Journal of Nonlinear and Convex Analysis
Volume 21, Number 1, 2020, 21–29

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

THE SCHATTEN p-NORM ON $\mathbb{R}^n$

CHIEN-HAO HUANG, JEIN-SHAN CHEN*, AND CHU-CHIN HU†

ABSTRACT. It is well known that the Schatten p-norm defined on the space of matrices is useful and possesses nice properties. In this paper, we explore the concept of Schatten p-norm on $\mathbb{R}^n$ via the structure of Euclidean Jordan algebra. Two types of Schatten p-norm on $\mathbb{R}^n$ are defined and the relationship between these two norms is also investigated.

1. Introduction

In mathematical analysis, we usually employ various norms to measure the distance of two elements in any arbitrary vector spaces. In particular, the following three specific norms are frequently used:

$$\begin{aligned} \|x\|_1 &= \textstyle\sum_{i=1}^n |x_i|, \\ \|x\|_2 &= \left(\textstyle\sum_{i=1}^n x_i^2\right)^{1/2}, \\ \|x\|_\infty &= \max_{1\le i\le n} |x_i|, \end{aligned}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. It is known that $\|x\|_2$ and $\|x\|_\infty$ are the so-called Euclidian norm and supremum norm, respectively.

Let A be a bounded linear operator on $\mathbb{R}^n$. Associated with each norm $\|\cdot\|$ on $\mathbb{R}^n$, one can define an operator norm of A, denoted by $\|A\|$, as below:

$$\|A\| := \sup_{\|x\|=1} \|Ax\|.$$

We denote by $|A|$ the positive operator $(A^*A)^{1/2}$ and by $s(A)$ the vector whose coordinates are the singular values of A (i.e., the eigenvalues of $|A|$), arranged as $s_1(A) \ge s_2(A) \ge \cdots \ge s_n(A)$. Then, there holds

$$\|A\| = \|\,|A|\,\| = s_1(A).$$

In addition, for any unitary operators U, V on $\mathbb{C}^n$, we know that

$$\|A\| = \|UAV\|.$$

In this case, we say that $\|\cdot\|$ is *unitary invariant* and is very useful in various analysis. Moreover, there are two classes of norms, which posses the aforementioned property. The first one is the so-called *Schatten p-norm* defined by

$$\|A\|_p := \begin{cases} \left[\sum_{i=1}^n (s_i(A))^p\right]^{1/p}, & \text{for} \quad 1 \le p < \infty; \\ s_1(A), & \text{for} \quad p = \infty. \end{cases}$$

Key words and phrases. Schatten p-norm, Euclidean Jordan algebra, second-order cone.

*The author's work is supported by Ministry of Science and Technology, Taiwan.

†Corresponding author.

We point out that the Schatten p-norm is different from the operator norm (matrix norm) induced by l_p-norm, which is given by $\|A\|_p := \sup_{\|x\|_p=1} \|Ax\|_p$, although these two norms share the same notation. The readers can distinguish them from the contexts, so we do not denote them by different symbols. It is also known that the Schatten p-norm satisfies

- $\|A_1 A_2\|_1 \leq \|A_1\|_p \cdot \|A_2\|_q$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $p \in [1, \infty]$ (Holder inequality).
- $\|A\|_1 \geq \|A\|_{p_1} \geq \|A\|_{p_2} \geq \|A\|_\infty$ where $1 \leq p_1 \leq p_2 \leq \infty$ (Monotonicity).

For more details and applications of the Schatten p-norm, please refer to [1, 2, 8, 14, 15] and references therein.

The second one is the so-called *Ky Fan k-norm* defined by

$$\|A\|_{(k)} := \sum_{i=1}^{k} (s_i(A)), \quad 1 \leq k \leq n.$$

In other words, $\|A\|_{(k)}$ is exactly the sum of the k largest singular values of A. In addition, the Ky Fan 1-norm is the operator norm induced by the Euclidean norm; and hence it is also called the operator 2-norm. For more details and applications of the Ky Fan k-norm, please refer to [1, 2, 6, 8, 15] and references therein.

