Multistratum Fractional Factorial Designs

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Abstract: Recent work on multistratum fractional factorial designs is set in a general and unified framework, and a criterion for selecting multistratum fractional factorial designs that takes stratum variances into account is proposed. Application of the general theory is illustrated on designs of experiments with multiple processing stages, including split-lot designs, blocked strip-plot designs, and post-fractionated strip-block designs. In particular, this helps elucidate the relationship between the three different design settings studied by Miller (1997), Bingham et al (2008), and Vivacqua and Bisgaard (2009). The construction and selection of two-stage designs in these settings are shown to be equivalent. Good designs based on our criterion are found and compared with those tabulated in Vivacqua and Bisgaard (2009).

Key words and phrases: Block structure, information capacity, minimum aberration, orthogonal design, post-fractionated strip-block design, split-lot design, split-plot design, strip-plot design.

1   Introduction

Multistratum experiments refer to those with multiple sources of errors. The error structure of an experiment is determined by the structure of experimental units, called the block structure. The block structure, treatment structure, and design (assignment of the treatments to experimental units) together specify a linear model based on which estimates of the treatment contrasts of interest are computed from the data. In two important papers, Nelder (1965a, b) developed a unified theory for the analysis of randomized experiments with what he called simple block structures, which cover most of the block structures encountered in practice. Speed and Bailey (1982) and Tjur (1984) further developed the theory to cover the more general orthogonal block structures. An excellent account can be found in Bailey (2008). In a series of papers, Brien and Bailey (2006, 2009, 2010) discussed experiments involving multiple randomizations.

Multistratum experiments are common in agriculture. Federer and King (2007) provided a comprehensive treatment of the design and analysis of split plot and split block experiments. In recent years, we have seen rising interest in design of industrial experiments with multiple strata. Some treatment factors may require larger experimental units than others since their levels are more difficult to change, or in experiments with multiple processing stages the levels of the treatment factors are assigned at different stages. Such constraints render complete randomization not possible, and the resulting restriction on randomization leads to the consideration of split-plots (Huang, Chen, and Voelkel (1998), Bingham and Sitter (1999a, 1999b, 2001), Bingham, Schoen, and Sitter (2004), McLeod and Brewster (2004)) and strip-plots (Miller (1997)). All of these involve block structures that are examples of Nelder's simple block structures. The split-lot designs of Mee and Bates (1998) for experiments with multiple processing stages, and the recent extensions in Bingham, Sitter, Kelly, Moore, and Olivas (2008) and Ranjan, Bingham, and Dean (2009) deal with block structures that are not simple block structures, but are still within the scope studied by Speed and Bailey (1982) and Tjur (1984). The highly relevant work of Nelder, Speed, Bailey, and Tjur has not been cited in the recent literature, although the general theory they developed has much to offer on issues ranging from analysis to design construction. One objective of this paper is to revisit this important line of work. We demonstrate its utility by showing how it provides a general framework for investigating factorial experiments with multiple processing stages. In particular, it helps elucidate the relationship between the three different design settings studied by Miller (1997), Bingham et al (2008), and Vivacqua and Bisgaard (2009). We also show how more general results on the analysis of experiments with multiple processing stages follow from the general theory.

Our second objective is to take up the issue of selecting good multistratum fractional factorial designs. Most of the existing work has used ad hoc modifications of the minimum aberration criterion (Fries and Hunter (1980)) that are not guided by a consistent principle. For example, block effects in blocked designs are usually treated as fixed effects while their counterparts in split-plot designs, the whole-plot effects, are considered as random, and the selection of split-plot designs is based on the usual minimum aberration criterion for unstructured

units. In particular, variances of the different sources of errors are not taken into account, whereas one would expect the relative sizes of such variances to play a role in design selection. Cheng and Tsai (2009) proposed a criterion for the selection of blocked and split-plot fractional factorial designs that incorporates the ratio of inter- and intra-block (whole-plot and subplot) variances. We extend this approach to general multistratum fractional factorial designs.

Block structures and the associated strata are reviewed in Section 2. The analysis of experiments with multiple processing stages is discussed in Section 3. Our criterion for selecting orthogonal multistratum fractional factorial designs is presented in Section 4, and applied to designs of experiments with two processing stages in the settings of Miller (1997), Bingham et al (2008), and Vivacqua and Bisgaard (2009). Better designs are reported.

