathcal{K}, \quad \langle F(\zeta), G(\zeta) \rangle_{\mathbb{V}} = 0. \tag{1}$$

This class of problems provides a unified framework for the classical nonlinear programming and complementarity problem [5] over the nonnegative orthant cone in $\mathbb{R}^n$, the second-order cone optimization and complementarity problem [1], and the semidefinite programming and complementarity problem [16,21], and becomes one of main research interests in the current optimization field; see, e.g., [4,6,10,12, 13,17–19,22].

Analogous to the three classes of special SCCPs above, the complementarity function associated with the symmetric cone plays a crucial role in the development of merit function methods and smoothing (nonsmooth) Newton methods for solving the SCCPs. Recall that $\phi : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ is called *a symmetric cone complementary function* if it satisfies the following equivalence:

$$\phi(x, y) = 0 \iff x \in \mathcal{K}, \quad y \in \mathcal{K}, \quad \langle x, y \rangle_{\mathbb{V}} = 0. \tag{2}$$

With such a function, the SCCP can be reformulated as an unconstrained minimization

$$\min_{\zeta \in \mathbb{V}} \Psi(\zeta) := \frac{1}{2} \|\phi(F(\zeta), G(\zeta))\|_{\mathbb{V}}^2, \tag{3}$$

in the sense that $\zeta^*$ solves (1) if and only if it is a solution of (3) with zero optimal value, where $\| \cdot \|_{\mathbb{V}}$ denotes the norm induced by the inner product $\langle \cdot, \cdot \rangle_{\mathbb{V}}$. If $\Psi$ is continuously differentiable, then the efficient unconstrained minimization methods can be applied for (3) to yield a solution of (1). This method is often known as the merit function approach.

Two most popular choices for $\phi$ are the natural residual (NR) symmetric cone complementarity function $\phi_{\rm NR}$ and the Fischer–Burmeister (FB) symmetric cone complementarity function $\phi_{\rm FB}$, respectively, defined as

$$\phi_{\rm NR}(x, y) := x - (x - y)_+ \quad \forall x, y \in \mathbb{V} \tag{4}$$

and

$$\phi_{\rm FB}(x, y) := (x + y) - (x^2 + y^2)^{1/2} \quad \forall x, y \in \mathbb{V}, \tag{5}$$

where $z_+$ means the metric projection of $z \in \mathbb{V}$ onto the symmetric cone $\mathcal{K}$, $x^2 = x \circ x$ denotes the Jordan product of $x$ and itself, and $x^{1/2}$ means the unique square root of $x \in \mathcal{K}$, i.e., $(x^{1/2})^2 = x$. The squared norm of $\phi_{\rm FB}$ induces a smooth merit function with global Lipschitz continuous gradients (see [9,15]). This implies that finding solutions to (1) is equivalent to seeking solutions of the unconstrained smooth minimization problem

$$\min_{\zeta \in \mathbb{V}} \Psi_{\rm FB}(\zeta) := \frac{1}{2} \left\|\phi_{\rm FB}(F(\zeta), G(\zeta))\right\|_{\mathbb{V}}^2. \tag{6}$$

However, in order to establish the convergence rate of the merit function method for the SCCPs based on (6), the key is to prove that $\phi_{\text{FB}}$ and $\phi_{\text{NR}}$ has the same order of growth, i.e., to show that there exist constants $c_1 > 0$ and $c_2 > 0$ such that for all $x, y \in \mathbb{V}$,

$$c_1\|\phi_{\text{NR}}(x, y)\|_{\mathbb{V}}^2 \le \|\phi_{\text{FB}}(x, y)\|_{\mathbb{V}}^2 \le c_2\|\phi_{\text{NR}}(x, y)\|_{\mathbb{V}}^2. \tag{7}$$

When $\mathbb{A}$ is the Euclidean Jordan algebra $\mathbb{R}$ equipped with the multiplication of real numbers, Tseng showed in [20, Lemma 3.1] that inequality (7) holds with $c_1 = 2-\sqrt{2}$ and $c_2 = 2+\sqrt{2}$; when $\mathbb{A}$ is the Jordan spin algebra $\mathscr{L}_n$ (see Example 2.3 in the next section), Pan et al. [14] recently established inequality (7) by contradiction. We note that for the case where $\mathbb{A}$ is the $n \times n$ real symmetric matrix algebra (see Example 2.2 in the next section), in 1998 Tseng [21] proposed an open question "whether the FB function $\left\|\phi_{\text{FB}}(F(\zeta), G(\zeta))\right\|_{\mathbb{V}}^2$ is bounded above and below by a constant multiple of the NR function $\left\|\phi_{\text{NR}}(F(\zeta), G(\zeta))\right\|_{\mathbb{V}}^2$", which is equivalent to asking whether or not inequality (7) holds under this case. To our best knowledge, until now this open question is not resolved.