Now, we consider the space $\mathbb{S}_n$ of $n \times n$ real symmetric matrices. Under the Jordan product $X \circ Y = \frac{1}{2}(XY + YX)$ and the bilinear form $\langle X, Y \rangle := \mathrm{tr}(XY)$, $(\mathbb{S}_n, \circ, \langle \cdot, \cdot \rangle)$ forms a Euclidean Jordan algebra whose definition will be elaborated in Section 2. Based on Spectral Decomposition Theorem [7, Theorem III.1.2], we also note that the eigenvalues of A coincide with the spectral values of $A \in \mathbb{S}_n$. It is also known that $\mathbb{R}^n$ can be viewed as a Euclidean Jordan algebra under appropriate Jordan product and inner product. This motivates us to study whether the Schatten p-norms can be defined on $\mathbb{R}^n$ or not.

In this short paper, we shall define two types of Schatten p-norm on $\mathbb{R}^n$ and investigate some inequalities about these two norms. The paper is organized as below. In Section 2, we recall some basic definitions and properties about Euclidean Jordan algebra. Under the standard inner product, two types of Euclidean Jordan algebra over $\mathbb{R}^n$ are also established in Section 3. Moreover, some relationship about these two norms are deduced as well.

2. Preliminary

In this section, we review the basic concepts and properties concerning Jordan algebras and symmetric cones from the book [7] which are needed in the subsequent analysis.

A *Euclidean Jordan algebra* is a finite dimensional inner product space $(\mathbb{V}, \langle \cdot, \cdot \rangle)$ ($\mathbb{V}$ for short) over the field of real numbers $\mathbb{R}$ equipped with a bilinear map $(x, y) \mapsto x \circ y : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$, which satisfies the following conditions:

(i) $x \circ y = y \circ x$ for all $x, y \in \mathbb{V}$;

(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathbb{V}$;

(iii) $\langle x \circ y, z\rangle = \langle x, y \circ z\rangle$ for all $x, y, z \in \mathbb{V}$,

where $x^2 := x \circ x$, and $x \circ y$ is called the *Jordan product* of x and y. If a Jordan product only satisfies the conditions (i) and (ii) in the above definition, the algebra $\mathbb{V}$ is said to be a *Jordan algebra*. Moreover, if there is an (unique) element $e \in \mathbb{V}$ such that $x \circ e = x$ for all $x \in \mathbb{V}$, the element e is called the *Jordan identity* in $\mathbb{V}$. Note that a Jordan algebra does not necessarily have an identity element. Throughout this paper, we assume that $\mathbb{V}$ is a Euclidean Jordan algebra with an identity element e.

In the Euclidean Jordan algebra $\mathbb{V}$, the set of squares $\mathcal{K} := \{x^2 : x \in \mathbb{V}\}$ is called a *symmetric cone* [7, Theorem III.2.1], which means $\mathcal{K}$ is a self-dual closed convex cone and, for any two elements $x, y \in \mathrm{int}(\mathcal{K})$, there exists an invertible linear transformation $\Gamma : \mathbb{V} \to \mathbb{V}$ such that $\Gamma(x) = y$ and $\Gamma(\mathcal{K}) = \mathcal{K}$. An element $c \in \mathbb{V}$ is called an *idempotent* if $c^2 = c$, and it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. Two idempotents c, d are said to be *orthogonal* if $c \circ d = 0$. In addition, we say that a finite set $\{e^{(1)}, e^{(2)}, \cdots, e^{(r)}\}$ of primitive idempotents in $\mathbb{V}$ is a *Jordan frame* if

$$e^{(i)} \circ e^{(j)} = 0 \quad \text{for} \quad i \neq j, \quad \text{and} \quad \sum_{i=1}^{r} e^{(i)} = e.$$

Note that $\langle e^{(i)}, e^{(j)}\rangle = \langle e^{(i)} \circ e^{(j)}, e\rangle$ whenever $i \neq j$. With the above, there has the spectral decomposition of an element x in $\mathbb{V}$.