We make the hierarchical assumption that lower-order effects are more important than higher order effects and effects of the same order are equally important, a situation in which minimum aberration is a suitable criterion. For simplicity, we further assume that three-factor and higher order interactions are negligible. Our study is focused on regular fractional factorial designs. We also assume that estimates of the main effects are required; thus no aliasing among the main effects is allowed.

2   Block structures, strata, and orthogonal designs

We denote the number of experimental units by N. A block structure can be described by a set $\mathfrak{F}$ of factors on the experimental units, called unit factors. Each unit factor $\mathcal{F}$ can be thought of as a partition of the experimental units into disjoint classes, called $\mathcal{F}$-classes or levels of $\mathcal{F}$. A unit factor is called uniform if all its classes are of the same size. If $\mathcal{F}_1 \neq \mathcal{F}_2$ and each $\mathcal{F}_1$-class is contained in some $\mathcal{F}_2$-class, then we write $\mathcal{F}_1 \prec \mathcal{F}_2$ and say that $\mathcal{F}_1$ is finer than $\mathcal{F}_2$ ($\mathcal{F}_2$ is coarser than $\mathcal{F}_1$, or $\mathcal{F}_1$ is nested in $\mathcal{F}_2$). We also write $\mathcal{F}_1 \preceq \mathcal{F}_2$ if $\mathcal{F}_1 \prec \mathcal{F}_2$ or $\mathcal{F}_1 = \mathcal{F}_2$. The coarsest unit factor has all the experimental units in one single class; at the other extreme, for the finest factor, each class consists of one single unit. The former is called the universal factor and the latter the equality factor, and they are denoted by $\mathcal{U}$ and $\mathcal{E}$, respectively.

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For example, the block structure of a split-plot experiment with w whole-plots each containing s subplots is determined by two uniform factors: the ws-level equality factor $\mathcal{E}$ and a w-level factor $\mathcal{P}$ that partitions the $N = ws$ units into w classes of equal size. For convenience, we always include the trivial factor $\mathcal{U}$ when describing a block structure since it corresponds to the single degree of freedom for the mean in the ANOVA. Then the block structure of a split-plot experiment consists of the three factors $\mathcal{U}$, $\mathcal{E}$, and $\mathcal{P}$ with $\mathcal{E} \prec \mathcal{P} \prec \mathcal{U}$. Such a block structure is often denoted by w/s, where / stands for nesting.

For a row-column experiment with $N = rc$ units arranged in r rows and c columns, the block structure consists of four factors $\mathcal{U}$, $\mathcal{R}$, $\mathcal{C}$, and $\mathcal{E}$, with $\mathcal{E} \prec \mathcal{R} \prec \mathcal{U}$ and $\mathcal{E} \prec \mathcal{C} \prec \mathcal{U}$, where $\mathcal{R}$ has r levels, $\mathcal{C}$ has c levels and $\mathcal{E}$ has rc levels. Such a block structure, denoted by $r \times c$, where $\times$ stands for crossing, is the block structure of a strip-plot (also called strip-block) experiment in which some treatment factors have their main effects confounded with rows or columns. This arises, e.g., in an experiment involving two processing stages, where at the first stage rc batches of material are grouped into r classes of size c, with the same level of each first-stage treatment factor assigned to all the batches in the same group, and at the second stage they are regrouped into c classes of size r, again with the same level of each second-stage factor assigned to all the batches in the same group. In a strip-plot experiment, the groupings at the two stages are based on the rows and columns.

Simple block structures are those that can be obtained by iterations of crossing and nesting operators. The blocked strip-plot designs studied in Miller (1997) have the simple block structure $b/(r \times c)$, with an $r \times c$ row-column structure within each of b blocks. Such a block structure consists of five uniform factors $\mathcal{E}$, $\mathcal{R}'$, $\mathcal{C}'$, $\mathcal{B}$, and $\mathcal{U}$ on brc units, with $\mathcal{E} \prec \mathcal{R}' \prec \mathcal{B} \prec \mathcal{U}$ and $\mathcal{E} \prec \mathcal{C}' \prec \mathcal{B} \prec \mathcal{U}$, where $\mathcal{B}$ has b levels, $\mathcal{R}'$ has br levels and $\mathcal{C}'$ has bc levels. A $2/(4 \times 4)$ block structure is depicted in Figure 2.1.