In this paper, we show that (7) holds with $c_1 = 2 - \sqrt{2}$ and $c_2 = 2 + \sqrt{2}$, which does not only offer an affirmative answer to the open question of [21], but also extends the results of [20, Lemma 3.1] and [14] to the setting of symmetric cones. Particularly, the proof is surprisingly simpler than that of [20, Lemma 3.1] and [14]. As a direct consequence of (7), we also establish the global error bound property of the FB merit function for SCCPs.

## 2 Preliminaries

This section recalls some results on Euclidean Jordan algebras that will be used in the next section. More detailed expositions of Euclidean Jordan algebras can be found in the monograph by Faraut and Korányi [3] and Koecher's lecture notes [7].

*A Euclidean Jordan algebra* is a triple $(\mathbb{V}, \circ, \langle\cdot,\cdot\rangle_{\mathbb{V}})$ where $(\mathbb{V}, \langle\cdot,\cdot\rangle_{\mathbb{V}})$ is a finite dimensional inner product space over the real number field $\mathbb{R}$ and $(x, y) \mapsto x \circ y : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ is a bilinear mapping satisfying the following three conditions:

(i) $x \circ y = y \circ x$ for all $x, y \in \mathbb{V}$;
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathbb{V}$, where $x^2 = x \circ x$;
(iii) $\langle x \circ y, z\rangle_{\mathbb{V}} = \langle y, x \circ z\rangle_{\mathbb{V}}$ for all $x, y, z \in \mathbb{V}$.

Henceforth, we assume that $\mathbb{A} = (\mathbb{V}, \circ, \langle\cdot,\cdot\rangle_{\mathbb{V}})$ is a Euclidean Jordan algebra with an element $e \in \mathbb{V}$ (called the *unit* element) such that $x \circ e = x$ for all $x \in \mathbb{V}$. By [3, Theorem III. 2.1], the set of squares $\mathcal{K} := \{x^2 \mid x \in \mathbb{V}\}$ is a symmetric cone. In the following, we present three common examples of Euclidean Jordan algebras.

*Example 2.1* Consider $\mathbb{R}^n$ with the (usual) inner product and Jordan product defined respectively as

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i \quad \text{and} \quad x \circ y = x * y \quad \forall x, y \in \mathbb{R}^n$$

where $x_i$ denotes the $i$th component of $x$, etc., and $x * y$ denotes the component-wise product of vectors $x$ and $y$. Then, $\mathbb{R}^n$ is a Euclidean Jordan algebra with the nonnegative orthant $\mathbb{R}^n_+$ as its cone of squares.

*Example 2.2* **The algebra $\mathscr{S}_n$ of $n \times n$ real symmetric matrices.** Let $\mathbb{S}^{n\times n}$ be the space of all $n \times n$ real symmetric matrices with the trace inner product and Jordan product, respectively, defined by

$$\langle X, Y\rangle_{\mathrm{T}} := \mathrm{Tr}(XY) \;\text{ and }\; X \circ Y := \frac{1}{2}(XY + YX) \quad \forall X, Y \in \mathbb{S}^{n\times n}.$$

Then, $(\mathbb{S}^{n\times n}, \circ, \langle\cdot,\cdot\rangle_{\mathrm{T}})$ is a Euclidean Jordan algebra, and we write it as $\mathscr{S}_n$. The cone of squares $\mathbb{S}^{n\times n}_+$ in $\mathscr{S}_n$ is the set of all positive semidefinite matrices in $\mathbb{S}^{n\times n}$.

