Exercise (Week 12)
November 24, 2022

1. Define a relation $\prec$ on $\mathbb{Z}$ by the rule: $a \prec b$ if and only if either $(ab < 0) \land (a < b)$ or $(ab \geq 0) \land (|a| < |b|)$.

   (a) Prove that $(\mathbb{Z}, \prec)$ is a strict total ordered set.

   (b) Let $-\mathbb{N}$ be the set of negative integers. Show that 1 is the least upper bound of $-\mathbb{N}$.

   (c) Prove that $(\mathbb{Z}, \prec)$ is a well ordered set (*i.e.* for any nonempty $S \subseteq \mathbb{Z}$, the least element of S exists).

2. Let $f \subseteq \mathbb{R} \times \mathbb{R}$ be a relation from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$f = \{(a,b) : b - a = \begin{cases} 1, & \text{if } a \geq 0; \\ -1, & \text{if } a \leq 0. \end{cases} \}$$

   Is f a function from $\mathbb{R}$ to $\mathbb{R}$? Show your reason.

3. Suppose that $f : X \to X$ is a function and consider f as a relation on X. Prove that if f is symmetric then $f \circ f = \mathrm{id}_X$.

4. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Suppose $\forall x, x' \in X$, $y, y' \in Y$ and $z, z' \in Z$ we have $x \star x' \in X$, $y \diamond y' \in Y$ and $z * z' \in Z$. Moreover, suppose we have

$$f(x \star x') = f(x) \diamond f(x') \text{ and } g(y \diamond y') = g(y) * g(y').$$

   Prove if $(g \circ f)(x) = z$ and $(g \circ f)(x') = z'$, then $(g \circ f)(x \star x') = z * z'$.