Figure 2.1. A $2/(4 \times 4)$ block structure

Given two factors $\mathcal{F}_1$ and $\mathcal{F}_2$, the finest factor $\mathcal{G}$ such that $\mathcal{F}_1 \preceq \mathcal{G}$ and $\mathcal{F}_2 \preceq \mathcal{G}$, denoted by $\mathcal{F}_1 \vee \mathcal{F}_2$, is called the supremum of $\mathcal{F}_1$ and $\mathcal{F}_2$. It is easy to see that for the $r \times c$ block structure, $\mathcal{R} \vee \mathcal{C} = \mathcal{U}$, and for $b/(r \times c)$, $\mathcal{R}' \vee \mathcal{C}' = \mathcal{B}$.

We say that two factors $\mathcal{F}_1$ and $\mathcal{F}_2$ have proportional frequencies if the size of the intersection of any $\mathcal{F}_1$-class and any $\mathcal{F}_2$-class is proportional to the product of the sizes of the relevant classes of $\mathcal{F}_1$ and $\mathcal{F}_2$. Clearly the two factors $\mathcal{R}$ and $\mathcal{C}$ in $r \times c$ have proportional frequencies since each $\mathcal{R}$-class meets with each $\mathcal{C}$-class at exactly one unit. The condition of proportional frequencies can be characterized geometrically as follows. For any factor $\mathcal{F}$, let $V_{\mathcal{F}}$ be the space $\{\boldsymbol{y} \in R^N : y_i = y_j$ if unit i and unit j are in the same $\mathcal{F}$-class$\}$. Then $\dim V_{\mathcal{F}} = n_{\mathcal{F}}$, where $n_{\mathcal{F}}$ is the number of levels of $\mathcal{F}$. A necessary and sufficient condition for $\mathcal{F}_1$ and $\mathcal{F}_2$ to have proportional frequencies is that

$$V_{\mathcal{F}_1} \ominus V_{\mathcal{U}} \text{ and } V_{\mathcal{F}_2} \ominus V_{\mathcal{U}} \text{ are orthogonal,.} \tag{2.1}$$

where $V_{\mathcal{F}_i} \ominus V_{\mathcal{U}}$ is the orthogonal complement of $V_{\mathcal{U}}$ relative to $V_{\mathcal{F}_i}$. We note that $V_{\mathcal{U}}$ is the one-dimensional space spanned by the vector of all 1's; thus $V_{\mathcal{F}} \ominus V_{\mathcal{U}}$ is the $(n_{\mathcal{F}} - 1)$-dimensional space $\{\boldsymbol{y} \in V_{\mathcal{F}} : \sum_{i=1}^{N} y_i = 0\}$. We denote this space by $C_{\mathcal{F}}$.

It can be shown that for any $\mathcal{F}_1$ and $\mathcal{F}_2$,

$$V_{\mathcal{F}_1 \vee \mathcal{F}_2} = V_{\mathcal{F}_1} \cap V_{\mathcal{F}_2}, \text{ and } C_{\mathcal{F}_1 \vee \mathcal{F}_2} = C_{\mathcal{F}_1} \cap C_{\mathcal{F}_2}. \tag{2.2}$$

We say that $\mathcal{F}_1$ and $\mathcal{F}_2$ are orthogonal if

$$V_{\mathcal{F}_1} \ominus (V_{\mathcal{F}_1} \cap V_{\mathcal{F}_2}) \text{ and } V_{\mathcal{F}_2} \ominus (V_{\mathcal{F}_1} \cap V_{\mathcal{F}_2}) \text{ are orthogonal.} \tag{2.3}$$

If (2.1) holds, i.e., $\mathcal{F}_1$ and $\mathcal{F}_2$ have proportional frequencies, then each $\mathcal{F}_1$-class has nonempty intersection with each $\mathcal{F}_2$-class. It follows that $\mathcal{F}_1 \vee \mathcal{F}_2 = \mathcal{U}$. By (2.2), (2.1) is a sufficient condition for (2.3).

Condition (2.3), generally weaker than (2.1), is equivalent to that within each $(\mathcal{F}_1 \vee \mathcal{F}_2)$-class, $\mathcal{F}_1$ and $\mathcal{F}_2$ have proportional frequencies. Thus, although the two factors $\mathcal{R}'$ and $\mathcal{C}'$ in the block structure $b/(r \times c)$ do not have proportional frequencies, they are orthogonal. All the block structures w/s, $r \times c$ and $b/(r \times c)$ we have seen so far are simple block structures and consist of pairwise orthogonal factors.