*Example 2.3* **The Jordan spin algebra $\mathscr{L}_n$.** Consider $\mathbb{R}^n$ $(n > 1)$ with the inner product $\langle\cdot,\cdot\rangle$ and Jordan product

$$x \circ y := \begin{bmatrix} \langle x, y\rangle \\ x_0\bar{y} + y_0\bar{x} \end{bmatrix}$$

for any $x = (x_0; \bar{x})$, $y = (y_0; \bar{y}) \in \mathbb{R}\times\mathbb{R}^{n-1}$. We denote the Euclidean Jordan algebra $(\mathbb{R}^n, \circ, \langle\cdot,\cdot\rangle)$ by $\mathscr{L}_n$. The cone of squares, called the Lorentz cone (or the second-order cone), is given by $\mathscr{L}_n^+ := \{(x_0; \bar{x}) \in \mathbb{R}\times\mathbb{R}^{n-1} \mid x_0 \geq \|\bar{x}\|\}$.

For $x \in \mathbb{V}$, let $m(x) := \min\{k : \{e, x, x^2, \ldots, x^k\} \text{ are linearly dependent}\}$ and define the *rank* of $\mathbb{A}$ by $r := \max\{m(x) : x \in \mathbb{V}\}$. Recall that an element $c \in \mathbb{V}$ is *idempotent* if $c^2 = c$, and it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. One says that a finite set $\{c_1, c_2, \ldots, c_k\}$ of primitive idempotents in $\mathbb{V}$ is a *Jordan frame* if

$$c_j \circ c_i = 0 \;\text{ if } j \neq i \text{ for all } j, i = 1, 2, \ldots, k, \quad \text{and} \quad \textstyle\sum_{j=1}^k c_j = e.$$

Now we may state the second version of the spectral decomposition theorem.

**Theorem 2.1** ([3, Theorem III.1.2]) *Let $\mathbb{A}$ be a Euclidean Jordan algebra with rank $r$. Then, for every $x \in \mathbb{V}$, there exist a Jordan frame $\{c_1, c_2, \ldots, c_r\}$ and real numbers $\lambda_1(x), \lambda_2(x), \ldots, \lambda_r(x)$, arranged in the decreasing order $\lambda_1(x) \geq \cdots \geq \lambda_r(x)$, such that*

$$x = \lambda_1(x)c_1 + \lambda_2(x)c_2 + \cdots + \lambda_r(x)c_r.$$

*The numbers $\lambda_j(x)$ (counting multiplicities), which are uniquely determined by $x$, are called the eigenvalues of $x$, and* $\mathrm{tr}(x) = \sum_{j=1}^r \lambda_j(x)$ *is called the trace of $x$.*

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a scalar valued function. Then, it is natural to define a vector valued function $\phi_{\mathbb{V}} : \mathbb{V} \to \mathbb{V}$ associated with the Euclidean Jordan algebra $\mathbb{A}$ [8,19] by

$$\phi_{\mathbb{V}}(x) := \phi(\lambda_1(x))c_1 + \phi(\lambda_2(x))c_2 + \cdots + \phi(\lambda_r(x))c_r,$$

where $x \in \mathbb{V}$ has the spectral decomposition $x = \sum_{j=1}^{r} \lambda_j(x)c_j$. This function is also called *Löwner's operator* in recognition of Löwner's contribution. When $\phi(t) = t_+ := \max\{0, t\}$ for $t \in \mathbb{R}$, $\phi_{\mathbb{V}}(x)$ becomes the metric projector operator over $\mathcal{K}$:

$$x_+ = (\lambda_1(x))_+ c_1 + (\lambda_1(x))_+ c_2 + \cdots + (\lambda_r(x))_+ c_r \quad \forall x \in \mathbb{V};$$

while $\phi(t) = t_- := \min\{0, t\}$ for $t \in \mathbb{R}$, it is the metric projector operator over $-\mathcal{K}$

$$x_- = (\lambda_1(x))_- c_1 + (\lambda_1(x))_- c_2 + \cdots + (\lambda_r(x))_- c_r \quad \forall x \in \mathbb{V}.$$

In the sequel, we let $|x|$ be Löwner's operator induced by $\phi(t) = |t|$ for $t \in \mathbb{R}$. Then,

$$|x| = x_+ - x_- = 2x_+ - x = x - 2x_- \quad \forall x \in \mathbb{V}. \tag{8}$$

Recall that a Euclidean Jordan algebra is said to be *simple* if it is not the direct sum of two Euclidean Jordan algebras. It is easy to see that $\mathscr{S}_n$ and $\mathscr{L}_n$ are simple Euclidean Jordan algebras, whereas the Euclidean Jordan algebra in Example 2.1 is not simple. Let $\mathbb{H}^{n\times n}$ denote the space of $n \times n$ complex Hermitian matrices, $\mathbb{Q}^{n\times n}$ the space of $n \times n$ quaternion Hermitian matrices, and $\mathbb{O}^{3\times 3}$ the space of $3 \times 3$ octonion Hermitian matrices.

