Journal of Nonlinear and Convex Analysis
Volume 18, Number 9, 2017, 1699–1705

# PROJECTION FORMULA AND ONE TYPE OF SPECTRAL FACTORIZATION ASSOCIATED WITH $p$-ORDER CONE

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Abstract. In this short paper, we establish the projection formula associated with $p$-order cone and further discover one type of spectral factorization associated with $p$-order cone. These expressions will be key bricks for further analysis and study about $p$-order cone optimization.

## 1. Introduction

Recently, there has been much attention on symmetric cone optimization, see [5, 12, 13, 15, 16] and references therein, but not much on non-symmetric cone optimization. In general, non-symmetric cones include $p$-order cone [1, 17], circular cone [3, 7, 18], $L^p$ cone [10], and copositive cone [8], etc. Unlike symmetric cone case in which the Euclidean Jordan algebra can unify the whole analysis, there has not been found a special unified Jordan algebra for non-symmetric cones until now. Nonetheless, analogous to tackling symmetric cone optimization, in which the spectral decomposition [9] plays a key role, we believe that in order to find out a way to deal with non-symmetric cone optimization problems, the first key step is to figure out their corresponding projection formulae and spectral factorization.

A good spectral factorization, like the eigenvalue decomposition in linear algebra, provides an efficient way for computer software to compute some special function, for instance, projection function. Moreover, the efficiency of computing projection formulae can help on designing some algorithms for solving non-symmetric cone optimization problems, for example, the so-called projection gradient method and merit function method, and so on. For circular cone case, its corresponding projection formula and spectral factorization are studied in [18]. However, there are no further investigations for other non-symmetric cone cases yet. In this paper, we characterize the projection formula of element $\mathbf{z}$ onto $p$-order cone, and establish one type of spectral factorization associated with $p$-order cone. We believe that these expressions are key bricks for further analysis and study about $p$-order cone optimization.

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2010 *Mathematics Subject Classification.* 49J52, 90C33, 65K05, 26B05.

*Key words and phrases.* $p$-Order cone, projection, spectral factorization.

*The author's work is supported by National Natural Science Foundation of China (No. 11471241).

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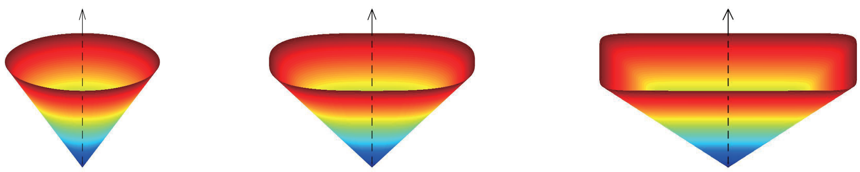
The $p$-order cone in $\mathbb{R}^n$, which is a generalization of the second-order cone [4,6], is defined as

$$\mathcal{K}_p := \left\{ \mathbf{x} \in \mathbb{R}^n \,\middle|\, x_1 \geq \left( \sum_{i=2}^{n} |x_i|^p \right)^{\frac{1}{p}} \right\} \quad (p > 1). \tag{1.1}$$

If we write $\mathbf{x} := (x_1, \mathbf{x_2}) \in \mathbb{R} \times \mathbb{R}^{(n-1)}$, the $p$-order cone $\mathcal{K}_p$ can be equivalently expressed as

$$\mathcal{K}_p = \left\{ \mathbf{x} = (x_1, \mathbf{x_2}) \in \mathbb{R} \times \mathbb{R}^{(n-1)} \,|\, x_1 \geq \|\mathbf{x_2}\|_p \right\}, \quad (p > 1).$$

The pictures of three different cones $\mathcal{K}_p$ in $\mathbb{R}^3$ are depicted in Figure 1.



(a) 2-order cone (b) 3-order cone (c) 10-order cone

FIGURE 1. Three different $p$-order cones in $\mathbb{R}^3$.

