

## EXAMPLES OF $r$ -CONVEX FUNCTIONS AND CHARACTERIZATIONS OF $r$ -CONVEX FUNCTIONS ASSOCIATED WITH SECOND-ORDER CONE

CHIEN-HAO HUANG, HONG-LIN HUANG, AND JEIN-SHAN CHEN\*

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ABSTRACT. In this paper, we revisit the concept of  $r$ -convex functions which were studied in 1970s. We present several novel examples of  $r$ -convex functions that are new to the existing literature. In particular, for any given  $r$ , we show examples which are  $r$ -convex functions. In addition, we extend the concepts of  $r$ -convexity and quasi-convexity to the setting associated with second-order cone. Characterizations about such new functions are established. These generalizations will be useful in dealing with optimization problems involved in second-order cones.

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### 1. INTRODUCTION

It is known that the concept of convexity plays a central role in many applications including mathematical economics, engineering, management science, and optimization theory. Moreover, much attention has been paid to its generalization, to the associated generalization of the results previously developed for the classical convexity, and to the discovery of necessary and/or sufficient conditions for a function to have generalized convexities. Some of the known extensions are quasiconvex functions,  $r$ -convex functions [1, 24], and so-called SOC-convex functions [7, 8]. Other further extensions can be found in [19, 23]. For a single variable continuous, the midpoint-convex function on  $\mathbb{R}$  is also a convex function. This result was generalized in [22] by relaxing continuity to lower-semicontinuity and replacing the number  $\frac{1}{2}$  with an arbitrary parameter  $\alpha \in (0, 1)$ . An analogous consequence was obtained in [18, 23] for quasiconvex functions.

To understand the main idea behind  $r$ -convex function, we recall some concepts that were independently defined by Martos [17] and Avriel [2], and has been studied by the latter author. Indeed, this concept relies on the classical definition of convex functions and some well-known results from analysis dealing with weighted means of positive numbers. Let  $w = (w_1, \dots, w_m) \in \mathbb{R}^m$ ,  $q = (q_1, \dots, q_m) \in \mathbb{R}^m$  be vectors

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\*Corresponding author. The author's work is supported by Ministry of Science and Technology, Taiwan.

whose components are positive and nonnegative numbers, respectively, such that  $\sum_{i=1}^m q_i = 1$ . Given the vector of weights  $q$ , the *weighted  $r$ -mean* of the numbers  $w_1, \dots, w_m$  is defined as below (see [13]):

$$(1.1) \quad M_r(w; q) = M_r(w_1, \dots, w_m; q) := \begin{cases} \left( \sum_{i=1}^m q_i (w_i)^r \right)^{1/r} & \text{if } r \neq 0, \\ \prod_{i=1}^m (w_i)^{q_i} & \text{if } r = 0. \end{cases}$$

It is well-known from [13] that for  $s > r$ , there holds

$$(1.2) \quad M_s(w_1, \dots, w_m; q) \geq M_r(w_1, \dots, w_m; q)$$

for all  $q_1, \dots, q_m \geq 0$  with  $\sum_{i=1}^m q_i = 1$ . The  $r$ -convexity is built based on the aforementioned weighted  $r$ -mean. For a convex set  $S \subseteq \mathbb{R}^n$ , a real-valued function  $f : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be  $r$ -convex if, for any  $x, y \in S$ ,  $\lambda \in [0, 1]$ ,  $q_2 = \lambda$ ,  $q_1 = 1 - q_2$ ,  $q = (q_1, q_2)$ , there has

$$f(q_1 x + q_2 y) \leq \ln \left\{ M_r(e^{f(x)}, e^{f(y)}; q) \right\}.$$

From (1.1), it can be verified that the above inequality is equivalent to

$$(1.3) \quad f((1 - \lambda)x + \lambda y) \leq \begin{cases} \ln[(1 - \lambda)e^{rf(x)} + \lambda e^{rf(y)}]^{1/r} & \text{if } r \neq 0, \\ (1 - \lambda)f(x) + \lambda f(y) & \text{if } r = 0. \end{cases}$$

Similarly,  $f$  is said to be  $r$ -concave on  $S$  if the inequality (1.3) is reversed. It is clear from the above definition that a real-valued function is convex (concave) if and only if it is 0-convex (0-concave). Besides, for  $r < 0$  ( $r > 0$ ), an  $r$ -convex ( $r$ -concave) function is called *superconvex* (*superconcave*); while for  $r > 0$  ( $r < 0$ ), it is called *subconvex* (*subconcave*). In addition, it can be verified that the  $r$ -convexity of  $f$  on  $C$  with  $r > 0$  ( $r < 0$ ) is equivalent to the convexity (concavity) of  $e^{rf}$  on  $S$ .

A function  $f : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be *quasiconvex* on  $S$  if, for all  $x, y \in S$ ,

$$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}, \quad 0 \leq \lambda \leq 1.$$

Analogously,  $f$  is said to be *quasiconcave* on  $S$  if, for all  $x, y \in S$ ,

$$f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}, \quad 0 \leq \lambda \leq 1.$$

From [13], we know that

$$\begin{aligned} \lim_{r \rightarrow +\infty} M_r(w_1, \dots, w_m; q) &\equiv M_\infty(w_1, \dots, w_m) = \max\{w_1, \dots, w_m\}, \\ \lim_{r \rightarrow -\infty} M_r(w_1, \dots, w_m; q) &\equiv M_{-\infty}(w_1, \dots, w_m) = \min\{w_1, \dots, w_m\}. \end{aligned}$$

Then, it follows from (1.2) that  $M_\infty(w_1, \dots, w_m) \geq M_r(w_1, \dots, w_m; q) \geq M_{-\infty}(w_1, \dots, w_m)$  for every real number  $r$ . Thus, if  $f$  is  $r$ -convex on  $S$ , it is also  $(+\infty)$ -convex, that is,  $f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}$  for every  $x, y \in S$  and  $\lambda \in [0, 1]$ . Similarly, if  $f$  is  $r$ -concave on  $S$ , it is also  $(-\infty)$ -concave, i.e.,  $f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}$ .

The following review some basic properties regarding  $r$ -convex function from [1] that will be used in the subsequent analysis.

**Property 1.1.** Let  $f : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ . Then, the followings hold.

- (a) If  $f$  is  $r$ -convex ( $r$ -concave) on  $S$ , then  $f$  is also  $s$ -convex ( $s$ -concave) on  $S$  for  $s > r$  ( $s < r$ ).
- (b) Suppose that  $f$  is twice continuously differentiable on  $S$ . For any  $(x, r) \in S \times \mathbb{R}$ , we define

$$\phi(x, r) = \nabla^2 f(x) + r \nabla f(x) \nabla f(x)^T.$$

Then,  $f$  is  $r$ -convex on  $S$  if and only if  $\phi$  is positive semidefinite for all  $x \in S$ .

- (c) Every  $r$ -convex ( $r$ -concave) function on a convex set  $S$  is also quasiconvex (quasiconcave) on  $S$ .
- (d)  $f$  is  $r$ -convex if and only if  $(-f)$  is  $(-r)$ -concave.
- (e) Let  $f$  be  $r$ -convex ( $r$ -concave),  $\alpha \in \mathbb{R}$  and  $k > 0$ . Then  $f + \alpha$  is  $r$ -convex ( $r$ -concave) and  $k \cdot f$  is  $(\frac{r}{k})$ -convex ( $(\frac{r}{k})$ -concave).
- (f) Let  $\phi, \psi : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  be  $r$ -convex ( $r$ -concave) and  $\alpha_1, \alpha_2 > 0$ . Then, the function  $\theta$  defined by

$$\theta(x) = \begin{cases} \ln [\alpha_1 e^{r\phi(x)} + \alpha_2 e^{r\psi(x)}]^{1/r} & \text{if } r \neq 0, \\ \alpha_1 \phi(x) + \alpha_2 \psi(x) & \text{if } r = 0, \end{cases}$$

is also  $r$ -convex ( $r$ -concave).