**Theorem 2.2** ([3, Theorem V.3.7]) *Suppose that* $\mathbb{A} = (\mathbb{V}, \circ, \langle\cdot,\cdot\rangle_{\mathbb{V}})$ *is a simple Euclidean Jordan algebra of rank* $r \geq 3$. *Then,* $\mathbb{A}$ *is isomorphic to one of the following*

(i) *The algebra* $\mathscr{S}_n$ *of* $n \times n$ *real symmetric matrices given by Example 2.2;*
(ii) *The algebra* $\mathscr{H}_n$ *of all* $n \times n$ *complex Hermitian matrices with trace inner product* $\langle x, y\rangle_{\mathrm{T}} := \Re\mathrm{Tr}(xy^*)$ *and Jordan product* $x \circ y := \frac{1}{2}(xy + yx)$ *for any* $x, y \in \mathbb{H}^{n\times n}$*;*
(iii) *The algebra* $\mathscr{Q}_n$ *of all* $n \times n$ *quaternionic Hermitian matrices with trace inner product* $\langle x, y\rangle_{\mathrm{T}} := \Re\mathrm{Tr}(xy^*)$ *and Jordan product* $x \circ y := \frac{1}{2}(xy + yx)$ *for any* $x, y \in \mathbb{Q}^{n\times n}$*;*
(iv) *The algebra* $\mathscr{O}_3$ *of all* $3 \times 3$ *octonionic Hermitian matrices with trace inner product* $\langle x, y\rangle_{\mathrm{T}} := \Re\mathrm{Tr}(xy^*)$ *and Jordan product* $x \circ y := \frac{1}{2}(xy + yx)$ *for any* $x, y \in \mathbb{O}^{3\times 3}$*;*
(v) *The Jordan spin algebra* $\mathscr{L}_n$ *given by Example 2.3.*

*where the notation "$*$" means the conjugate transpose,* $\mathrm{Tr}(xy)$ *denotes the trace of* $xy$ *which is the multiplication of matrices* $x$ *and* $y$*, and* $\Re a$ *means the real part of* $a$.

Unless otherwise stated, in the rest of this paper, we assume that $\mathbb{A} = (\mathbb{V}, \circ, \langle\cdot,\cdot\rangle_{\mathbb{V}})$ is a simple Euclidean Jordan algebra, and denote $\|\cdot\|_{\mathbb{V}}$, $\|\cdot\|$ and $\|\cdot\|_{\mathrm{T}}$ the norm induced by the inner product $\langle\cdot,\cdot\rangle_{\mathbb{V}}$, $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{\mathrm{T}}$, respectively. We also write $x \succeq_{\mathcal{K}} y$ (respectively, $x \succ_{\mathcal{K}} y$) to mean $x - y \in \mathcal{K}$ (respectively, $x - y \in \mathrm{int}\mathcal{K}$).

## 3 Main result

To establish the main result of this paper, the following lemma plays an important role.

**Lemma 3.1** (a) *For any* $x, y \in \mathbb{V}$, *if* $x \succeq_{\mathcal{K}} 0$, $y \succeq_{\mathcal{K}} 0$ *and* $x \succeq_{\mathcal{K}} y$, *then* $x^{1/2} \succeq_{\mathcal{K}} y^{1/2}$.
(b) *For any* $u, v, w \in \mathbb{V}$, *if* $w \succeq_{\mathcal{K}} 0$ *and* $2w^2 = u^2 + v^2$, *then there holds that*

$$w \succeq_{\mathcal{K}} \frac{1}{2}(u+v).$$

*Proof* (a) This is result of [6, Prop. 8], which is also implied by [8].
(b) Since $u^2 + v^2 - 2u \circ v = (u-v) \circ (u-v) \in \mathcal{K}$, using $2w^2 = u^2 + v^2$ yields

$$w^2 = \frac{1}{2}(u^2+v^2) \succeq_{\mathcal{K}} \frac{1}{4}(u^2+v^2) + \frac{1}{2} u \circ v = \frac{1}{4}(u+v)^2.$$