From (1.1) and Figure 1, it is clear to see that when $p = 2$, $\mathcal{K}_2$ is exactly the second-order cone $\mathcal{K}^n = \left\{ \mathbf{x} = (x_1, \mathbf{x_2}) \in \mathbb{R} \times \mathbb{R}^{(n-1)} \,|\, x_1 \geq \|\mathbf{x_2}\| \right\}$, which confirms that the second-order cone is a special case of $p$-order cone.

It is well known that $\mathcal{K}_p$ is a convex cone and its dual cone is given by

$$\mathcal{K}_p^* = \left\{ \mathbf{y} \in \mathbb{R}^n \,\middle|\, y_1 \geq \left( \sum_{i=2}^{n} |y_i|^q \right)^{\frac{1}{q}} \right\}$$

or equivalently

$$\mathcal{K}_p^* = \left\{ \mathbf{y} = (y_1, \mathbf{y_2}) \in \mathbb{R} \times \mathbb{R}^{(n-1)} \,|\, y_1 \geq \|\mathbf{y_2}\|_q \right\} = \mathcal{K}_q,$$

where $q > 1$ and satisfies $\frac{1}{p} + \frac{1}{q} = 1$. In addition, the dual cone $\mathcal{K}_p^*$ is also a convex cone.

For an application of $p$-order cone programming, we refer the readers to [17], in which a primal-dual potential reduction algorithm for $p$-order cone constrained optimization problems is studied. Besides, in [17], a special optimization problem called sum of $p$-norms is transformed into an $p$-order cone constrained optimization problems.

To end this section, we say a few words about the notations used in this paper. We consider the Euclidean space $\mathbb{R}^n$ equipped with the standard inner product

$\langle \cdot, \cdot \rangle$. The Euclidean norm is defined as $\|\mathbf{z}\| := \sqrt{\langle \mathbf{z}, \mathbf{z} \rangle}$. Let $\mathcal{K}$ be any closed convex cone. We denote its dual cone by

$$\mathcal{K}^* = \{\mathbf{y} \mid \langle \mathbf{y}, \mathbf{x} \rangle \geq 0 \ \ \forall \mathbf{x} \in \mathcal{K}\},$$

and denote its polar cone by

$$\mathcal{K}^\circ = \{\mathbf{y} \mid \langle \mathbf{y}, \mathbf{x} \rangle \leq 0 \ \ \forall \mathbf{x} \in \mathcal{K}\}.$$

Moreover, $\partial\mathcal{K}$ means the boundary of $\mathcal{K}$ and $\Pi_{\mathcal{K}}(\mathbf{z})$ is the projection of $\mathbf{z}$ onto $\mathcal{K}$.

## 2. Projection formula and spectral factorization

In [18], we see that the spectral factorization associated with circular cone is figured out first and then the projection onto circular cone is characterized. For the $p$-order cone case, the procedure is totally opposite. More specifically, we need to characterize the projection onto such cone, and then figure out its corresponding spectral factorization. In particular, one type of spectral factorization associated with $p$-order cone are provided.

First, we start with the general Orthogonal Projection Theorem associated with any closed convex cone in Hilbert space (see [14, Theorem II.3]). The Orthogonal Projection Theorem is also known in the optimization community as the Moreau Decomposition(see [11]), which says for any $\mathbf{z} \in \mathbb{R}^n$, $\mathbf{z}$ can be decomposed as

$$\mathbf{z} = \Pi_{\mathcal{K}}(\mathbf{z}) + \Pi_{\mathcal{K}^\circ}(\mathbf{z}) = \Pi_{\mathcal{K}}(\mathbf{z}) + \Pi_{-\mathcal{K}^*}(\mathbf{z}) \tag{2.1}$$

where $\mathcal{K}$ is any closed convex cone with polar cone $\mathcal{K}^\circ$ and dual cone $\mathcal{K}^*$. When $\mathcal{K}$ represents the special structure of the $p$-order cone $\mathcal{K}_p$, the explicit expression (2.1) is characterized in following theorem.