- (g) Let  $\phi : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  be  $r$ -convex ( $r$ -concave) such that  $r \leq 0$  ( $r \geq 0$ ) and let the real valued function  $\psi$  be nondecreasing  $s$ -convex ( $s$ -concave) on  $\mathbb{R}$  with  $s \in \mathbb{R}$ . Then, the composite function  $\theta = \psi \circ \phi$  is also  $s$ -convex ( $s$ -concave).
- (h)  $\phi : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is  $r$ -convex ( $r$ -concave) if and only if, for every  $x, y \in S$ , the function  $\psi$  given by

$$\psi(\lambda) = \phi((1 - \lambda)x + \lambda y)$$

is an  $r$ -convex ( $r$ -concave) function of  $\lambda$  for  $0 \leq \lambda \leq 1$ .

- (i) Let  $\phi$  be a twice continuously differentiable real quasiconvex function on an open convex set  $S \subseteq \mathbb{R}^n$ . If there exists a real number  $r^*$  satisfying

$$(1.4) \quad r^* = \sup_{x \in S, \|z\|=1} \frac{-z^T \nabla^2 \phi(x) z}{[z^T \nabla \phi(x)]^2}$$

whenever  $z^T \nabla \phi(x) \neq 0$ , then  $\phi$  is  $r$ -convex for every  $r \geq r^*$ . We obtain the  $r$ -concave analog of the above theorem by replacing supremum in (1.4) by infimum.

In this paper, we will present new examples of  $r$ -convex functions in Section 2. Meanwhile, we extend the  $r$ -convexity and quasi-convexity concepts to the setting associated with second-order cone in Section 4 and Section 5. Applications of  $r$ -convexity to optimization theory can be found in [2, 12, 15]. In general,  $r$ -convex functions can be viewed as the functions between convex functions and quasi-convex

functions. We believe that the aforementioned extensions will be beneficial for dealing optimization problems involved second-order constraints. We point out that extending the concepts of  $r$ -convex and quasi-convex functions to the setting associated with second-order cone, which belongs to symmetric cones, is not easy and obvious since any two vectors in the Euclidean Jordan algebra cannot be compared under the partial order  $\preceq_{\mathcal{K}^n}$ , see [8]. Nonetheless, using the projection onto second-order cone pave a way to do such extensions, more details will be seen in Sections 4 and 5.

To close this section, we recall some background materials regarding second-order cone. The second-order cone (SOC for short) in  $\mathbb{R}^n$ , also called the Lorentz cone, is defined by

$$\mathcal{K}^n = \{x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\| \leq x_1\}.$$

For  $n = 1$ ,  $\mathcal{K}^n$  denotes the set of nonnegative real number  $\mathbb{R}_+$ . For any  $x, y$  in  $\mathbb{R}^n$ , we write  $x \succeq_{\mathcal{K}^n} y$  if  $x - y \in \mathcal{K}^n$  and write  $x \succ_{\mathcal{K}^n} y$  if  $x - y \in \text{int}(\mathcal{K}^n)$ . In other words, we have  $x \succeq_{\mathcal{K}^n} 0$  if and only if  $x \in \mathcal{K}^n$  and  $x \succ_{\mathcal{K}^n} 0$  if and only if  $x \in \text{int}(\mathcal{K}^n)$ . The relation  $\succeq_{\mathcal{K}^n}$  is a partial ordering but not a linear ordering in  $\mathcal{K}^n$ , i.e., there exist  $x, y \in \mathcal{K}^n$  such that neither  $x \succeq_{\mathcal{K}^n} y$  nor  $y \succeq_{\mathcal{K}^n} x$ . To see this, for  $n = 2$ , let  $x = (1, 1)$  and  $y = (1, 0)$ , we have  $x - y = (0, 1) \notin \mathcal{K}^n$ ,  $y - x = (0, -1) \notin \mathcal{K}^n$ .

For dealing with second-order cone programs (SOCP) and second-order cone complementarity problems (SOCCP), we need *spectral decomposition* associated with SOC [9]. More specifically, for any  $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ , the vector  $x$  can be decomposed as

$$x = \lambda_1 u_x^{(1)} + \lambda_2 u_x^{(2)},$$

where  $\lambda_1, \lambda_2$  and  $u_x^{(1)}, u_x^{(2)}$  are the spectral values and the associated spectral vectors of  $x$ , respectively, given by

$$\lambda_i = x_1 + (-1)^i \|x_2\|,$$

$$u_x^{(i)} = \begin{cases} \frac{1}{2}(1, (-1)^i \frac{x_2}{\|x_2\|}) & \text{if } x_2 \neq 0, \\ \frac{1}{2}(1, (-1)^i w) & \text{if } x_2 = 0. \end{cases}$$

for  $i = 1, 2$  with  $w$  being any vector in  $\mathbb{R}^{n-1}$  satisfying  $\|w\| = 1$ . If  $x_2 \neq 0$ , the decomposition is unique.

For any function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , the following vector-valued function associated with  $\mathcal{K}^n$  ( $n \geq 1$ ) was considered in [7, 8]:

$$(1.5) \quad f^{\text{soc}}(x) = f(\lambda_1)u_x^{(1)} + f(\lambda_2)u_x^{(2)}, \quad \forall x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}.$$

If  $f$  is defined only on a subset of  $\mathbb{R}$ , then  $f^{\text{soc}}$  is defined on the corresponding subset of  $\mathbb{R}^n$ . The definition (1.5) is unambiguous whether  $x_2 \neq 0$  or  $x_2 = 0$ . The cases of  $f^{\text{soc}}(x) = x^{1/2}$ ,  $x^2$ ,  $\exp(x)$  are discussed in [10]. In fact, the above definition (1.5) is analogous to the one associated with positive semidefinite cone  $S_+^n$  [20, 21].

Throughout this paper,  $\mathbb{R}^n$  denotes the space of  $n$ -dimensional real column vectors,  $C$  denotes a convex subset of  $\mathbb{R}$ ,  $S$  denotes a convex subset of  $\mathbb{R}^n$ , and  $\langle \cdot, \cdot \rangle$  means the Euclidean inner product, whereas  $\| \cdot \|$  is the Euclidean norm. The notation “:=” means “define”. For any  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\nabla f(x)$  denotes the gradient of  $f$  at  $x$ .  $C^{(i)}(J)$  denotes the family of functions which are defined on  $J \subseteq \mathbb{R}^n$  to  $\mathbb{R}$  and have the  $i$ -th continuous derivative, while  $T$  means transpose.

2. EXAMPLES OF  $r$ -FUNCTIONS

In this section, we try to discover some new  $r$ -convex functions which is verified by applying Property 1.1. With these examples, we have a more complete picture about characterizations of  $r$ -convex functions. Moreover, for any given  $r$ , we also provide examples which are  $r$ -convex functions.

**Example 2.1.** For any real number  $p$ , let  $f : (0, \infty) \rightarrow \mathbb{R}$  be defined by  $f(t) = t^p$ .

- (a) If  $p > 0$ , then  $f$  is convex for  $p \geq 1$ , and  $(+\infty)$ -convex for  $0 < p < 1$ .
- (b) If  $p < 0$ , then  $f$  is convex.

To see this, we first note that  $f'(t) = pt^{p-1}$ ,  $f''(t) = p(p-1)t^{p-2}$  and

$$\sup_{s \cdot f'(t) \neq 0, |s|=1} \frac{-s \cdot f''(t) \cdot s}{[s \cdot f'(t)]^2} = \sup_{p \neq 0} \frac{(1-p)t^{-p}}{p} = \begin{cases} \infty & \text{if } 0 < p < 1, \\ 0 & \text{if } p > 1 \text{ or } p < 0. \end{cases}$$

Then, applying Property 1.1 yields the desired result.

**Example 2.2.** Suppose that  $f$  is defined on  $(-\frac{\pi}{2}, \frac{\pi}{2})$ .

- (a) The function  $f(t) = \sin t$  is  $\infty$ -convex.
- (b) The function  $f(t) = \tan t$  is 1-convex.
- (c) The function  $f(t) = \ln(\sec t)$  is  $(-1)$ -convex.
- (d) The function  $f(t) = \ln|\sec t + \tan t|$  is 1-convex.