Theorem 2.1 (Spectral Decomposition Theorem ([7, Theorem III.1.2]))**.** *Let $\mathbb{V}$ be a Euclidean Jordan algebra. Then there is a number r such that, for every $x \in \mathbb{V}$, there exists a Jordan frame $\{e^{(1)}, \cdots, e^{(r)}\}$ and real numbers $\lambda_1(x), \cdots, \lambda_r(x)$ with*

$$x = \lambda_1(x)e^{(1)} + \cdots + \lambda_r(x)e^{(r)}.$$

Here, the numbers $\lambda_i(x)$ $(i = 1, \cdots, r)$ are called the spectral values of x, the expression $\lambda_1(x)e^{(1)} + \cdots + \lambda_r(x)e^{(r)}$ is called the spectral decomposition of x. Moreover, $\mathrm{tr}(x) := \sum_{i=1}^{r} \lambda_i(x)$ is called the trace of x, $\det(x) := \lambda_1(x)\lambda_2(x)\cdots\lambda_r(x)$ is called the determinant of x, and r is called the rank of $\mathbb{V}$.

3. Main results

In this section, we introduce two types of Euclidean Jordan algebra over $\mathbb{R}^n$ under the standard inner product

$$\langle x, y\rangle := x_1y_1 + x_2y_2 + \cdots + x_ny_n,$$

for any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

(I). The first type of Jordan product is defined as

$$x \bullet y = (x_1y_1, x_2y_2, \ldots, x_ny_n).$$

It is easy to check $(\mathbb{R}^n, \bullet, \langle\cdot,\cdot\rangle)$ forms a Euclidean Jordan algebra with the identity element $e = (1, 1, \ldots, 1)$. Under this case, it is not a simple Euclidean Jordan algebra. Indeed, it is a Cartesian product of simple Euclidean Jordan algebras. In view of this, for all $i = 1, 2, \ldots, n$, we denote e_i the vector with the i-th component is 1 and the others are all zeros. Then, each e_i is a primitive idempotent in $\mathbb{R}^n$ and the set $\{e_1, e_2, \ldots, e_n\}$ forms a Jordan frame. The induced symmetric cone is

$$\mathbb{R}^n_+ := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, i = 1, 2, \ldots, n\}.$$

Moreover, it is clear that for any $x \in \mathbb{R}^n$,

$$x = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n.$$

On the other hand, from the Spectral Decomposition Theorem, we know the spectral values of x are $x_1, x_2, \ldots, x_n$. In light of all the above observations, the first type of Schatten p-norm $\|\cdot\|_p$ is defined as

$$\|x\|_p := \left[\sum_{i=1}^n |x_i|^p\right]^{1/p}, \quad 1 \leq p < \infty, \tag{3.1}$$

which coincides with the well-known l_p-norm. Moreover, for $p = \infty$, the Schatten p-norm becomes

$$\|x\|_\infty = \max_{1\leq i\leq n} |x_i|,$$

which coincides with the well-known supremum norm. From the coincidence, the properties of the Schatten p-norm on $\mathbb{R}^n$ space can be easily obtained. As below, we only state two of them, and the proofs are omitted because in this case, they are the l_p-norm exactly. In the literature, there are various nice proofs for these two properties, see [11, 12].

Proposition 3.1. *For any fixed $x \in \mathbb{R}^n$, let $\|x\|_p$ denote the first type of Schatten p-norm on $\mathbb{R}^n$ given as in* (3.1). *Then, the function $p \mapsto \|x\|_p$ is a decreasing function on* $[1, \infty)$.

Proposition 3.2. *For any fixed $x \in \mathbb{R}^n$, let $\|x\|_p$ denote the first type of Schatten p-norm on $\mathbb{R}^n$ given as in* (3.1). *Then, for $1 \leq p \leq q \leq \infty$, there holds*

$$\|x\|_q \leq \|x\|_p \leq n^{\frac{1}{p}-\frac{1}{q}} \cdot \|x\|_q. \tag{3.2}$$

(II). Now, we consider the second type of Jordan product which the induced symmetric cone is the so-called second-order cone (or Lorentz cone). For the simplicity of notation, we denote $x = (x_1, \bar{\mathbf{x}}) \in \mathbb{R} \times \mathbb{R}^{n-1}$ the vector in $\mathbb{R}^n$, i.e., $\bar{\mathbf{x}} := (x_2, x_3, \ldots, x_n)$ is a vector in $\mathbb{R}^{n-1}$. For any $x = (x_1, \bar{\mathbf{x}})$, $y = (y_1, \bar{\mathbf{y}}) \in \mathbb{R} \times \mathbb{R}^{n-1}$, the second type of Jordan product is defined as

$$x \circ y = (\langle x, y\rangle, y_1\bar{\mathbf{x}} + x_1\bar{\mathbf{y}}).$$

We note that $e = (1, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$ acts as the Jordan identity. Besides, this Jordan product is *not associative* and $\mathcal{K}^n$ is *not closed* under this Jordan product.