**Theorem 2.1.** *Let $\mathbf{z} = (z_1, \mathbf{z_2}) \in \mathbb{R} \times \mathbb{R}^{(n-1)}$. Then, the projection of $\mathbf{z}$ onto $\mathcal{K}_p$ is given by*

$$\Pi_{\mathcal{K}_p}(\mathbf{z}) = \begin{cases} \mathbf{z}, & \mathbf{z} \in \mathcal{K}_p \\ \mathbf{0}, & \mathbf{z} \in -\mathcal{K}_p^* = -\mathcal{K}_q \\ \mathbf{u}, & \textit{otherwise} \ \ (\textit{i.e.}, -\|\mathbf{z_2}\|_q < z_1 < \|\mathbf{z_2}\|_p) \end{cases} \tag{2.2}$$

*where $\mathbf{u} = (u_1, \bar{\mathbf{u}})$ with $\bar{\mathbf{u}} = (u_2, u_3, \cdots, u_n)^T \in \mathbb{R}^{(n-1)}$ satisfying*

$$u_1 = \|\bar{\mathbf{u}}\|_p = (|u_2|^p + |u_3|^p + \cdots + |u_n|^p)^{\frac{1}{p}}$$

*and*

$$u_i - z_i + \frac{u_1 - z_1}{u_1^{p-1}} |u_i|^{p-2} u_i = 0, \qquad \forall i = 2, \cdots, n.$$

*Proof.* From Projection Theorem [2, Prop. 2.2.1], we know that, for every $\mathbf{z} \in \mathbb{R}^n$, a vector $\mathbf{u} \in \mathcal{K}_p$ is equal to the projection point $\Pi_{\mathcal{K}_p}(\mathbf{z})$ if and only if

$$\mathbf{u} \in \mathcal{K}_p, \quad \mathbf{z} - \mathbf{u} \in \mathcal{K}_p^\circ \ \text{ and } \ \langle \mathbf{z} - \mathbf{u}, \mathbf{u} \rangle = 0.$$

With this, the first two cases of (2.2) are obvious. Hence, we only need to consider the third case. Based on the expression of the element $\mathbf{u}$, it is easy to verify that $\mathbf{u} \in \partial\mathcal{K}_p$. Moreover, we have

$$\mathbf{z} - \mathbf{u} = \begin{bmatrix} z_1 - u_1 \\ \mathbf{z_2} - \bar{\mathbf{u}} \end{bmatrix} := \begin{bmatrix} z_1 - u_1 \\ \bar{\mathbf{h}} \end{bmatrix},$$

where $\bar{\mathbf{h}} = (h_2, h_3, \cdots, h_n)^T$ with

$$h_i = \frac{u_1 - z_1}{u_1^{p-1}} |u_i|^{p-2} u_i, \qquad \forall i = 2, \cdots, n.$$

Noting that

$$\begin{aligned} \|\bar{\mathbf{h}}\|_q &= \left| \frac{u_1 - z_1}{u_1^{p-1}} \right| \left( \sum_{i=2}^n |u_i|^{(p-1)q} \right)^{\frac{1}{q}} \\ &= \left| \frac{u_1 - z_1}{u_1^{p-1}} \right| \left( \|\bar{\mathbf{u}}\|_p^p \right)^{\frac{1}{q}} \\ &= |u_1 - z_1|, \end{aligned} \tag{2.3}$$

where the second equality holds due to $\frac{1}{p} + \frac{1}{q} = 1$, and the last equality holds because of $\|\bar{\mathbf{u}}\|_p = u_1$. Noting that

$$\begin{aligned} \langle \mathbf{z} - \mathbf{u}, \mathbf{u} \rangle &= (z_1 - u_1)u_1 + \langle \bar{\mathbf{h}}, \bar{\mathbf{u}} \rangle \\ &= (z_1 - u_1)u_1 + \frac{u_1 - z_1}{u_1^{p-1}} \left( \sum_{i=2}^n |u_i|^p \right) \\ &= (z_1 - u_1)u_1 + \frac{u_1 - z_1}{u_1^{p-1}} \|\bar{\mathbf{u}}\|_p^p \\ &= (z_1 - u_1)u_1 + (u_1 - z_1)u_1 \\ &= 0, \end{aligned}$$