To see (a), we note that  $f'(t) = \cos t$ ,  $f''(t) = -\sin t$ , and

$$\sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}, |s|=1} \frac{-s \cdot f''(t) \cdot s}{[s \cdot f'(t)]^2} = \sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}} \frac{\sin t}{\cos^2 t} = \infty.$$

Hence  $f(t) = \sin t$  is  $\infty$ -convex.

To see (b), we note that  $f'(t) = \sec^2 t$ ,  $f''(t) = 2 \sec^2 t \cdot \tan t$ , and

$$\sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}} \frac{-f''(t)}{[f'(t)]^2} = \sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}} \frac{-2 \sec^2 t \cdot \tan t}{\sec^4 t} = \sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}} (-\sin 2t) = 1.$$

This says that  $f(t) = \tan t$  is 1-convex.

To see (c), we note that  $f'(t) = \tan t$ ,  $f''(t) = \sec^2 t$ , and

$$\sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}} \frac{-f''(t)}{[f'(t)]^2} = \sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}} \frac{-\sec^2 t}{\tan^2 t} = \sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}} (-\csc^2 t) = -1.$$

Then, it is clear to see that  $f(t) = \ln(\sec t)$  is  $(-1)$ -convex.

To see (d), we note that  $f'(t) = \sec t$ ,  $f''(t) = \sec t \cdot \tan t$ , and

$$\sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}} \frac{-f''(t)}{[f'(t)]^2} = \sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}} \frac{-\sec t \cdot \tan t}{\sec^2 t} = \sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}} (-\sin t) = 1.$$

Thus,  $f(t) = \ln |\sec t + \tan t|$  is 1-convex.

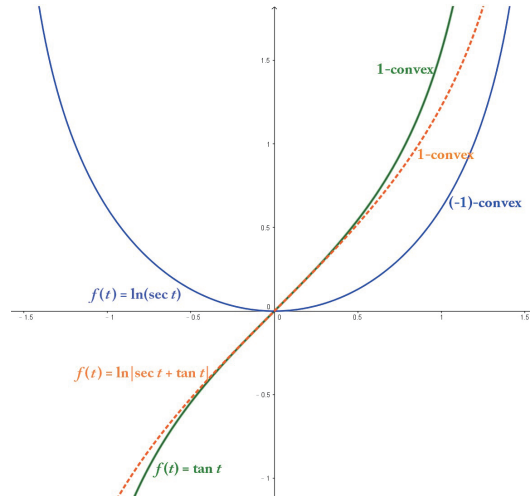


FIGURE 1. Graphs of  $r$ -convex functions with various values of  $r$ .

In light of Example 2.2(b)-(c) and Property 1.1(e), the next example indicates that for any given  $r \in \mathbb{R}$  (no matter positive or negative), we can always construct an  $r$ -convex function accordingly. The graphs of various  $r$ -convex functions are depicted in Figure 1.

**Example 2.3.** For any  $r \neq 0$ , let  $f$  be defined on  $(-\frac{\pi}{2}, \frac{\pi}{2})$ .

(a) The function  $f(t) = \frac{\tan t}{r}$  is  $|r|$ -convex.

(b) The function  $f(t) = \frac{\ln(\sec t)}{r}$  is  $(-r)$ -convex.

(a) First, we compute that  $f'(t) = \frac{\sec^2 t}{r}$ ,  $f''(t) = \frac{2 \sec^2 t \cdot \tan t}{r}$ , and

$$\sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}} \frac{-f''(t)}{[f'(t)]^2} = \sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}} (-r \sin 2t) = |r|.$$

This says that  $f(t) = \frac{\tan t}{r}$  is  $|r|$ -convex.

(b) Similarly, from  $f'(t) = \frac{\tan t}{r}$ ,  $f''(t) = \frac{\sec^2 t}{r}$ , and

$$\sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}} \frac{-f''(t)}{[f'(t)]^2} = \sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}} (-r \csc^2 t) = -r.$$

Then, it is easy to see that  $f(t) = \frac{\ln(\sec t)}{r}$  is  $(-r)$ -convex.

**Example 2.4.** The function  $f(x) = \frac{1}{2} \ln(\|x\|^2 + 1)$  defined on  $\mathbb{R}^2$  is 1-convex.

For  $x = (s, t) \in \mathbb{R}^2$ , and any real number  $r \neq 0$ , we consider the function

$$\begin{aligned} \phi(x, r) &= \nabla^2 f(x) + r \nabla f(x) \nabla f(x)^T \\ &= \frac{1}{(\|x\|^2 + 1)^2} \begin{bmatrix} t^2 - s^2 + 1 & -2st \\ -2st & s^2 - t^2 + 1 \end{bmatrix} + \frac{r}{(\|x\|^2 + 1)^2} \begin{bmatrix} s^2 & st \\ st & t^2 \end{bmatrix} \\ &= \frac{1}{(\|x\|^2 + 1)^2} \begin{bmatrix} (r-1)s^2 + t^2 + 1 & (r-2)st \\ (r-2)st & s^2 + (r-1)t^2 + 1 \end{bmatrix}. \end{aligned}$$

Applying Property 1.1(b), we know that  $f$  is  $r$ -convex if and only if  $\phi$  is positive semidefinite, which is equivalent to

$$(2.1) \quad (r-1)s^2 + t^2 + 1 \geq 0$$

$$(2.2) \quad \begin{vmatrix} (r-1)s^2 + t^2 + 1 & (r-2)st \\ (r-2)st & s^2 + (r-1)t^2 + 1 \end{vmatrix} \geq 0.$$

It is easy to verify the inequality (2.1) holds for all  $x \in \mathbb{R}^2$  if and only if  $r \geq 1$ . Moreover, we note that

$$\begin{aligned} &\begin{vmatrix} (r-1)s^2 + t^2 + 1 & (r-2)st \\ (r-2)st & s^2 + (r-1)t^2 + 1 \end{vmatrix} \geq 0 \\ \iff &s^2 t^2 + s^2 + t^2 + 1 + (r-1)^2 s^2 t^2 + (r-1)(s^4 + s^2 + t^4 + t^2) - (r-2)^2 s^2 t^2 \geq 0 \\ \iff &s^2 + t^2 + 1 + (2r-2)s^2 t^2 + (r-1)(s^4 + s^2 + t^4 + t^2) \geq 0, \end{aligned}$$

and hence the inequality (2.2) holds for all  $x \in \mathbb{R}^2$  whenever  $r \geq 1$ . Thus, we conclude by Property 1.1(b) that  $f$  is 1-convex on  $\mathbb{R}^2$ .

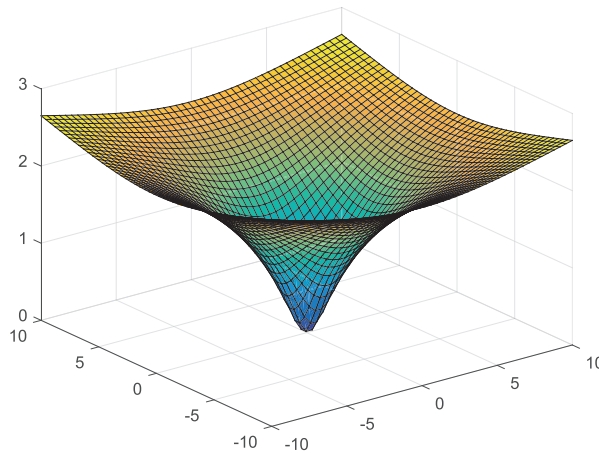


FIGURE 2. Graphs of 1-convex functions  $f(x) = \frac{1}{2} \ln(\|x\|^2 + 1)$ .