The induced symmetric cone is an important example of symmetric cones, which is defined as follows:

$$\mathcal{K}^n = \left\{x = (x_1, \bar{\mathbf{x}}) \in \mathbb{R} \times \mathbb{R}^{n-1} \,|\, x_1 \geq \|\bar{\mathbf{x}}\|_2\right\}.$$

For any vector $x = (x_1, \bar{\mathbf{x}}) \in \mathbb{R} \times \mathbb{R}^{n-1}$, it can be decomposed as

$$x = \lambda_1(x) u_x^{(1)} + \lambda_2(x) u_x^{(2)},$$

where $\lambda_1(x), \lambda_2(x)$ and $u_x^{(1)}, u_x^{(2)}$ are the spectral values and the associated spectral vectors of x, respectively, given by

$$\lambda_i(x) = x_1 + (-1)^i \|\bar{\mathbf{x}}\|_2,$$

$$u_x^{(i)} = \begin{cases} \frac{1}{2}\left(1, (-1)^i \frac{\bar{\mathbf{x}}}{\|\bar{\mathbf{x}}\|_2}\right) & \text{if} \quad \bar{\mathbf{x}} \neq 0, \\ \frac{1}{2}\left(1, (-1)^i \bar{\mathbf{w}}\right) & \text{if} \quad \bar{\mathbf{x}} = 0. \end{cases}$$

for $i = 1, 2$ with $\bar{\mathbf{w}}$ being any vector in $\mathbb{R}^{n-1}$ satisfying $\|\bar{\mathbf{w}}\|_2 = 1$. If $\bar{\mathbf{x}} \neq 0$, the decomposition is unique. Here, we remark that for $n = 1$, the second-order cone $\mathcal{K}^1$ reduces to the nonnegative real number $\mathbb{R}_+$ and the Jordan product is the basic multiplication on $\mathbb{R}$.

We say a few words about the discrepancy between $(\mathbb{R}^n, \circ, \langle\cdot,\cdot\rangle)$ and $(\mathbb{R}^n, \bullet, \langle\cdot,\cdot\rangle)$. Note that $(\mathbb{R}^n, \circ, \langle\cdot,\cdot\rangle)$ is a **simple** Euclidean Jordan algebra, whereas the Euclidean Jordan algebra $(\mathbb{R}^n, \bullet, \langle\cdot,\cdot\rangle)$ is not simple. As mentioned earlier, $(\mathbb{R}^n, \bullet, \langle\cdot,\cdot\rangle)$ can be written as the direct sum of $(\mathbb{R}, \cdot)$. In fact, there are only five types of simple Euclidean Jordan algebra [7]. For more details for second-order cones, please refers to [3, 4, 5].

According to the structure of $(\mathbb{R}^n, \circ, \langle\cdot,\cdot\rangle)$, the second type of Schatten p-norm on $\mathbb{R}^n$ space ought to be defined as

$$|||x|||_p := \left[\sum_{i=1}^{2} |\lambda_i(x)|^p\right]^{1/p}. \tag{3.3}$$

In a recent paper, Huang et. al [9, Theorem 3.6-3.7] (also see [10, 13] for Euclidean Jordan algebra) established a trace version inequality via the trace version of Young inequality:

$$[\mathrm{tr}(|x+y|^p)]^{1/p} \leq [\mathrm{tr}(|x|^p)]^{1/p} + [\mathrm{tr}(|y|^p)]^{1/p},$$

where $x, y \in \mathbb{R}^n$ and $p \geq 1$. By the definition of the second type of Schatten p-norm on $\mathbb{R}^n$ given as in (3.3), it is not hard to verify that this inequality is equivalent to

$$|||x+y|||_p \leq |||x|||_p + |||y|||_p. \tag{3.4}$$

Hence, the functional $x \mapsto |||x|||_p$ can actually define a norm on $\mathbb{R}^n$ space for $p \geq 1$. In particular, we note

$$|||x|||_2 = \left[\sum_{i=1}^{2} |\lambda_i(x)|^2\right]^{1/2} = \sqrt{2} \cdot \|x\|_2. \tag{3.5}$$

Moreover, this norm $||| \cdot |||_p$ can be viewed as a norm by applying the first type of Schatten p-norm to the vector $(x_1 - \|\bar{\mathbf{x}}\|, x_1 + \|\bar{\mathbf{x}}\|) \in \mathbb{R}^2$ which consists of the spectral values of x. Accordingly we immediately have the following inequality by applying (3.2).