On the other hand,

$$\begin{aligned} \langle \mathbf{z} - \mathbf{u}, \mathbf{u} \rangle &= (z_1 - u_1)z_1 + \langle \bar{\mathbf{h}}, \bar{\mathbf{u}} \rangle \\ &= (z_1 - u_1)u_1 - \|\bar{\mathbf{h}}\|_q \|\bar{\mathbf{u}}\|_p \\ &= (z_1 - u_1)u_1 - |z_1 - u_1| \|\bar{\mathbf{u}}\|_p \\ &= ((z_1 - u_1) - |z_1 - u_1|) \|\bar{\mathbf{u}}\|_p, \end{aligned}$$

where the second equality holds due to the equal case of Hölder inequality, This implies that $(z_1 - u_1) - |z_1 - u_1| = 0$. Hence, we have $z_1 - u_1 < 0$. Together with (2.3) again, this leads to $\|\bar{\mathbf{h}}\|_q = u_1 - z_1$, which implies $\mathbf{z} - \mathbf{u} \in \mathcal{K}_p^\circ$. Hence, the desired result is obtained. Furthermore, the projection of $\mathbf{z}$ onto $\mathcal{K}_p$ is expressed as in (2.2). □

In the sequel, for the sake of simplicity, we denote $\mathbf{z}^+ := \Pi_{\mathcal{K}_p}(\mathbf{z})$. Moreover, because $\mathcal{K}_p^\circ = -\mathcal{K}_p^* = -\mathcal{K}_q$, we know

$$\Pi_{-\mathcal{K}_p^*}(\mathbf{z}) = \Pi_{-\mathcal{K}_q}(\mathbf{z}) = -\Pi_{\mathcal{K}_q}(-\mathbf{z}).$$

This together with (2.2) and the proof of Theorem 2.1 gives

$$\mathbf{z}^- := -\Pi_{\mathcal{K}_q}(-\mathbf{z}) = \begin{cases} \mathbf{z}, & -\mathbf{z} \in \mathcal{K}_q \\ \mathbf{0}, & -\mathbf{z} \in -\mathcal{K}_q^* = -\mathcal{K}_p \\ \mathbf{v}, & \text{otherwise} \ \ (i.e., -\|\mathbf{z_2}\|_p < -z_1 < \|\mathbf{z_2}\|_q) \end{cases} \tag{2.4}$$

where $\mathbf{v} = (v_1, \bar{\mathbf{v}})$ with $\bar{\mathbf{v}} = (v_2, v_3, \cdots, v_n)^T \in \mathbb{R}^{(n-1)}$ satisfying

$$-v_1 = \|\bar{\mathbf{v}}\|_q = (|v_2|^q + |v_3|^q + \cdots + |v_n|^q)^{\frac{1}{q}}$$

and

$$v_i - z_i - (-1)^{q-1}\frac{v_1 - z_1}{v_1^{q-1}}|v_i|^{q-2}v_i = 0, \qquad \forall i = 2, \cdots, n.$$

By the definition of $\mathbf{z}^+$ and $\mathbf{z}^-$, it follows that $\langle \mathbf{z}^+, \mathbf{z}^- \rangle = 0$. Together the expression of $\mathbf{u}$ in (2.2) with the expression of $\mathbf{v}$ in (2.4) again, we obtain

$$\begin{cases} v_1 = z_1 - u_1 \\ v_i = z_i - u_i = \dfrac{u_1 - z_1}{u_1^{q-1}}|u_i|^{p-2}u_i, \qquad \forall i = 2, 3, \cdots, n. \end{cases} \tag{2.5}$$

**Remark 2.2.** *Unfortunately, from the formula (2.2) in Theorem 2.1 and the formula (2.4), we can not obtain the spectral factorization for* $\mathbf{z} = (z_1, \mathbf{z_2}) \in \mathbb{R} \times \mathbb{R}^{(n-1)}$. *This is different from the case of second-order cone. In order to get the goal, we develop one type of factorization for* $\mathbf{z}$ *as below. Such factorization is called the* **spectral factorization**.