3. PROPERTIES OF SOC-FUNCTIONS

As mentioned in Section 1, another contribution of this paper is extending the concept of  $r$ -convexity to the setting associated with second-order cone. To this end, we recall what SOC-convex function means. For any  $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$  and  $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ , we define their *Jordan product* as

$$x \circ y = (x^T y, y_1 x_2 + x_1 y_2).$$

We write  $x^2$  to mean  $x \circ x$  and write  $x + y$  to mean the usual componentwise addition of vectors. Then,  $\circ, +$ , together with  $e' = (1, 0, \dots, 0)^T \in \mathbb{R}^n$  and for any  $x, y, z \in \mathbb{R}^n$ , the following basic properties [10, 11] hold: (1)  $e' \circ x = x$ , (2)  $x \circ y = y \circ x$ , (3)  $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ , (4)  $(x + y) \circ z = x \circ z + y \circ z$ . Notice that the Jordan product is *not associative* in general. However, it is power associative, i.e.,  $x \circ (x \circ x) = (x \circ x) \circ x$  for all  $x \in \mathbb{R}^n$ . Thus, we may, without loss of ambiguity, write  $x^m$  for the product of  $m$  copies of  $x$  and  $x^{m+n} = x^m \circ x^n$  for all positive integers  $m$  and  $n$ . Here, we set  $x^0 = e'$ . Besides,  $\mathcal{K}^n$  is *not closed* under Jordan product.

For any  $x \in \mathcal{K}^n$ , it is known that there exists a unique vector in  $\mathcal{K}^n$  denoted by  $x^{1/2}$  such that  $(x^{1/2})^2 = x^{1/2} \circ x^{1/2} = x$ . Indeed,

$$x^{1/2} = \left( s, \frac{x_2}{2s} \right), \quad \text{where } s = \sqrt{\frac{1}{2} \left( x_1 + \sqrt{x_1^2 - \|x_2\|^2} \right)}.$$

In the above formula, the term  $x_2/s$  is defined to be the zero vector if  $x_2 = 0$  and  $s = 0$ , i.e.,  $x = 0$ . For any  $x \in \mathbb{R}^n$ , we always have  $x^2 \in \mathcal{K}^n$ , i.e.,  $x^2 \succeq_{\mathcal{K}^n} 0$ . Hence, there exists a unique vector  $(x^2)^{1/2} \in \mathcal{K}^n$  denoted by  $|x|$ . It is easy to verify that  $|x| \succeq_{\mathcal{K}^n} 0$  and  $x^2 = |x|^2$  for any  $x \in \mathbb{R}^n$ . It is also known that  $|x| \succeq_{\mathcal{K}^n} x$ . For any  $x \in \mathbb{R}^n$ , we define  $[x]_+$  to be the nearest point projection of  $x$  onto  $\mathcal{K}^n$ , which is the same definition as in  $\mathbb{R}_+^n$ . In other words,  $[x]_+$  is the optimal solution of the parametric SOCP:  $[x]_+ = \arg \min \{ \|x - y\| \mid y \in \mathcal{K}^n \}$ . In addition, it can be verified that  $[x]_+ = (x + |x|)/2$ ; see [10, 11].

**Property 3.1** ([11, Proposition 3.3]). For any  $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ , we have

- (a)  $|x| = (x^2)^{1/2} = |\lambda_1|u_x^{(1)} + |\lambda_2|u_x^{(2)}$ .
- (b)  $[x]_+ = [\lambda_1]_+u_x^{(1)} + [\lambda_2]_+u_x^{(2)} = \frac{1}{2}(x + |x|)$ .

Next, we review the concepts of SOC-monotone and SOC-convex functions which are introduced in [7].

**Definition 3.2.** For a real valued function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

- (a)  $f$  is said to be *SOC-monotone* of order  $n$  if its corresponding vector-valued function  $f^{\text{soc}}$  defined as in (1.5) satisfies

$$x \succeq_{\mathcal{K}^n} y \implies f^{\text{soc}}(x) \succeq_{\mathcal{K}^n} f^{\text{soc}}(y).$$

The function  $f$  is said to be SOC-monotone if  $f$  is SOC-monotone of all order  $n$ .



(b)  $f$  is said to be *SOC-convex* of order  $n$  if its corresponding vector-valued function  $f^{\text{soc}}$  defined as in (1.5) satisfies

$$(3.1) \quad f^{\text{soc}}((1 - \lambda)x + \lambda y) \preceq_{\mathcal{K}^n} (1 - \lambda)f^{\text{soc}}(x) + \lambda f^{\text{soc}}(y)$$

for all  $x, y \in \mathbb{R}^n$  and  $0 \leq \lambda \leq 1$ . Similarly,  $f$  is said to be *SOC-concave* of order  $n$  on  $C$  if the inequality (3.1) is reversed. The function  $f$  is said to be *SOC-convex* (respectively, *SOC-concave*) if  $f$  is *SOC-convex* of all order  $n$  (respectively, *SOC-concave* of all order  $n$ ).

The concepts of *SOC-monotone* and *SOC-convex* functions are analogous to matrix monotone and matrix convex functions [5, 14], and are special cases of operator monotone and operator convex functions [3, 6, 16]. Examples of *SOC-monotone* and *SOC-convex* functions are given in [7]. It is clear that the set of *SOC-monotone* functions and the set of *SOC-convex* functions are both closed under linear combinations and under pointwise limits.

**Property 3.3** ([8, Theorem 3.1]). Let  $f \in C^{(1)}(J)$  with  $J$  being an open interval and  $\text{dom}(f^{\text{soc}}) \subseteq \mathbb{R}^n$ . Then, the following hold.

- (a)  $f$  is *SOC-monotone* of order 2 if and only if  $f'(\tau) \geq 0$  for any  $\tau \in J$ ;
- (b)  $f$  is *SOC-monotone* of order  $n \geq 3$  if and only if the  $2 \times 2$  matrix

$$\begin{bmatrix} f^{(1)}(t_1) & \frac{f(t_2) - f(t_1)}{t_2 - t_1} \\ \frac{f(t_2) - f(t_1)}{t_2 - t_1} & f^{(1)}(t_2) \end{bmatrix} \succeq O \quad \text{for all } t_1, t_2 \in J \text{ and } t_1 \neq t_2.$$

**Property 3.4** ([8, Theorem 4.1]). Let  $f \in C^{(2)}(J)$  with  $J$  being an open interval in  $\mathbb{R}$  and  $\text{dom}(f^{\text{soc}}) \subseteq \mathbb{R}^n$ . Then, the following hold.

- (a)  $f$  is *SOC-convex* of order 2 if and only if  $f$  is convex;
- (b)  $f$  is *SOC-convex* of order  $n \geq 3$  if and only if  $f$  is convex and the inequality

$$(3.2) \quad \frac{1}{2}f^{(2)}(t_0) \frac{[f(t_0) - f(t) - f^{(1)}(t)(t_0 - t)]}{(t_0 - t)^2} \geq \frac{[f(t) - f(t_0) - f^{(1)}(t_0)(t - t_0)]}{(t_0 - t)^4}$$

holds for any  $t_0, t \in J$  and  $t_0 \neq t$ .

**Property 3.5** ([4, Theorem 3.3.7]). Let  $f : S \rightarrow \mathbb{R}$  where  $S$  is a nonempty open convex set in  $\mathbb{R}^n$ . Suppose  $f \in C^2(S)$ . Then,  $f$  is convex if and only if  $\nabla^2 f(x) \succeq O$ , for all  $x \in S$ .

**Property 3.6** ([7, Proposition 4.1]). Let  $f : [0, \infty] \rightarrow [0, \infty]$  be continuous. If  $f$  is *SOC-concave*, then  $f$  is *SOC-monotone*.

**Property 3.7** ([11, Proposition 3.2]). Suppose that  $f(t) = e^t$  and  $g(t) = \ln t$ . Then, the corresponding *SOC-functions* of  $e^t$  and  $\ln t$  are given as below.

- (a) For any  $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ ,

$$f^{\text{soc}}(x) = e^x = \begin{cases} e^{x_1} \left( \cosh(\|x_2\|), \sinh(\|x_2\|) \frac{x_2}{\|x_2\|} \right) & \text{if } x_2 \neq 0, \\ (e^{x_1}, 0) & \text{if } x_2 = 0, \end{cases}$$

where  $\cosh(\alpha) = (e^\alpha + e^{-\alpha})/2$  and  $\sinh(\alpha) = (e^\alpha - e^{-\alpha})/2$  for  $\alpha \in \mathbb{R}$ .