Now suppose there are t treatments in an experiment with block structure $\mathfrak{F}$. For each $i = 1, \ldots, N$, let y_i be the observation on the ith unit and $\mathcal{T}(i) \in \{1, \ldots, t\}$ be the treatment assigned to unit i, and for each $\mathcal{F} \in \mathfrak{F}$, let $\mathcal{F}(i) \in \{1, \ldots, n_{\mathcal{F}}\}$ be the $\mathcal{F}$-class to which unit i belongs. Then we assume that

$$y_i = \mu + \alpha_{\mathcal{T}(i)} + \sum_{\mathcal{F} \in \mathfrak{F}} \beta^{\mathcal{F}}_{\mathcal{F}(i)},$$

where μ and the treatment effects $\alpha_1, \ldots, \alpha_t$ are unknown constants, and $\{\beta^{\mathcal{F}}_1, \ldots, \beta^{\mathcal{F}}_{n_{\mathcal{F}}}\}_{\mathcal{F} \in \mathfrak{F}}$ are uncorrelated random variables with $\mathrm{E}(\beta^{\mathcal{F}}_k) = 0$ and $\mathrm{var}(\beta^{\mathcal{F}}_k) = \sigma^2_{\mathcal{F}}$. We further assume that $\sigma^2_{\mathcal{F}} > 0$ for all $\mathcal{F} \neq \mathcal{U}$. Allowing $\sigma^2_{\mathcal{U}} = 0$ covers the models that do not contain a random term corresponding to the mean.

Let $\boldsymbol{X}_{\mathcal{T}}$ be the $N \times t$ (0,1)-matrix with the (i,j)th entry equal to 1 if and only if the jth treatment is observed on the ith unit, and for each $\mathcal{F}$, let $\boldsymbol{X}_{\mathcal{F}}$ be the $N \times n_{\mathcal{F}}$ (0,1)-matrix in which the (i,j)th entry is equal to 1 if and only if the ith unit belongs to the jth $\mathcal{F}$-class. Then we can write $\boldsymbol{y} = (y_1, \ldots, y_N)^T$ as

$$\boldsymbol{y} = \mu \boldsymbol{1}_N + \boldsymbol{X}_{\mathcal{T}} \boldsymbol{\alpha} + \sum_{\mathcal{F} \in \mathfrak{F}} \boldsymbol{X}_{\mathcal{F}} \boldsymbol{\beta}^{\mathcal{F}}, \tag{2.4}$$

where $\boldsymbol{1}_N$ is the $N \times 1$ vector of 1's, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_t)^T$ and $\boldsymbol{\beta}^{\mathcal{F}} = (\beta^{\mathcal{F}}_1, \ldots, \beta^{\mathcal{F}}_{n_{\mathcal{F}}})^T$.

For simple block structures, Nelder (1965a) provided an algorithm for determining orthogonal projections of $\boldsymbol{y}$ onto the eigenspaces of $\mathrm{cov}(\boldsymbol{y})$, which is crucial for carrying out the ANOVA. Speed and Bailey (1982) and Tjur (1984) extended this result to more general block structures. We assume that $\mathfrak{F}$ satisfies the following.

(a) All the factors in $\mathfrak{F}$ are uniform and pairwise orthogonal. (2.5)

(b) $\mathcal{E} \in \mathfrak{F}$ and $\mathcal{U} \in \mathfrak{F}$. (2.6)

(c) $\mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{F} \Rightarrow \mathcal{F}_1 \vee \mathcal{F}_2 \in \mathfrak{F}$. (2.7)

For a block structure satisfying (2.5)-(2.7), $\mathrm{cov}(\boldsymbol{y})$ can be written in the spectral form

$$\mathrm{cov}(\boldsymbol{y}) = \sum_{\mathcal{F} \in \mathfrak{F}} \xi_{\mathcal{F}} \boldsymbol{P}_{W_{\mathcal{F}}}, \tag{2.8}$$

where $\{W_{\mathcal{F}}\}_{\mathcal{F} \in \mathfrak{F}}$ are mutually orthogonal eigenspaces of $\boldsymbol{V}$ such that $R^N = \oplus_{\mathcal{F} \in \mathfrak{F}} W_{\mathcal{F}}$, and $\boldsymbol{P}_{W_{\mathcal{F}}}$ is the orthogonal projection matrix onto $W_{\mathcal{F}}$. Furthermore,