From part(a) and $w \succeq_{\mathcal{K}} 0$, this implies that $w \succeq_{\mathcal{K}} \frac{1}{2}|u+v| \succeq_{\mathcal{K}} \frac{1}{2}(u+v)$. □

**Proposition 3.1** *Let* $\mathscr{L}_n$ *be the Euclidean Jordan algebra in Example* 2.3. *Then,*

$$(2-\sqrt{2})\|\phi_{\text{NR}}(x,y)\| \le \|\phi_{\text{FB}}(x,y)\| \le (2+\sqrt{2})\|\phi_{\text{NR}}(x,y)\|, \quad \forall x, y \in \mathscr{L}_n.$$

*Proof* Fix any $x, y \in \mathbb{V}$. If $\phi_{\text{NR}}(x,y) = 0$, then we also have $\phi_{\text{FB}}(x,y) = 0$, and the desired result is immediate. Therefore, in the following arguments we assume that $\phi_{\text{NR}}(x,y) \neq 0$. Using Eq. (8) and the definition of $\phi_{\text{NR}}$, it is not hard to see that

$$\phi_{\text{NR}}(x,y) \;=\; \frac{1}{2}\left[(x+y) - |x-y|\right].$$

This together with the definition of $\phi_{\text{FB}}$ gives

$$\begin{aligned} \phi_{\text{FB}}(x,y) &= 2\phi_{\text{NR}}(x,y) + |x-y| - (x^2+y^2)^{1/2} \\ &= 2\phi_{\text{NR}}(x,y) + z(x,y) \end{aligned} \tag{9}$$

where $z(x,y) \equiv |x-y| - (x^2+y^2)^{1/2}$. By Eq. (9) and the triangle inequality, it suffices to argue $\|z(x,y)\| \le \sqrt{2}\|\phi_{\text{NR}}(x,y)\|$, that is,

$$\|z(x,y)\|^2 \le \frac{1}{2}\left\|x+y-|x-y|\right\|^2. \tag{10}$$

Substituting the expression of $z(x, y)$ into (10), we obtain that (10) is equivalent to

$$\begin{aligned}
&\left\| |x-y| - (x^2+y^2)^{1/2} \right\|^2 \leq \frac{1}{2}\, \|x+y-|x-y|\|^2 \\
&\Longleftrightarrow \|x-y\|^2 - 2\langle |x-y|, (x^2+y^2)^{1/2}\rangle + \|(x^2+y^2)^{1/2}\|^2 \\
&\quad \leq \frac{1}{2}\left[\|x+y\|^2 + \|x-y\|^2 - 2\langle |x-y|, x+y\rangle\right] \\
&\Longleftrightarrow \|c(x,y)\|^2 - \langle 2c(x,y) - (x+y), |x-y|\rangle - 2\langle x, y\rangle \leq 0 \qquad (11)
\end{aligned}$$

where $c(x, y) \equiv (x^2 + y^2)^{1/2}$. Thus, to prove the desired result, it suffices to argue that inequality (11) holds. Indeed, since

$$\|c(x,y)\|^2 = \langle c(x,y)^2, e\rangle = \langle x^2+y^2, e\rangle = \|x\|^2 + \|y\|^2,$$

we have

$$\begin{aligned}
&\|c(x,y)\|^2 - \langle 2c(x,y) - (x+y), |x-y|\rangle - 2\langle x, y\rangle \\
&\quad = \|x-y\|^2 - \langle 2c(x,y) - (x+y), |x-y|\rangle \\
&\quad = \langle |x-y|, -2c(x,y) + (x+y) + |x-y|\rangle. \qquad (12)
\end{aligned}$$

Applying Lemma 3.1 with $w = c(x, y)$, $u = (x + y)$ and $v = |x - y|$, we know

$$-2c(x,y) + (x+y) + |x-y| \in -\mathscr{L}_n^+.$$

This, along with $|x - y| \in \mathscr{L}_n^+$ and Eq. (12), shows that inequality (11) holds. □