**Theorem 2.3** (Spectral factorization)**.** *Let* $\mathbf{z} = (z_1, \mathbf{z_2}) \in \mathbb{R} \times \mathbb{R}^{(n-1)}$. *Then,* $\mathbf{z}$ *can be decomposed as*

$$\mathbf{z} = \alpha_1(\mathbf{z}) \cdot \mathbf{v}^{(1)}(\mathbf{z}) + \alpha_2(\mathbf{z}) \cdot \mathbf{v}^{(2)}(\mathbf{z}),$$

*where*

$$\begin{cases} \alpha_1(\mathbf{z}) &= \dfrac{z_1 + \|\mathbf{z_2}\|_p}{2} \\ \alpha_2(\mathbf{z}) &= \dfrac{z_1 - \|\mathbf{z_2}\|_p}{2} \end{cases}$$

*and*

$$\begin{cases} \mathbf{v}^{(1)}(\mathbf{z}) &= \begin{bmatrix} 1 \\ \mathbf{w_2} \end{bmatrix} \\ \mathbf{v}^{(2)}(\mathbf{z}) &= \begin{bmatrix} 1 \\ -\mathbf{w_2} \end{bmatrix} \end{cases}$$

*with* $\mathbf{w_2} = \frac{\mathbf{z_2}}{\|\mathbf{z_2}\|_p}$ *when* $\mathbf{z_2} \neq \mathbf{0}$*; while* $\mathbf{w_2}$ *being an arbitrary element satisfying* $\|\mathbf{w_2}\|_p = 1$ *when* $\mathbf{z_2} = \mathbf{0}$.

*Proof.* For $\mathbf{z_2} \neq \mathbf{0}$, we define $\widetilde{\mathbf{u}}(\mathbf{z}) := \begin{bmatrix} \tau\|\mathbf{z_2}\|_p \\ \tau\mathbf{z_2} \end{bmatrix} \in \partial\mathcal{K}_p$ such that $\widetilde{\mathbf{u}}(\mathbf{z}) - \mathbf{z} \in \partial\mathcal{K}_p$, where $\tau$ is an undetermined coefficient. From $\widetilde{\mathbf{u}}(\mathbf{z}) - \mathbf{z} \in \mathcal{K}_p$, we have

$$\tau\|\mathbf{z_2}\|_p - z_1 = \|(\tau - 1)\mathbf{z_2}\|_p$$

which yields

$$\tau = \frac{z_1 + \|\mathbf{z_2}\|_p}{2\|\mathbf{z_2}\|_p}.$$

This further implies

$$\widetilde{\mathbf{u}}(\mathbf{z}) = \begin{bmatrix} \left(\dfrac{z_1 + \|\mathbf{z_2}\|_p}{2\|\mathbf{z_2}\|_p}\right)\|\mathbf{z_2}\|_p \\ \left(\dfrac{z_1 + \|\mathbf{z_2}\|_p}{2\|\mathbf{z_2}\|_p}\right)\mathbf{z_2} \end{bmatrix}.$$