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for each $\mathcal{F} \in \mathfrak{F}$,

$$W_{\mathcal{F}} = V_{\mathcal{F}} \ominus \left(\sum_{\mathcal{F} \prec \mathcal{G}} V_{\mathcal{G}} \right) \text{ and } \xi_{\mathcal{F}} = \sum_{\mathcal{H} \preceq \mathcal{F}} \frac{N}{n_{\mathcal{H}}} \sigma_{\mathcal{H}}^2. \tag{2.9}$$

Each eigenspace $W_{\mathcal{F}}$ is called a stratum, and $\xi_{\mathcal{F}}$ is called a stratum variance. It is easy to see that if $\boldsymbol{c} \in W_{\mathcal{F}}$, then $\mathrm{var}(\boldsymbol{c}^T \boldsymbol{y}) = (\boldsymbol{c}^T \boldsymbol{c}) \xi_{\mathcal{F}}$. It follows from (2.9) that

$$\mathcal{F}_1 \preceq \mathcal{F}_2 \Rightarrow \xi_{\mathcal{F}_1} \leq \xi_{\mathcal{F}_2}. \tag{2.10}$$

Then for any $\mathcal{F}$, since $\mathcal{E} \preceq \mathcal{F}$, we have $\xi_{\mathcal{E}} \leq \xi_{\mathcal{F}}$. We call $W_{\mathcal{E}}$ the bottom or unit stratum.

Let $V_{\mathcal{T}}$ be the column space of $\boldsymbol{X}_{\mathcal{T}}$ and $C_{\mathcal{T}}$ be $V_{\mathcal{T}} \ominus V_{\mathcal{U}}$. Then $\{\mathrm{E}(\boldsymbol{c}^T \boldsymbol{y})\}_{\boldsymbol{c} \in C_{\mathcal{T}}}$ are all the estimable treatment contrasts. We call a design orthogonal if $C_{\mathcal{T}}$ can be decomposed as $\oplus_i C_i$, such that each C_i is contained in a certain $W_{\mathcal{F}}$ (but different C_i's may be in different $W_{\mathcal{F}}$'s), and $\mathrm{E}(\boldsymbol{c}^T \boldsymbol{y}), \boldsymbol{c} \in C_i$, are contrasts of interest. Clearly none of these $W_{\mathcal{F}}$ can be $W_{\mathcal{U}}$, which we call the mean stratum. Under an orthogonal design, all the treatment contrasts $\mathrm{E}(\boldsymbol{c}^T \boldsymbol{y})$ with $\boldsymbol{c} \in C_i$ have simple estimators: $\boldsymbol{c}^T \boldsymbol{y}$ is the best linear unbiased estimator of $\mathrm{E}(\boldsymbol{c}^T \boldsymbol{y})$, with $\mathrm{var}(\boldsymbol{c}^T \boldsymbol{y}) = (\boldsymbol{c}^T \boldsymbol{c}) \xi_{\mathcal{F}}$. Also, $\mathrm{cov}(\boldsymbol{c}_1^T \boldsymbol{y}, \boldsymbol{c}_2^T \boldsymbol{y}) = 0$ if $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ belong to different strata. In factorial experiments, typically these are contrasts representing main effects and interactions or, in the case of fractional factorial designs, linear combinations of aliased factorial effects. In this paper, we focus on such orthogonal multistratum designs, under which each factorial effect is estimated in one single stratum.

Remark 2.1. *The proofs of (2.8) and (2.9) can be found in Tjur (1984). We mostly adopt the notation in Bailey (2008); in Tjur (1984), $V_{\mathcal{F}}$, $W_{\mathcal{F}}$, and $N/n_{\mathcal{F}}$ are denoted by $L_{\mathcal{F}}$, $V_{\mathcal{F}}$, and $n_{\mathcal{F}}$, respectively, uniform factors are called balanced factors, $\mathcal{F}_1 \preceq \mathcal{F}_2$ is denoted by $\mathcal{F}_1 \geq \mathcal{F}_2$, and therefore the supremum of factors defined here is called infimum there. Tjur (1984) does not require that $\mathcal{U} \in \mathfrak{F}$, but it can be dealt with by assuming $\sigma_{\mathcal{U}}^2 = 0$.*