**Proposition 3.2** *Suppose that $\mathbb{A} = (\mathbb{V}, \circ, \langle\cdot,\cdot\rangle_{\mathbb{V}})$ is a simple Euclidean Jordan algebra with the rank $r \geq 3$. Then, it holds that*

$$(2-\sqrt{2})\|\phi_{\rm NR}(x,y)\|_{\rm T} \leq \|\phi_{\rm FB}(x,y)\|_{\rm T} \leq (2+\sqrt{2})\|\phi_{\rm NR}(x,y)\|_{\rm T} \quad \forall x, y \in \mathbb{V}.$$

*Proof* By Theorem 2.2, it suffices to prove that the desired result holds for $\mathbb{A} = \mathscr{S}_n$, or $\mathscr{H}_n$, or $\mathscr{Q}_n$, or $\mathscr{O}_3$. Fix any $x, y \in \mathbb{V}$ with $\mathbb{V} = \mathbb{S}^{n\times n}$, or $\mathbb{H}^{n\times n}$, or $\mathbb{Q}^{n\times n}$, or $\mathbb{O}^{3\times 3}$. If $\phi_{\rm NR}(x, y) = 0$, the result is direct. Thus, it suffices to consider the case of $\phi_{\rm NR}(x, y) \neq 0$. Note that for the simple Euclidean Jordan algebra $\mathscr{S}_n$, or $\mathscr{H}_n$, or $\mathscr{Q}_n$, or $\mathscr{O}_3$, we still have

$$\phi_{\rm FB}(x,y) \;=\; 2\phi_{\rm NR}(x,y) + z(x,y)$$

with $z(x, y) \equiv |x - y| - (x^2 + y^2)^{1/2}$. By the triangle inequality, it suffices to prove

$$\|z(x,y)\|_{\rm T}^2 \leq 2\|\phi_{\rm NR}(x,y)\|_T^2 = \frac{1}{2}\, \|x+y-|x-y|\|_{\rm T}^2. \qquad (13)$$

Using the definition of $\|\cdot\|_{\mathrm{T}}^2$ and noting that $\Re\mathrm{Tr}(uv^*) = \Re\mathrm{Tr}(uv) = \Re\mathrm{Tr}(vu)$ for all $u, v \in \mathbb{V}$, an elementary computation yields that (13) is equivalent to

$$\begin{aligned}
&\Re\mathrm{Tr}\left[(x-y)^2 + c(x,y)^2 - 2|x-y|c(x,y)\right]\\
&\quad\le \frac{1}{2}\Re\mathrm{Tr}\left[(x+y)^2 + (x-y)^2 - 2|x-y|(x+y)\right]\\
&\Longleftrightarrow \Re\mathrm{Tr}\left[-|x-y|(2c(x,y)-(x+y)) + (x-y)^2\right] \le 0\\
&\Longleftrightarrow \frac{1}{2}\Re\mathrm{Tr}\left[|x-y|\left(c(x,y) - \frac{(x+y)+|x-y|}{2}\right)\right] \ge 0\\
&\Longleftrightarrow \frac{1}{2}\left\langle |x-y|,\ c(x,y) - \frac{(x+y)+|x-y|}{2}\right\rangle_{\mathrm{T}} \ge 0
\end{aligned} \tag{14}$$

where $c(x, y) \equiv (x^2 + y^2)^{1/2}$. Applying Lemma 3.1 with $w = c(x, y)$, $u = (x + y)$ and $v = |x - y|$ yields that $c(x, y) - ((x + y) + |x - y|)/2 \succeq_{\mathcal{K}} 0$. This together with $|x - y| \succeq_{\mathcal{K}} 0$ implies that inequality (14) holds. Thus, we complete the proof. □

Combining Propositions 3.1 with 3.2, we readily obtain the main result of this paper.