Proposition 3.3. *For any $x \in \mathbb{R}^n$, let $|||x|||_p$ denote the second type of Schatten p-norm on $\mathbb{R}^n$ given as in* (3.3)*. Then, for $1 \le p \le q \le \infty$, there holds*

$$|||x|||_q \le |||x|||_p \le 2^{\frac{1}{p}-\frac{1}{q}} \cdot |||x|||_q.$$

It is natural to ask if there is any relationship between these two types of Schatten p-norm on $\mathbb{R}^n$. We observe that each spectral value of x includes the term $\|\bar{\mathbf{x}}\|$. This leads us to separate the discussion into two cases: $1 \le p < 2$ and $2 \le p$.

Theorem 3.4. *For any $x \in \mathbb{R}^n$, let $\|x\|_p$ denote the first type of Schatten p-norm on $\mathbb{R}^n$ given as in* (3.1) *and $|||x|||_p$ denote the second type of Schatten p-norm on $\mathbb{R}^n$ given as in* (3.3)*. Then, for $1 \le p \le \infty$, the following hold.*

(a) *For $1 \le p < 2$, there holds*

$$\sqrt{2}\, n^{\frac{1}{2}-\frac{1}{p}} \cdot \|x\|_p \le |||x|||_p \le 2^{\frac{1}{p}} \cdot \|x\|_p.$$

(b) *For $p \ge 2$, there holds*

$$2^{\frac{1}{p}} \cdot \|x\|_p \le |||x|||_p \le \sqrt{2}\, n^{\frac{1}{2}-\frac{1}{p}} \cdot \|x\|_p.$$

Proof. (a) For $1 \le p < 2$ and $x \in \mathbb{R}^n$, we have

$$|||x|||_p \ge |||x|||_2 = \sqrt{2} \cdot \|x\|_2 \ge \sqrt{2} n^{\frac{1}{2}-\frac{1}{p}} \cdot \|x\|_p,$$

and

$$|||x|||_p \le 2^{\frac{1}{p}-\frac{1}{2}} \cdot |||x|||_2 = 2^{\frac{1}{p}} \cdot \|x\|_2 \le 2^{\frac{1}{p}} \cdot \|x\|_p,$$

where the inequalities hold by Proposition 3.3 and inequalities (3.2) and (3.5).

(b) For $p \ge 2$ and $x \in \mathbb{R}^n$, we have

$$|||x|||_p \le |||x|||_2 = \sqrt{2} \cdot \|x\|_2 \le \sqrt{2} n^{\frac{1}{2}-\frac{1}{p}} \cdot \|x\|_p.$$

and

$$|||x|||_p \ge 2^{\frac{1}{p}-\frac{1}{2}} \cdot |||x|||_2 = 2^{\frac{1}{p}} \cdot \|x\|_2 \ge 2^{\frac{1}{p}} \cdot \|x\|_p.$$

where the inequalities hold by Proposition 3.3 and inequalities (3.2) and (3.5) as well. □

Now, we recall that a function $\Phi : \mathbb{R}^n \to \mathbb{R}$ is called a *symmetric gauge function* if

(i) Φ is a norm on the real space $\mathbb{R}^n$;
(ii) $\Phi(\sigma_n(x)) = \Phi(x)$ for all $x \in \mathbb{R}^n$, where $\sigma_n(x)$ is a permutation of the coordinate of x;
(iii) $\Phi(\delta_1 x_1, \delta_2 x_2, \dots, \delta_n x_n) = \Phi(x_1, x_2, \dots, x_n)$ for $\delta_j = \pm 1$;
(iv) $\Phi(1, 0, \dots, 0) = 1$.