3   Factorial experiments with multiple processing stages

Many industrial experiments involve a sequence of processing stages, where at each stage the experimental units are partitioned into disjoint classes, with those

in the same class assigned the same level of certain treatment factors. An experiment with s processing stages can be thought to have the block structure $\mathfrak{F} = \{\mathcal{U},$ $\mathcal{E}, \mathcal{F}_1, \ldots, \mathcal{F}_s\}$, where $\mathcal{F}_1, \ldots, \mathcal{F}_s$ define the partitions of the experimental units at the s stages. Model (2.4) with one set of random effects for each factor in $\mathfrak{F}$ is the same as that adopted by Mee and Bates (1998), Bingham et al (2008), and Ranjan et al (2009), except that they do not have a random term for $\mathcal{U}$. This, as commented in Remark 2.1, is the same as to assume that $\sigma_{\mathcal{U}}^2 = 0$ in (2.4).

If (2.5)-(2.7) hold for $\mathfrak{F}$, then there is a stratum associated with each factor in $\mathfrak{F}$, which can be determined by the rule given in (2.9). We have

$$W_{\mathcal{F}_i} = V_{\mathcal{F}_i} \ominus (\sum_{j:\mathcal{F}_i \prec \mathcal{F}_j} V_{\mathcal{F}_j}), \ \xi_{\mathcal{F}_i} = \sigma_{\mathcal{E}}^2 + \sum_{j:\mathcal{F}_j \preceq \mathcal{F}_i} \frac{N}{n_{\mathcal{F}_j}} \sigma_{\mathcal{F}_j}^2,$$

$$\dim(W_{\mathcal{F}_i}) = n_{\mathcal{F}_i} - 1 - \sum_{j:\mathcal{F}_i \prec \mathcal{F}_j} \dim(W_{\mathcal{F}_j}), \ i = 1, \ldots, s,$$

$$W_{\mathcal{E}} = \left[\sum_{i=1}^{s} V_{\mathcal{F}_i} \right]^{\perp}, \ \xi_{\mathcal{E}} = \sigma_{\mathcal{E}}^2, \ \dim(W_{\mathcal{E}}) = N - 1 - \sum_{i=1}^{s} \dim(W_{\mathcal{F}_i}).$$

In the special case where the factors $\mathcal{F}_1, \ldots, \mathcal{F}_s$ are not nested in one another, the above reduces to

$$W_{\mathcal{F}_i} = V_{\mathcal{F}_i} \ominus V_{\mathcal{U}} = C_{\mathcal{F}_i}, \ \xi_{\mathcal{F}_i} = \frac{N}{n_{\mathcal{F}_i}} \sigma_{\mathcal{F}_i}^2 + \sigma_{\mathcal{E}}^2, \ \dim(W_{\mathcal{F}_i}) = n_{\mathcal{F}_i} - 1, \ i = 1, \ldots, s,$$

$$W_{\mathcal{E}} = \left[\sum_{i=1}^{s} V_{\mathcal{F}_i} \right]^{\perp}, \ \xi_{\mathcal{E}} = \sigma_{\mathcal{E}}^2, \ \dim(W_{\mathcal{E}}) = N - 1 - \sum_{i=1}^{s} (n_{\mathcal{F}_i} - 1).$$

We show that this special case is equivalent to that $\mathcal{F}_1, \ldots, \mathcal{F}_s$ define an orthogonal array of strength two on the experimental units. Note that this refers to an orthogonal array with variable numbers of symbols as defined by Rao (1973) if $\mathcal{F}_1, \ldots, \mathcal{F}_s$ have different numbers of levels.

Proposition 3.1. *$\mathcal{F}_1, \ldots, \mathcal{F}_s$ define an orthogonal array of strength two on the units if and only if they are not nested in one another and $\mathfrak{F} = \{\mathcal{U}, \mathcal{E}, \mathcal{F}_1, \ldots, \mathcal{F}_s\}$ satisfies (2.5)-(2.7)*

Proof. Strength two of the array is equivalent to the uniformity and pairwise proportional frequencies of $\mathcal{F}_1, \ldots, \mathcal{F}_s$. Since pairwise proportional frequencies

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implies pairwise orthogonality and $\mathcal{F}_i \vee \mathcal{F}_j = \mathcal{U}$ for all $i \neq j$, if $\mathcal{F}_1, \ldots, \mathcal{F}_s$ define an orthogonal array of strength two on the units, then $\mathfrak{F}$ satisfies (2.5)-(2.7). Furthermore, pairwise proportional frequencies of $\mathcal{F}_1, \ldots, \mathcal{F}_s$ precludes any of them from nesting any others.