**Theorem 3.1** *Suppose that $\mathbb{A} = (\mathbb{V}, \circ, \langle\cdot,\cdot\rangle_{\mathbb{V}})$ is a simple Euclidean Jordan algebra. Let $\phi_{\mathrm{NR}}$ and $\phi_{\mathrm{FB}}$ be defined by* (4) *and* (5)*, respectively. Then, it holds that*

$$(2-\sqrt{2})\|\phi_{\mathrm{NR}}(x,y)\|_{\mathbb{V}} \le \|\phi_{\mathrm{FB}}(x,y)\|_{\mathbb{V}} \le (2+\sqrt{2})\|\phi_{\mathrm{NR}}(x,y)\|_{\mathbb{V}} \quad \forall x, y \in \mathbb{V}.$$

By Theorem 3.1, we may establish the global error bound property for the FB merit function of SCCPs under the jointly uniform Cartesian $P$-property of $F$ and $G$. To this end, we next assume that $\mathbb{A}$ is a direct product of simple Euclidean Jordan algebras:

$$\mathbb{A} = \mathbb{A}_1 \times \mathbb{A}_2 \times \cdots \times \mathbb{A}_m,$$

where each $\mathbb{A}_i = (\mathbb{V}_i, \circ, \langle\cdot,\cdot\rangle_{\mathbb{V}_i})$ is a simple Euclidean Jordan algebra with $\sum_{i=1}^m \dim(\mathbb{V}_i) = \dim(\mathbb{V})$. Then, $\mathcal{K} = \mathcal{K}^1 \times \mathcal{K}^2 \times \cdots \times \mathcal{K}^m$ with $\mathcal{K}^i$ being a symmetric cone in $\mathbb{V}_i$. For any $x, y \in \mathbb{V}$, we write $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m)$ with $x_i, y_i \in \mathbb{V}_i$. Then,

$$x \circ y = (x_1 \circ y_1, \ldots, x_m \circ y_m) \ \text{ and } \ \langle x, y\rangle_{\mathbb{V}} = \langle x_1, y_1\rangle_{\mathbb{V}_1} + \cdots + \langle x_m, y_m\rangle_{\mathbb{V}_m}.$$

Consequently, the SCCP (1) is equivalent to finding a vector $\zeta \in \mathbb{V}$ such that

$$F_i(\zeta) \in \mathcal{K}^i, \quad G_i(\zeta) \in \mathcal{K}^i, \quad \langle F_i(\zeta), G_i(\zeta)\rangle_{\mathbb{V}_i} = 0, \quad i = 1, 2, \ldots, m \tag{15}$$

where $F = (F_1, \ldots, F_m)$ and $G = (G_1, \ldots, G_m)$ with $F_i, G_i : \mathbb{V} \to \mathbb{V}_i$.

**Definition 3.1** [2] The mappings $F$ and $G$ are said to have the *jointly uniform Cartesian $P$-property* if there exists a constant $\rho > 0$ such that for any $\zeta, \xi \in \mathbb{V}$, there is an index $\nu \in \{1, \ldots, m\}$ such that

$$\langle F_\nu(\zeta) - F_\nu(\xi), G_\nu(\zeta) - G_\nu(\xi)\rangle_{\mathbb{V}_\nu} \geq \rho \|\zeta - \xi\|_{\mathbb{V}}^2.$$

**Theorem 3.2** *Suppose that $F$ and $G$ have the jointly uniform Cartesian $P$-property and are globally Lipschitz continuous with constants $L_1 > 0$ and $L_2 > 0$, respectively. If the SCCP (1) has an optimal solution, say $\zeta^*$, then*

$$\frac{2-\sqrt{2}}{(2L_1+L_2)^2}\Psi_{\rm FB}(\zeta) \leq \|\zeta - \zeta^*\|_{\mathbb{V}}^2 \leq \frac{(2+\sqrt{2})(L_1+L_2)^2}{\rho^2}\Psi_{\rm FB}(\zeta) \quad \forall \zeta \in \mathbb{V}$$

*where the constant $\rho$ is same as in Definition 3.1.*

*Proof* Fix any $\zeta \in \mathbb{V}$. Let $R(\zeta) \equiv \big(\phi_{\rm NR}(F_1(\zeta), G_1(\zeta)), \ldots, \phi_{\rm NR}(F_m(\zeta), G_m(\zeta))\big) \in \mathbb{V}$. Then, using Theorem 3.1 and noting that $\Psi_{\rm FB}(\zeta) \equiv \frac{1}{2}\sum_{i=1}^m \|\phi_{\rm FB}(F_i(\zeta), G_i(\zeta))\|_{\mathbb{V}_i}^2$, we get

$$\frac{2-\sqrt{2}}{2}\|R(\zeta)\|_{\mathbb{V}}^2 \leq \Psi_{\rm FB}(\zeta) \leq \frac{2+\sqrt{2}}{2}\|R(\zeta)\|_{\mathbb{V}}^2.$$