(b) For any  $x = (x_1, x_2) \in \text{int}(\mathcal{K}^n)$ ,  $\ln x$  is well-defined and

$$g^{\text{soc}}(x) = \ln x = \begin{cases} \frac{1}{2} \left( \ln(x_1^2 - \|x_2\|^2), \ln \left( \frac{x_1 + \|x_2\|}{x_1 - \|x_2\|} \right) \frac{x_2}{\|x_2\|} \right) & \text{if } x_2 \neq 0, \\ (\ln x_1, 0) & \text{if } x_2 = 0. \end{cases}$$

With these, we have the following technical lemmas that will be used in the subsequent analysis.

**Lemma 3.8.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be  $f(t) = e^t$  and  $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ ,  $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ . Then, the following hold.*

- (a)  $f$  is SOC-monotone of order 2 on  $\mathbb{R}$ .
- (b)  $f$  is not SOC-monotone of order  $n \geq 3$  on  $\mathbb{R}$ .
- (c) If  $x_1 - y_1 \geq \|x_2\| + \|y_2\|$ , then  $e^x \succeq_{\mathcal{K}^n} e^y$ . In particular, if  $x \in \mathcal{K}^n$ , then  $e^x \succeq_{\mathcal{K}^n} e^{(0,0)}$ .

*Proof.* (a) By applying Property 3.3(a), it is clear that  $f$  is SOC-monotone of order 2 since  $f'(\tau) = e^\tau \geq 0$  for all  $\tau \in \mathbb{R}$ .

(b) Take  $x = (2, 1.2, -1.6)$ ,  $y = (-1, 0, -4)$ , then we have  $x - y = (3, 1.2, 2.4) \succeq_{\mathcal{K}^n} 0$ . But, we compute that

$$\begin{aligned} e^x - e^y &= e^2 \left( \cosh(2), \sinh(2) \frac{(1.2, -1.6)}{2} \right) - e^{-1} \left( \cosh(4), \sinh(4) \frac{(0, -4)}{4} \right) \\ &= \frac{1}{2} [(e^4 + 1, .6(e^4 - 1), -.8(e^4 - 1)) - (e^3 + e^{-5}, 0, -e^3 + e^{-5})] \\ &= (17.7529, 16.0794, -11.3999) \not\succeq_{\mathcal{K}^n} 0. \end{aligned}$$

The last inequality is because  $\|(16.0794, -11.3999)\| = 19.7105 > 17.7529$ .

We also present an alternative argument for part(b) here. First, we observe that

$$(3.3) \quad \det \begin{bmatrix} f^{(1)}(t_1) & \frac{f(t_2) - f(t_1)}{t_2 - t_1} \\ \frac{f(t_2) - f(t_1)}{t_2 - t_1} & f^{(1)}(t_2) \end{bmatrix} = e^{t_1 + t_2} - \left( \frac{e^{t_2} - e^{t_1}}{t_2 - t_1} \right)^2 \geq 0$$

if and only if  $1 \geq \left( \frac{e^{(t_2 - t_1)/2} - e^{(t_1 - t_2)/2}}{t_2 - t_1} \right)^2$ . Denote  $s := (t_2 - t_1)/2$ , then the above inequality holds if and only if  $1 \geq (\sinh(s)/s)^2$ . In light of Taylor Theorem, we know  $\sinh(s)/s = 1 + s^2/6 + s^4/120 + \dots > 1$  for  $s \neq 0$ . Hence, (3.3) does not hold. Then, applying Property 3.3(b) says  $f$  is not SOC-monotone of order  $n \geq 3$  on  $\mathbb{R}$ .

(c) The desired result follows by the following implication:

$$\begin{aligned}
 & e^x \succeq_{\mathcal{K}^n} e^y \\
 \iff & e^{x_1} \cosh(\|x_2\|) - e^{y_1} \cosh(\|y_2\|) \\
 & \geq \left\| e^{x_1} \sinh(\|x_2\|) \frac{x_2}{\|x_2\|} - e^{y_1} \sinh(\|y_2\|) \frac{y_2}{\|y_2\|} \right\| \\
 \iff & [e^{x_1} \cosh(\|x_2\|) - e^{y_1} \cosh(\|y_2\|)]^2 \\
 & - \left\| e^{x_1} \sinh(\|x_2\|) \frac{x_2}{\|x_2\|} - e^{y_1} \sinh(\|y_2\|) \frac{y_2}{\|y_2\|} \right\|^2 \\
 = & e^{2x_1} + e^{2y_1} \\
 & - 2e^{x_1+y_1} \left[ \cosh(\|x_2\|) \cosh(\|y_2\|) - \sinh(\|x_2\|) \sinh(\|y_2\|) \frac{\langle x_2, y_2 \rangle}{\|x_2\| \|y_2\|} \right] \\
 \geq & 0 \\
 \iff & e^{2x_1} + e^{2y_1} - 2e^{x_1+y_1} \cosh(\|x_2\| + \|y_2\|) \geq 0 \\
 \iff & \cosh(\|x_2\| + \|y_2\|) \leq \frac{e^{2x_1} + e^{2y_1}}{2e^{x_1+y_1}} = \frac{e^{x_1-y_1} + e^{y_1-x_1}}{2} = \cosh(x_1 - y_1) \\
 \iff & x_1 - y_1 \geq \|x_2\| + \|y_2\|.
 \end{aligned}$$

□

**Lemma 3.9.** *Let  $f(t) = e^t$  be defined on  $\mathbb{R}$ , then  $f$  is SOC-convex of order 2. However,  $f$  is not SOC-convex of order  $n \geq 3$ .*

*Proof.* (a) By applying Property 3.4 (a), it is clear that  $f$  is SOC-convex since exponential function is a convex function on  $\mathbb{R}$ .

(b) As below, it is a counterexample which shows  $f(t) = e^t$  is not SOC-convex of order  $n \geq 3$ . To see this, we compute that

$$\begin{aligned}
 e^{[(2,0,-1)+(6,-4,-3)]/2} &= e^{(4,-2,-2)} \\
 &= e^4 \left( \cosh(2\sqrt{2}), \sinh(2\sqrt{2}) \cdot (-2, -2)/(2\sqrt{2}) \right) \\
 &\doteq (463.48, -325.45, -325.45)
 \end{aligned}$$

and

$$\begin{aligned}
 & \frac{1}{2} \left( e^{(2,0,-1)} + e^{(6,-4,-3)} \right) \\
 = & \frac{1}{2} \left[ e^2(\cosh(1), 0, -\sinh(1)) + e^6(\cosh(5), \sinh(5) \cdot (-4, -3)/5) \right] \\
 = & (14975, -11974, -8985).
 \end{aligned}$$

We see that  $14975 - 463.48 = 14511.52$ , but

$$\|(-11974, -8985) - (-325.4493, -325.4493)\| = 14515 > 14511.52$$

which is a contradiction. □

**Lemma 3.10** ([8, Proposition 5.1]). *The function  $g(t) = \ln t$  is SOC-monotone of order  $n \geq 2$  on  $(0, \infty)$ .*

In general, to verify the SOC-convexity of  $e^t$  (as shown in Proposition 3.1), we observe that the following fact

$$0 \prec_{\mathcal{K}^n} e^{rf^{\text{soc}}(\lambda x + (1-\lambda)y)} \preceq_{\mathcal{K}^n} w \implies rf^{\text{soc}}(\lambda x + (1-\lambda)y) \preceq_{\mathcal{K}^n} \ln(w)$$

is important and often needed. Note for  $x_2 \neq 0$ , we also have some observations as below.

- (a)  $e^x \succ_{\mathcal{K}^n} 0 \iff \cosh(\|x_2\|) \geq |\sinh(\|x_2\|)| \iff e^{-\|x_2\|} > 0$ .
- (b)  $0 \prec_{\mathcal{K}^n} \ln(x) \iff \ln(x_1^2 - \|x_2\|^2) > \left| \ln\left(\frac{x_1 + \|x_2\|}{x_1 - \|x_2\|}\right) \right| \iff \ln(x_1 - \|x_2\|) > 0 \iff x_1 - \|x_2\| > 1$ . Hence  $(1, 0) \prec_{\mathcal{K}^n} x$  implies  $0 \prec_{\mathcal{K}^n} \ln(x)$ .
- (c)  $\ln(1, 0) = (0, 0)$  and  $e^{(0,0)} = (1, 0)$ .