We note from Problem II.5.12(iv) of [2, page 53] that for any $x, y \in \mathbb{R}^n_+$, we have

$$\Phi(x) \leq \Phi(y) \quad \text{whenever} \quad x^{\downarrow} \prec y^{\downarrow},$$

where $x^{\downarrow} = (x_1^{\downarrow}, x_2^{\downarrow}, \ldots, x_n^{\downarrow})$, $y^{\downarrow} = (y_1^{\downarrow}, y_2^{\downarrow}, \ldots, y_n^{\downarrow})$ denotes the vectors obtained by the rearranging the coordinates of x, y in the decreasing orders, respectively, and $x^{\downarrow} \prec y^{\downarrow}$ means $\sum_{j=1}^k x_j^{\downarrow} \leq \sum_{j=1}^k y_j^{\downarrow}$ for all $1 \leq k \leq n$.

It is easy to check that the functional $(\lambda_1(x), \lambda_2(x)) \mapsto |||x|||_p$ is a symmetric gauge function on $\mathbb{R}^2$. For any $x, y \in \mathbb{R}^n$, we say x is weakly majorized by y, denoted by $x \prec_w y$ if $\lambda(x)^{\downarrow} \prec \lambda(y)^{\downarrow}$, that is,

$$\lambda_2(x) \leq \lambda_2(y) \quad \text{and} \quad \sum_{i=1}^2 \lambda_i(x) \leq \sum_{i=1}^2 \lambda_i(y).$$

In light of these concepts, we achieve the following norm inequality.

Theorem 3.5. *Let $|||\cdot|||_p$ denote the second type of Schatten p-norm on $\mathbb{R}^n$ given as in* (3.3). *For any $x, y \in \mathcal{K}^n$, there holds*

$$x \prec_w y \quad \Longrightarrow \quad |||x|||_p \leq |||y|||_p.$$

Proof. For any $x, y \in \mathcal{K}^n$ with $x \prec_w y$, we note that both $(\lambda_1(x), \lambda_2(x))$ and $(\lambda_1(y), \lambda_2(y))$ lie in $\mathbb{R}^2_+$, and $\lambda(x)^{\downarrow} \prec \lambda(y)^{\downarrow}$. Then, using Problem II.5.12(iv) [2, page 53] we obtain the desired inequality since $(\lambda_1(x), \lambda_2(x)) \mapsto |||x|||_p$ is a symmetric gauge function. $\square$

In linear algebra, functional analysis and some other related areas of mathematics, a quasinorm is often used in the analysis, which is similar to a norm except that the triangle inequality is replaced by

$$\|x + y\| \leq K(\|x\| + \|y\|)$$

for some $K > 0$. It is already known that for $0 < p < 1$, the functional $x \mapsto (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ defines a quasinorm $\|\cdot\|_p$ on $\mathbb{R}^n$ and satisfies

$$\|x + y\|_p \leq 2^{\frac{1}{p}}(\|x\|_p + \|y\|_p).$$

Meanwhile, a question arises from the above discussion. Is the second type of Schatten p-norm $|||\cdot|||_p$ a quasinorm for $0 < p < 1$? To answer this question, we need the following technical lemma.

Lemma 3.6. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers and $0 < p < 1$. Then, we have*

$$\left(\sum_{i=1}^n a_i \right)^p \leq \sum_{i=1}^n a_i^p.$$

Proof. This is a fundamental result, please refer to [11] for a proof. $\square$

Theorem 3.7. *Let $|||\cdot|||_p$ denote the second type of Schatten p-norm on $\mathbb{R}^n$ given as in* (3.3)*. For any $x, y \in \mathbb{R}^n$ and $0 < p < 1$, there holds*

$$|||x+y|||_p \leq 2^{\frac{1}{p}-1}\left(|||x|||_p + |||y|||_p\right).$$

Proof. We note that

$$\begin{aligned} |||x+y|||_p &= \left(|\lambda_1(x+y)|^p + |\lambda_2(x+y)|^p\right)^{\frac{1}{p}} \\ &\leq 2^{\frac{1}{p}-1}\left(|\lambda_1(x+y)| + |\lambda_2(x+y)|\right) \\ &\leq 2^{\frac{1}{p}-1}\left(|\lambda_1(x)| + |\lambda_2(x)| + |\lambda_1(y)| + |\lambda_2(y)|\right) \\ &\leq 2^{\frac{1}{p}-1}\left(\left(|\lambda_1(x)|^p + |\lambda_2(x)|^p\right)^{\frac{1}{p}} + \left(|\lambda_1(y)|^p + |\lambda_2(y)|^p\right)^{\frac{1}{p}}\right) \\ &= 2^{\frac{1}{p}-1}\left(|||x|||_p + |||y|||_p\right), \end{aligned}$$

where the three inequalities hold by the convexity of the function $t \mapsto t^{1/p}$, the inequality (3.4) for $p = 1$, and Lemma 3.6, respectively. □