Conversely, suppose $\mathfrak{F}$ satisfies (2.5)-(2.7) and $\mathcal{F}_1, \ldots, \mathcal{F}_s$ are not nested in one another. Then by (2.7), $\mathcal{F}_i \vee \mathcal{F}_j \in \mathfrak{F}$ for all $i \neq j$. Since $\mathcal{F}_1, \ldots, \mathcal{F}_s$ are not nested in one another, we must have $\mathcal{F}_i \vee \mathcal{F}_j = \mathcal{U}$, and hence $V_{\mathcal{F}_i} \cap V_{\mathcal{F}_j} = V_{\mathcal{U}}$, for all $i \neq j$. This and the pairwise orthogonality of $\mathcal{F}_1, \ldots, \mathcal{F}_s$ imply that they have pairwise proportional frequencies and therefore define an orthogonal array of strength two on the units. $\square$

The split-lot designs studied in Mee and Bates (1998) fall in this category. For example, suppose 16 batches of material are to be partitioned into four groups of equal size at each of two stages. Then, as discussed in Section 2, one can arrange the 16 batches in four rows and four columns, and the grouping at the two stages can be done according to the rows and columns, respectively. This yields the simple block structure 4×4. If there are four stages, then the partitions at the two additional stages can be based on the Latin and Greek letters in the following Graeco-Latin square:

$A\alpha$	$B\beta$	$C\gamma$	$D\delta$
$B\gamma$	$A\delta$	$D\alpha$	$C\beta$
$C\delta$	$D\gamma$	$A\beta$	$B\alpha$
$D\beta$	$C\alpha$	$B\delta$	$A\gamma$

This is an example where the experimental units form a fraction of all possible combinations of the unit factors ($N < \prod_{i=1}^{s} n_{\mathcal{F}_i}$). The four factors, rows, columns, Latin letters and Greek letters, define an orthogonal array of strength two on the 16 units. It follows that in this case there are five strata in addition to the mean stratum: the bottom stratum and one stratum associated with each of the four unit factors. We note that Section 4 of Ranjan et al (2009) essentially addresses the existence and construction of orthogonal arrays of strength two.

Proposition 3.1 implies that if $\mathcal{F}_1, \ldots, \mathcal{F}_s$ are uniform, pairwise orthogonal, and are not nested in one another, but do not define an orthogonal array of

strength two, then (2.7) cannot hold, and hence (2.9) does not apply, causing complications in the design and analysis. This phenomenon arose in Bingham et al (2008) and Vivacqua and Bisgaard (2009).

In order to have enough degrees of freedom when using normal or half normal plots to judge significance of the effects estimated in the same stratum, Bingham et al (2008) advocated the partition of experimental units into large number of groups at each stage, which results in large $\dim(C_{\mathcal{F}_i})$. Geometrically, it may cause some of the $C_{\mathcal{F}_i}$'s to have nontrivial intersections. Bingham et al (2008) observed that if $C_{\mathcal{F}_i}$ and $C_{\mathcal{F}_j}$ have a nontrivial intersection, then for $\boldsymbol{a} \in C_{\mathcal{F}_i} \cap C_{\mathcal{F}_j}$, $\boldsymbol{a} \neq \boldsymbol{0}$, $\mathrm{var}(\boldsymbol{a}^T \boldsymbol{y})$ is greater than if $\boldsymbol{a} \in C_{\mathcal{F}_i}$ or $\boldsymbol{a} \in C_{\mathcal{F}_j}$ when $C_{\mathcal{F}_i} \cap C_{\mathcal{F}_j} = \{\boldsymbol{0}\}$. Combinatorially, having a large number of groups at each stage may make proportional frequencies not possible for some of the $\mathcal{F}_i$'s. For example, if we have to partition 32 experimental units into eight groups of size four at each of two stages then, since $8^2 > 32$, not all $\mathcal{F}_1$-classes can meet with every $\mathcal{F}_2$-class. As in Mee and Bates (1998), the 32 units also form a fraction of complete crossing of the two unit factors (rows and columns); an important difference, however, is that since there are only two unit factors in two-stage experiments, no proper fraction of their combinations can be an orthogonal array of strength two, and (2.7) would fail to hold.