#### 4. SOC- $r$ -CONVEX FUNCTIONS

In this section, we define the so-called SOC- $r$ -convex functions which is viewed as the natural extension of  $r$ -convex functions to the setting associated with second-order cone.

**Definition 4.1.** Suppose that  $r \in \mathbb{R}$  and  $f : C \subseteq \mathbb{R} \rightarrow \mathbb{R}$  where  $C$  is a convex subset of  $\mathbb{R}$ . Let  $f^{\text{soc}} : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  be its corresponding SOC-function defined as in (1.5). The function  $f$  is said to be SOC- $r$ -convex of order  $n$  on  $C$  if, for  $x, y \in S$  and  $\lambda \in [0, 1]$ , there holds

$$(4.1) \quad f^{\text{soc}}(\lambda x + (1-\lambda)y) \preceq_{\mathcal{K}^n} \begin{cases} \frac{1}{r} \ln(\lambda e^{rf^{\text{soc}}(x)} + (1-\lambda)e^{rf^{\text{soc}}(y)}) & r \neq 0, \\ \lambda f^{\text{soc}}(x) + (1-\lambda)f^{\text{soc}}(y) & r = 0. \end{cases}$$

Similarly,  $f$  is said to be SOC- $r$ -concave of order  $n$  on  $C$  if the inequality (4.1) is reversed. We say  $f$  is SOC- $r$ -convex (respectively, SOC- $r$ -concave) on  $C$  if  $f$  is SOC- $r$ -convex (respectively, SOC- $r$ -concave) of all order  $n$  (respectively, SOC- $r$ -concave of all order  $n$ ) on  $C$ .

It is clear from the above definition that a real function is SOC-convex (SOC-concave) if and only if it is SOC-0-convex (SOC-0-concave). In addition, a function  $f$  is SOC- $r$ -convex if and only if  $-f$  is SOC- $(-r)$ -concave. From [1, Theorem 4.1], it is shown that  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  is  $r$ -convex with  $r \neq 0$  if and only if  $e^{r\phi}$  is convex whenever  $r > 0$  and concave whenever  $r < 0$ . However, we observe that the exponential function  $e^t$  is not SOC-convex for  $n \geq 3$  by Lemma 3.9. This is a hurdle to build parallel result for general  $n$  in the setting of SOC case. As seen in Proposition 4.5, the parallel result is true only for  $n = 2$ . Indeed, for  $n \geq 3$ , only one direction holds which can be viewed as a weaker version of [1, Theorem 4.1].

**Proposition 4.2.** *Let  $f : [0, \infty) \rightarrow [0, \infty)$  be continuous. If  $f$  is SOC- $r$ -concave with  $r \geq 0$ , then  $f$  is SOC-monotone.*

*Proof.* For any  $0 < \lambda < 1$ , we can write  $\lambda x = \lambda y + \frac{(1-\lambda)\lambda}{(1-\lambda)}(x - y)$ . If  $r = 0$ , then  $f$  is SOC-concave and SOC-monotone by Property 3.6. If  $r > 0$ , then

$$\begin{aligned} f^{\text{soc}}(\lambda x) &\succeq_{\mathcal{K}^n} \frac{1}{r} \ln \left( \lambda e^{r f^{\text{soc}}(y)} + (1 - \lambda) e^{r f^{\text{soc}}(\frac{\lambda}{1-\lambda}(x-y))} \right) \\ &\succeq_{\mathcal{K}^n} \frac{1}{r} \ln \left( \lambda e^{r(0,0)} + (1 - \lambda) e^{r(0,0)} \right) \\ &= \frac{1}{r} \ln (\lambda(1, 0) + (1 - \lambda)(1, 0)) \\ &= 0, \end{aligned}$$

where the second inequality is due to  $x - y \succeq_{\mathcal{K}^n} 0$  and Lemmas 3.8-3.10. Letting  $\lambda \rightarrow 1$ , we obtain that  $f^{\text{soc}}(x) \succeq_{\mathcal{K}^n} f^{\text{soc}}(y)$ , which says that  $f$  is SOC-monotone.  $\square$

In fact, in light of Lemma 3.8-3.10, we have the following Lemma which is useful for subsequent analysis.

**Lemma 4.3.** *Let  $z \in \mathbb{R}^n$  and  $w \in \text{int}(\mathcal{K}^n)$ . Then, the following hold.*

- (a) *For  $n = 2$  and  $r > 0$ ,  $z \preceq_{\mathcal{K}^n} \ln(w)/r \iff rz \preceq_{\mathcal{K}^n} \ln(w) \iff e^{rz} \preceq_{\mathcal{K}^n} w$ .*
- (b) *For  $n = 2$  and  $r > 0$ ,  $z \preceq_{\mathcal{K}^n} \ln(w)/r \iff rz \succeq_{\mathcal{K}^n} \ln(w) \iff e^{rz} \succeq_{\mathcal{K}^n} w$ .*
- (c) *For  $n \geq 2$ , if  $e^{rz} \preceq_{\mathcal{K}^n} w$ , then  $rz \preceq_{\mathcal{K}^n} \ln(w)$ .*

**Proposition 4.4.** *For  $n = 2$  and let  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Then, the following hold.*

- (a) *The function  $f(t) = t$  is SOC- $r$ -convex (SOC- $r$ -concave) on  $\mathbb{R}$  for  $r > 0$  ( $r < 0$ ).*
- (b) *If  $f$  is SOC-convex, then  $f$  is SOC- $r$ -convex (SOC- $r$ -concave) for  $r > 0$  ( $r < 0$ ).*

*Proof.* (a) For  $r > 0$ ,  $x, y \in \mathbb{R}^n$  and  $\lambda \in [0, 1]$ , we note that the corresponding vector-valued SOC-function of  $f(t) = t$  is  $f^{\text{soc}}(x) = x$ . Therefore, to prove the desired result, we need to verify that

$$f^{\text{soc}}(\lambda x + (1 - \lambda)y) \preceq_{\mathcal{K}^n} \frac{1}{r} \ln \left( \lambda e^{r f^{\text{soc}}(x)} + (1 - \lambda) e^{r f^{\text{soc}}(y)} \right).$$

To this end, we see that

$$\begin{aligned} \lambda x + (1 - \lambda)y &\preceq_{\mathcal{K}^n} \frac{1}{r} \ln (\lambda e^{rx} + (1 - \lambda)e^{ry}) \\ \iff \lambda rx + (1 - \lambda)ry &\preceq_{\mathcal{K}^n} \ln (\lambda e^{rx} + (1 - \lambda)e^{ry}) \\ \iff e^{\lambda rx + (1-\lambda)ry} &\preceq_{\mathcal{K}^n} \lambda e^{rx} + (1 - \lambda)e^{ry}, \end{aligned}$$

where the first “ $\iff$ ” is true due to Lemma 4.3, whereas the second “ $\iff$ ” holds because  $e^t$  and  $\ln t$  are SOC-monotone of order 2 by Lemma 3.8 and Lemma 3.10. Then, using the fact that  $e^t$  is SOC-convex of order 2 gives the desired result.