4. Concluding remark

In this paper, we have successfully extended the concept of Schatten p-norm on matrices space to the setting of $\mathbb{R}^n$ space via Euclidean Jordan algebra. Two types of Schatten p-norm on $\mathbb{R}^n$ space are defined and their connection is discussed. As a matter of fact, Tao et al. [13, Theorem 4.1] establish that for $p \geq 1$, the functional

$$x \mapsto \left[\sum_{i=1}^{r} |\lambda_i(x)|^p\right]^{1/p}$$

forms a trace p-norm in any Euclidean Jordan algebra with rank r. In view of Theorem 3.7, we suspect that there is possibility to improve it although we cannot get the proof done yet. Hence, we make a conjecture as below, which is for our future study.

Conjecture 1. Let $(\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$ be a Euclidean Jordan algebra of rank r. Then for $0 < p < 1$, the functional

$$x \mapsto \left[\sum_{i=1}^{r} |\lambda_i(x)|^p\right]^{1/p}$$

is a quasinorm on $\mathbb{V}$.

References

[1] R. Bhatia, *Matrix Analysis*, Springer-Verlag, New York, 1997.

[2] R. Bhatia, *Positive Definite Matrices*, Princeton university press, 2005.

[3] J.-S. Chen, *The convex and monotone functions associated with second-order cone*, Optimization **55** (2006), 363–385.

[4] J.-S. Chen, X. Chen, S.-H. Pan and J. Zhang, *Some characterizations for SOC-monotone and SOC-convex functions*, J. Global Optim. **45** (2009), 259–279.

[5] J.-S. Chen, X. Chen and P. Tseng, *Analysis of nonsmooth vector-valued functions associated with second-order cones*, Math. Program. **101** (2004), 95–117.

[6] C. Ding, *Variational analysis of the Ky Fan k-norm*, Set-Valued and Var. Anal. **25** (2017), 265–296.

[7] J. Faraut and A. Korányi, *Analysis on Symmetric Cones*, Oxford Mathematical Monographs, Oxford University Press, New York, 1994.

[8] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[9] C.-H. Huang, J.-S. Chen and C.-C. Hu, *Trace versions of Young inequality and its applications*, to appear in Journal of Nonlinear and Convex Analysis (2018).

[10] S. Kum and Y. Lim *Penalized complementarity functions on symmetric cones*, J. Global Optim. **46** (2010), 475–485.

[11] M. Raïssouli and I. H. Jebril, *Various proofs for the decrease monotonicity of the Schatten's power norm, various families of* $\mathbb{R}^n$*-norms and some open problems*, Int. J. Open Problems Compt. Math. **3** (2010), 164–174.

[12] W. Rudin, *Real and Complex Analysis*, 3rd ed., New York, McGraw-Hill, 1987.

[13] J. Tao, L. Kong, Z. Luo and N. Xiu, *Some majorization inequalities in Euclidean Jordan algebras*, Linear Algebra Appl. **461** (2014), 92–122.

[14] Y. Xie, S. Gu, Y. Liu, W. Zuo, W. Zhang and L. Zhang, *Weighted Schatten p-norm minimization for image denoising and background subtraction*, IEEE T. Image Process. **25** (2016), 4842–4857.

[15] X. Zhan, *Matrix Inequalities*, Lecture Notes in Mathematics **1790**, Springer-Verlag, Berlin, 2002.

Manuscript received September 10, 2018
revised December 8, 2018

C. H. Hunag
General Education Center Wenzao Ursuline University of Languages Kaohsiung 80793, Taiwan
E-mail address: `q100300@mail.wzu.edu.tw`

J. S. Chen
Department of Mathematics, National Taiwan Normal University, Taipei 11677, Taiwan
E-mail address: `jschen@math.ntnu.edu.tw`

C. C. Hu
Department of Mathematics, National Taiwan Normal University, Taipei 11677, Taiwan
E-mail address: `cchu@ntnu.edu.tw`