In addition, using the same arguments as in [9, Theorem 6.3], we have

$$\frac{1}{2L_1+L_2}\|R(\zeta)\|_{\mathbb{V}} \leq \|\zeta - \zeta^*\|_{\mathbb{V}} \leq \frac{L_1+L_2}{\rho}\|R(\zeta)\|_{\mathbb{V}}.$$

From the last two inequalities, we immediately obtain the desired result. □

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## References

1. Chen, J.-S., Tseng, P.: An unconstrained smooth minimization reformulation of the second-order cone complementarity problem. Math. Program. **104**, 293–327 (2005)
2. Chen, X., Qi, H.D.: Cartesian $P$-property and its applications to the semidefinite linear complementarity problem. Math. Program. **106**, 177–201 (2006)
3. Faraut, J., Korányi, A.: Analysis on symmetric cones. In: Oxford Mathematical Monographs. Oxford University Press, New York (1994)
4. Faybusovich, L.: Euclidean Jordan algebras and interior point algorithms. Positivity **1**, 331–357 (1997)
5. Facchinei, F., Pang, J.-S.: Finite-Dimensional Variational Inequalities and Complementarity Problems, vols. I and II. Springer, New York (2003)
6. Gowda, M.S., Sznajder, R., Tao, J.: Some P-properties for linear transformations on Euclidean Jordan algebras. Linear Algebra Appl. **393**, 203–232 (2004)
7. Koecher, M.: The Minnesota Notes on Jordan Algebras and Their Applications. Edited and annotated by A. Brieg and S. Walcher. Springer, Berlin (1999)

8. Korányi, A.: Monotone functions on formally real Jordan algebras. Math. Ann. **269**, 73–76 (1984)
9. Kong, L.C., Xiu, N.H.: The Fischer-Burmeister complementarity function on Euclidean Jordan algebras. Pac. J. Optim. (2009, published online)
10. Kong, L.C., Sun, J., Xiu, N.H.: A regularized smoothing Newton method for symmetric cone complementarity problems. SIAM J. Optim. **19**, 1028–1047 (2008)
11. Kong, L.C., Tuncel, L., Xiu, N.H.: Vector-valued implicit Lagrangian for symmetric cone complementarity problems. Asia-Pac. J. Oper. Res. **26**, 199–233 (2009)
12. Li, Y.M., Wang, X.T., Wei, D.Y.: A new class of complementarity functions for symmetric cone complementarity problems. Optim. Lett. (2010, to appear)
13. Liu, Y., Zhang, L., Wang, Y.: Some propeties of a class of merit functions for symmetric cone complementarity problems. Asia-Pac. J. Oper. Res. **23**, 473–496 (2006)
14. Pan, S.-H., Chen, J.-S., Li, J.-F.: The same growth of FB-SOC merit function and NR-SOC merit function. Technical report (2009)
15. Pan, S.-H., Chen, J.-S.: A one-parametric class of merit functions for the symmetric cone complementarity problem. J. Math. Anal. Appl. **355**, 195–215 (2009)
16. Pardalos, P.M., Ramana, M. : Semidefinite programming. In: Terlaky, T. (ed.) Interior Point Methods of Mathematical Programming, pp. 369–398. Kluwer, Dordrecht (1996)
17. Qin, L.-X., Kong, L.C., Han, J.-Y.: Sufficiency of linear transformations on Euclidean Jordan algebras. Optim. Lett. **3**, 265–276 (2009)
18. Schmieta, S.H., Alizadeh, F.: Extension of primal-dual interior point algorithms for symmetric cones. Math. Program. **96**, 409–438 (2003)
19. Sun, D., Sun, J.: Löwner's operator and spectral functions on Euclidean Jordan algebras. Math. Oper. Res. **33**, 421–445 (2008)
20. Tseng, P.: Growth behavior of a class of merit functions for the nonlinear complementarity problems. J. Optim. Theory Appl. **89**, 17–37 (1996)
21. Tseng, P.: Merit function for semidefinite complementarity problems. Math. Program. **83**, 159–185 (1998)
22. Yoshise, A.: Interior point trajectories and a homogenous model for nonlinear complementarity problems over symmetric cones. SIAM J. Optim. **17**, 1129–1153 (2006)