Therefore, we can rewrite $\mathbf{z}$ as

$$\begin{aligned} \mathbf{z} &= \widetilde{\mathbf{u}}(\mathbf{z}) + (\mathbf{z} - \widetilde{\mathbf{u}}(\mathbf{z})) \\ &= \begin{bmatrix} \left(\dfrac{z_1 + \|\mathbf{z_2}\|_p}{2\|\mathbf{z_2}\|_p}\right) \|\mathbf{z_2}\|_p \\ \left(\dfrac{z_1 + \|\mathbf{z_2}\|_p}{2\|\mathbf{z_2}\|_p}\right) \mathbf{z_2} \end{bmatrix} + \begin{bmatrix} \left(\dfrac{z_1 - \|\mathbf{z_2}\|_p}{2\|\mathbf{z_2}\|_p}\right) \|\mathbf{z_2}\|_p \\ \left(\dfrac{\|\mathbf{z_2}\|_p - z_1}{2\|\mathbf{z_2}\|_p}\right) \mathbf{z_2} \end{bmatrix} \\ &= \left(\dfrac{z_1 + \|\mathbf{z_2}\|_p}{2}\right) \begin{bmatrix} 1 \\ \frac{\mathbf{z_2}}{\|\mathbf{z_2}\|_p} \end{bmatrix} + \left(\dfrac{z_1 - \|\mathbf{z_2}\|_p}{2}\right) \begin{bmatrix} 1 \\ -\frac{\mathbf{z_2}}{\|\mathbf{z_2}\|_p} \end{bmatrix} \\ &:= \alpha_1(\mathbf{z}) \cdot \mathbf{v}^{(1)}(\mathbf{z}) + \alpha_2(\mathbf{z}) \cdot \mathbf{v}^{(2)}(\mathbf{z}) \end{aligned}$$

which gives the desired spectral factorization.
For $\mathbf{z_2} = \mathbf{0}$, it is easy to verify that $\mathbf{z} = \alpha_1(\mathbf{z}) \cdot \mathbf{v}^{(1)}(\mathbf{z}) + \alpha_2(\mathbf{z}) \cdot \mathbf{v}^{(2)}(\mathbf{z})$ with

$$\mathbf{v}^{(1)}(\mathbf{z}) = \begin{bmatrix} 1 \\ \mathbf{w_2} \end{bmatrix} \quad \text{and} \quad \mathbf{v}^{(2)}(\mathbf{z}) = \begin{bmatrix} 1 \\ -\mathbf{w_2} \end{bmatrix},$$

where $\mathbf{w_2}$ is an arbitrary element satisfying $\|\mathbf{w_2}\|_p = 1$. Then, the desired factorization holds. □

**Remark 2.4.** *Theorem 2.3 can be proved by verifying the equality directly. Nonetheless, we provide the constructive way to show how to obtain* $\mathbf{v}^1(\mathbf{z})$, $\mathbf{v}^2(\mathbf{z})$ *and* $\alpha_1(\mathbf{z})$, $\alpha_2(\mathbf{z})$. *Moreover, from Theorem 2.3, we also know that* $\alpha_1(\mathbf{z}) \geq \alpha_2(\mathbf{z})$.

As a consequence of Theorem 2.3 and Remark 2.4, we have the following corollary.

**Corollary 2.5.** *Let* $\mathbf{z} = \alpha_1(\mathbf{z}) \cdot \mathbf{v}^{(1)}(\mathbf{z}) + \alpha_2(\mathbf{z}) \cdot \mathbf{v}^{(2)}(\mathbf{z})$ *be the spectral factorization of type II for* $\mathbf{z}$ *given as in Theorem 2.3. Then,* $\mathbf{v}^{(i)}(\mathbf{z}) \in \mathcal{K}_p$ *for* $i = 1, 2$. *Moreover, the following hold*

$$\mathbf{z} \in \mathcal{K}_p \iff \alpha_2(\mathbf{z}) \geq 0.$$

## 3. Concluding remarks

In this short paper, we have characterized the projection formula of any element $\mathbf{z}$ onto $p$-order cone, and have established one type of spectral factorization associated with $p$-order cone. As mentioned, this expression will be key bricks for further analysis and study about $p$-order cone optimization.

One may ask what the advantages and disadvantages of the spectral factorization are? To answer this question, we say a few words for this point. The advantage of the spectral factorization is that the vectors $\mathbf{v}^{(i)}(\mathbf{z})$ $(i = 1, 2)$ both lie in $\mathcal{K}_p$, which implies that any $\mathbf{z}$ in $\mathbb{R}^n$ can be expressed by two vectors in $p$-order cone $\mathcal{K}_p$. However, to the contrast, this factorization for $\mathbf{z}$ is not an orthogonal decomposition, which is different from the case in the second-order cone setting.

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*Manuscript received April 1, 2016*

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