(b) For any  $x, y \in \mathbb{R}^n$  and  $0 \leq \lambda \leq 1$ , it can be verified that

$$\begin{aligned} f^{\text{soc}}(\lambda x + (1 - \lambda)y) &\preceq_{\mathcal{K}^n} \lambda f^{\text{soc}}(x) + (1 - \lambda)f^{\text{soc}}(y) \\ &\preceq_{\mathcal{K}^n} \frac{1}{r} \ln \left( \lambda e^{r f^{\text{soc}}(x)} + (1 - \lambda) e^{r f^{\text{soc}}(y)} \right), \end{aligned}$$

where the second inequality holds according to the proof of (a). Thus, the desired result follows.  $\square$

**Proposition 4.5.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Then  $f$  is SOC- $r$ -convex if  $e^{rf}$  is SOC-convex (SOC-concave) for  $n \geq 2$  and  $r > 0$  ( $r < 0$ ). For  $n = 2$ , we can replace “if” by “if and only if”.*

*Proof.* Suppose that  $e^{rf}$  is SOC-convex. For any  $x, y \in \mathbb{R}^n$  and  $0 \leq \lambda \leq 1$ , using that fact that  $\ln t$  is SOC-monotone (Lemma 3.10) yields

$$\begin{aligned} & e^{rf^{\text{SOC}}(\lambda x + (1-\lambda)y)} \preceq_{\mathcal{K}^n} \lambda e^{rf^{\text{SOC}}(x)} + (1-\lambda)e^{rf^{\text{SOC}}(y)} \\ \implies & rf^{\text{SOC}}(\lambda x + (1-\lambda)y) \preceq_{\mathcal{K}^n} \ln \left( \lambda e^{rf^{\text{SOC}}(x)} + (1-\lambda)e^{rf^{\text{SOC}}(y)} \right) \\ \iff & f^{\text{SOC}}(\lambda x + (1-\lambda)y) \preceq_{\mathcal{K}^n} \frac{1}{r} \ln \left( \lambda e^{rf^{\text{SOC}}(x)} + (1-\lambda)e^{rf^{\text{SOC}}(y)} \right). \end{aligned}$$

When  $n = 2$ ,  $e^t$  is SOC-monotone as well, which implies that the “ $\implies$ ” can be replaced by “ $\iff$ ”. Thus, the proof is complete.  $\square$

Combining with Property 3.4, we can characterize the SOC- $r$ -convexity as follows.

**Proposition 4.6.** *Let  $f \in C^{(2)}(J)$  with  $J$  being an open interval in  $\mathbb{R}$  and  $\text{dom}(f^{\text{SOC}}) \subseteq \mathbb{R}^n$ . Then, for  $r > 0$ , the followings hold.*

- (a)  $f$  is SOC- $r$ -convex of order 2 if and only if  $e^{rf}$  is convex;
- (b)  $f$  is SOC- $r$ -convex of order  $n \geq 3$  if  $e^{rf}$  is convex and satisfies the inequality (3.2).

Next, we present several examples of SOC- $r$ -convex and SOC- $r$ -concave functions of order 2. For examples of SOC- $r$ -convex and SOC- $r$ -concave functions (of order  $n$ ), we are still unable to discover them.

**Example 4.7.** For  $n = 2$ , the following hold.

- (a) The function  $f(t) = t^2$  is SOC- $r$ -convex on  $\mathbb{R}$  for  $r \geq 0$ .
- (b) The function  $f(t) = t^3$  is SOC- $r$ -convex on  $[0, \infty)$  for  $r > 0$ , while it is SOC- $r$ -concave on  $(-\infty, 0]$  for  $r < 0$ .
- (c) The function  $f(t) = 1/t$  is SOC- $r$ -convex on  $[-r/2, 0)$  or  $(0, \infty)$  for  $r > 0$ , while it is SOC- $r$ -concave on  $(-\infty, 0)$  or  $(0, -r/2]$  for  $r < 0$ .
- (d) The function  $f(t) = \sqrt{t}$  is SOC- $r$ -convex on  $[1/r^2, \infty)$  for  $r > 0$ , while it is SOC- $r$ -concave on  $[0, \infty)$  for  $r < 0$ .
- (e) The function  $f(t) = \ln t$  is SOC- $r$ -convex (SOC- $r$ -concave) on  $(0, \infty)$  for  $r > 0$  ( $r < 0$ ).

*Proof.* (a) First, we denote  $h(t) := e^{rt^2}$ . Then, we have  $h'(t) = 2rte^{rt^2}$  and  $h''(t) = (1 + 2rt^2)2re^{rt^2}$ . From Property 3.5, we know  $h$  is convex if and only if  $h''(t) \geq 0$ . Thus, the desired result holds by applying Property 3.4 and Proposition 4.5. The arguments for other cases are similar and we omit them.  $\square$

5. SOC-QUASICONVEX FUNCTIONS

In this section, we define the so-called SOC-quasiconvex functions which is a natural extension of quasiconvex functions to the setting associated with second-order cone.

Recall that a function  $f : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be *quasiconvex* on  $S$  if, for any  $x, y \in S$  and  $0 \leq \lambda \leq 1$ , there has

$$f(\lambda x + (1 - \lambda)y) \leq \max \{f(x), f(y)\}.$$

We point out that the relation  $\succeq_{\mathcal{K}^n}$  is not a linear ordering. Hence, it is not possible to compare any two vectors (elements) via  $\succeq_{\mathcal{K}^n}$ . Nonetheless, we note that

$$\max\{a, b\} = b + [a - b]_+ = \frac{1}{2}(a + b + |a - b|), \quad \text{for any } a, b \in \mathbb{R}.$$

This motivates us to define SOC-quasiconvex functions in the setting of second-order cone.

**Definition 5.1.** Let  $f : C \subseteq \mathbb{R} \rightarrow \mathbb{R}$  and  $0 \leq \lambda \leq 1$ . The function  $f$  is said to be SOC-quasiconvex of order  $n$  on  $C$  if, for any  $x, y \in \mathbb{R}^n$ , there has

$$f^{\text{soc}}(\lambda x + (1 - \lambda)y) \preceq_{\mathcal{K}^n} f^{\text{soc}}(y) + [f^{\text{soc}}(x) - f^{\text{soc}}(y)]_+$$

where

$$f^{\text{soc}}(y) + [f^{\text{soc}}(x) - f^{\text{soc}}(y)]_+ = \begin{cases} f^{\text{soc}}(x) & \text{if } f^{\text{soc}}(x) \succeq_{\mathcal{K}^n} f^{\text{soc}}(y), \\ f^{\text{soc}}(y) & \text{if } f^{\text{soc}}(x) \prec_{\mathcal{K}^n} f^{\text{soc}}(y), \\ \frac{1}{2}(f^{\text{soc}}(x) + f^{\text{soc}}(y) + |f^{\text{soc}}(x) - f^{\text{soc}}(y)|) & \text{if } f^{\text{soc}}(x) - f^{\text{soc}}(y) \notin \mathcal{K}^n \cup (-\mathcal{K}^n). \end{cases}$$

Similarly,  $f$  is said to be SOC-quasiconcave of order  $n$  if

$$f^{\text{soc}}(\lambda x + (1 - \lambda)y) \succeq_{\mathcal{K}^n} f^{\text{soc}}(x) - [f^{\text{soc}}(x) - f^{\text{soc}}(y)]_+.$$

The function  $f$  is called SOC-quasiconvex (SOC-quasiconcave) if it is SOC-quasiconvex of all order  $n$  (SOC-quasiconcave of all order  $n$ ).

**Proposition 5.2.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be  $f(t) = t$ . Then,  $f$  is SOC-quasiconvex on  $\mathbb{R}$ .

*Proof.* First, for any  $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ ,  $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ , and  $0 \leq \lambda \leq 1$ , we have

$$\begin{aligned} f^{\text{soc}}(y) \preceq_{\mathcal{K}^n} f^{\text{soc}}(x) &\iff (1 - \lambda)f^{\text{soc}}(y) \preceq_{\mathcal{K}^n} (1 - \lambda)f^{\text{soc}}(x) \\ &\iff \lambda f^{\text{soc}}(x) + (1 - \lambda)f^{\text{soc}}(y) \preceq_{\mathcal{K}^n} f^{\text{soc}}(x). \end{aligned}$$

Recall that the corresponding SOC-function of  $f(t) = t$  is  $f^{\text{soc}}(x) = x$ . Thus, for all  $x \in \mathbb{R}^n$ , this implies  $f^{\text{soc}}(\lambda x + (1 - \lambda)y) = \lambda f^{\text{soc}}(x) + (1 - \lambda)f^{\text{soc}}(y) \preceq_{\mathcal{K}^n} f^{\text{soc}}(x)$  under this case:  $f^{\text{soc}}(y) \preceq_{\mathcal{K}^n} f^{\text{soc}}(x)$ . The argument is similar to the case of  $f^{\text{soc}}(x) \preceq_{\mathcal{K}^n} f^{\text{soc}}(y)$ . Hence, it remains to consider the case of  $f^{\text{soc}}(x) - f^{\text{soc}}(y) \notin \mathcal{K}^n \cup (-\mathcal{K}^n)$ , i.e., it suffices to show that

$$\lambda f^{\text{soc}}(x) + (1 - \lambda)f^{\text{soc}}(y) \preceq_{\mathcal{K}^n} \frac{1}{2}(f^{\text{soc}}(x) + f^{\text{soc}}(y) + |f^{\text{soc}}(x) - f^{\text{soc}}(y)|).$$

To this end, we note that

$$|f^{\text{soc}}(x) - f^{\text{soc}}(y)| \succeq_{\mathcal{K}^n} f^{\text{soc}}(x) - f^{\text{soc}}(y)$$

and

$$|f^{\text{soc}}(x) - f^{\text{soc}}(y)| \succeq_{\mathcal{K}^n} f^{\text{soc}}(y) - f^{\text{soc}}(x),$$

which respectively implies

$$(5.1) \quad \frac{1}{2} (f^{\text{soc}}(x) + f^{\text{soc}}(y) + |f^{\text{soc}}(x) - f^{\text{soc}}(y)|) \succeq_{\mathcal{K}^n} x,$$

$$(5.2) \quad \frac{1}{2} (f^{\text{soc}}(x) + f^{\text{soc}}(y) + |f^{\text{soc}}(x) - f^{\text{soc}}(y)|) \succeq_{\mathcal{K}^n} y.$$

Then, adding up (5.1)  $\times \lambda$  and (5.2)  $\times (1 - \lambda)$  yields the desired result.  $\square$

**Proposition 5.3.** *If  $f : C \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is SOC-convex on  $C$ , then  $f$  is also SOC-quasiconvex on  $C$ .*

*Proof.* For any  $x, y \in \mathbb{R}^n$  and  $0 \leq \lambda \leq 1$ , it can be verified that

$$f^{\text{soc}}(\lambda x + (1 - \lambda)y) \preceq_{\mathcal{K}^n} \lambda f^{\text{soc}}(x) + (1 - \lambda)f^{\text{soc}}(y) \preceq_{\mathcal{K}^n} f^{\text{soc}}(y) + [f^{\text{soc}}(x) - f^{\text{soc}}(y)]_+,$$

where the second inequality holds according to the proof of Proposition 5.2. Thus, the desired result follows.  $\square$

From Proposition 5.3, we can easily construct examples of SOC-quasiconvex functions. More specifically, all the SOC-convex functions which were verified in [7] are SOC-quasiconvex functions, for instances,  $t^2$  on  $\mathbb{R}$ , and  $t^3, \frac{1}{t}, t^{1/2}$  on  $(0, \infty)$ .

## 6. FINAL REMARKS

In this paper, we revisit the concept of  $r$ -convex functions and provide a way to construct  $r$ -convex functions for any given  $r \in \mathbb{R}$ . We also extend such concept to the setting associated with SOC which will be helpful in dealing with optimization problems involved in second-order cones. In particular, we obtain some characterizations for SOC- $r$ -convexity and SOC-quasiconvexity.

Indeed, this is just the first step and there still have many things to clarify. For example, in Section 4, we conclude that SOC-convexity implies SOC- $r$ -convexity for  $n = 2$  only. The key role therein relies particularly on the SOC-convexity and SOC-monotonicity of  $e^t$ . However, for  $n > 2$ , the expressions of  $e^x$  and  $\ln(x)$  associated with second-order cone are very complicated so that it is hard to compare any two elements. In other words, when  $n = 2$ , the SOC-convexity and SOC-monotonicity of  $e^t$  make things much easier than the general case  $n \geq 3$ . To conquer this difficulty, we believe that we have to derive more properties of  $e^x$ . In particular, ‘‘Does SOC- $r$ -convex function have similar results as shown in Property 1.1?’’ is an important future direction.



## REFERENCES

- [1] M. Avriel, *r-convex functions*, Math. Program. **2** (1972), 309–323.
- [2] M. Avriel, *Solution of certain nonlinear programs involving r-convex functions* J. Optim. Theory Appl. **11** (1973), 159–174.
- [3] J. S. Aujla and H. L. Vasudeva, *Convex and monotone operator functions*, Annal. Polonici Math. **62** (1995), 1–11.
- [4] M. S. Bazaraa, H. D. Sherali and C. M. Shetty, *Nonlinear Programming*, John Wiley and Sons, 3rd edition, 2006.
- [5] R. Bhatia, *Matrix Analysis*, Springer-Verlag, New York, 1997.
- [6] P. Bhatia and K. P. Parthasarathy, *Positive definite functions and operator inequalities*, Bull. London Math. Soc. **32** (2000), 214–228.
- [7] J.-S. Chen, *The convex and monotone functions associated with second-order cone*, Optim. **55** (2006), 363–385.
- [8] J.-S. Chen, X. Chen, S.-H. Pan and J. Zhang, *Some characterizations for SOC-monotone and SOC-convex functions*, J. Global Optim. **45** (2009), 259–279.
- [9] J.-S. Chen, X. Chen and P. Tseng, *Analysis of nonsmooth vector-valued functions associated with second-order cones*, Math. Program. **101** (2004), 95–117.
- [10] J. Faraut and A. Korányi, *Analysis on Symmetric Cones*, Oxford Mathematical Monographs, Oxford University Press, New York, 1994.
- [11] M. Fukushima, Z.-Q. Luo, and P. Tseng, *Smoothing functions for second-order-cone complementarity problems*, SIAM J. Optim. **12** (2002), 436–460.
- [12] E. Galewska and M. Galewski, *r-convex transformability in nonlinear programming problems*, Comment. Math. Univ. Carol. **46** (2005), 555–565.
- [13] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, Cambridge University Press, 1967.
- [14] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.
- [15] A. Klinger and O.L. Mangasarian, *Logarithmic convexity and geometric programming*, J. Math. Anal. Appl. **24** (1968), 388–408.
- [16] A. Korányi, *Monotone functions on formally real Jordan algebras*, Math. Annal. **269** (1984), 73–76.
- [17] B. Martos, *The Power of Nonlinear Programming Methods* (In Hungarian). MTA Közgazdaságtudományi Intézetének Közleményei, No. 20, Budapest, Hungary, 1966.
- [18] R. N. Mukherjee and L. V. Keddy, *Semicontinuity and quasiconvex functions*, J. Optim. Theory Appl. **94** (1997), 715–720.
- [19] J. Ponstein, *Seven kinds of convexity*, SIAM Review **9** (1967), 115–119.
- [20] D. Sun and J. Sun, *Semismooth matrix valued functions*, Math. Oper. Res. **27** (2002), 150–169.
- [21] P. Tseng, *Merit function for semidefinite complementarity problems*, Math. Program. **83** (1998), 159–185.
- [22] X. M. Yang, *Convexity of semicontinuous functions*, Oper. Res. Soc. India **31** (1994), 309–317.
- [23] Y. M. Yang and S. Y. Liu, *Three kinds of generalized convexity*, J. Optim. Theory Appl. **86** (1995), 501–513.
- [24] Y. X. Zhao, S. Y. Wang and U. L. Coladas, *Characterizations of r-convex functions*, J. Optim. Theory Appl. **145** (2010), 186–195.

CHIEN-HAO HUANG

Department of Mathematics, National Taiwan Normal University, Taipei 11677, Taiwan

*E-mail address:* `qqnick0719@ntnu.edu.tw`

HONG-LIN HUANG

Department of Mathematics, National Taiwan Normal University, Taipei 11677, Taiwan

*E-mail address:* `buddin5678@gmail.com`

JEIN-SHAN CHEN

Department of Mathematics, National Taiwan Normal University, Taipei 11677, Taiwan

*E-mail address:* `jschen@math.ntnu.edu.tw`