Suppose, for example, the 32 units are the starred cells of the following 8×8 square, with the eight rows of size 4 constituting the first-stage groups, and the eight columns constituting the second-stage groups:

*	*	*	*				
*	*	*	*				
*	*	*	*				
*	*	*	*				
				*	*	*	*
				*	*	*	*
				*	*	*	*
				*	*	*	*

Figure 3.1 Incomplete crossing

We can see that both $\mathcal{F}_1$ and $\mathcal{F}_2$ are nested in a two-level factor that divides the

32 units into two "blocks" of size 16. Let this factor be $\mathcal{B}$. Then $\mathcal{F}_1 \vee \mathcal{F}_2 = \mathcal{B} \notin \mathfrak{F}$. Thus (2.7) does not hold.

In general, suppose $\mathcal{F}_1, \ldots, \mathcal{F}_s$ are uniform and pairwise orthogonal, but (2.7) does not hold. Then one can expand $\mathfrak{F} = \{\mathcal{U}, \mathcal{E}, \mathcal{F}_1, \ldots, \mathcal{F}_s\}$ by adding the supremums of existing factors in $\mathfrak{F}$, if needed, to a larger $\tilde{\mathfrak{F}}$ that satisfies (2.7). In other words, $\tilde{\mathfrak{F}}$ is the smallest block structure with $\mathfrak{F} \subset \tilde{\mathfrak{F}}$ that satisfies (2.7). In general, if $\cap_{j=1}^{k} C_{\mathcal{F}_{i_j}} \neq \{\mathbf{0}\}$, then $\vee_{j=1}^{k} \mathcal{F}_{i_j} \neq \mathcal{U}$, and we need to include $\vee_{j=1}^{k} \mathcal{F}_{i_j}$ in $\tilde{\mathfrak{F}}$, if it is not already in $\mathfrak{F}$. Such an $\tilde{\mathfrak{F}}$ still has pairwise orthogonal factors since it can be shown that if $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ are pairwise orthogonal, then $\mathcal{F}_1 \vee \mathcal{F}_2$ is orthogonal to each of $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$; see Corollary 10.4 of Bailey (2008). There is no guarantee that the added factors are uniform, but if all the added factors are indeed uniform, then (2.5)-(2.7) hold for $\tilde{\mathfrak{F}}$. We call the added factors pseudo factors since they are not in the originally intended block structure. Model (2.4) for $\tilde{\mathfrak{F}}$ remains the same as the original one for $\mathfrak{F}$ as long as we make $\sigma_{\mathcal{F}}^2 = 0$ for all the added pseudo factors. Then it follows that there is one stratum for each factor in $\tilde{\mathfrak{F}}$, and (2.9) can be applied to $\tilde{\mathfrak{F}}$ to determine the strata and stratum variances. We have

$$\xi_{\mathcal{F}_{i_1} \vee \cdots \vee \mathcal{F}_{i_k}} > \xi_{\mathcal{F}_{i_j}}, \text{ for all } j = 1, \ldots, k. \tag{3.1}$$

If $\mathfrak{F}$ satisfies (2.7), in particular, if $\mathcal{F}_1, \ldots, \mathcal{F}_s$ define an orthogonal array of strength two, then for all the factors whose levels are assigned at the ith stage, their main effects and interactions are estimated in stratum $W_{\mathcal{F}_i}$. On the other hand, some of these effects may be estimated in $W_{\mathcal{F}_{i_1} \vee \cdots \vee \mathcal{F}_{i_k}}$, resulting in larger variances if $\mathfrak{F}$ does not satisfy (2.7), and $\mathcal{F}_{i_1} \vee \cdots \vee \mathcal{F}_{i_k}$ is a pseudo factor in $\tilde{\mathfrak{F}}$ with $i = i_j$ for some j.

In Section 3 of Ranjan et al (2009), analysis of designs when some of the $C_{\mathcal{F}_i}$'s have nontrivial intersections was derived for the case where the $n_{\mathcal{F}_i}$'s are powers of 2, and it was commented that the result could be extended to the case where the $n_{\mathcal{F}_i}$'s are powers of prime numbers. This can be seen to follow from the general theory presented in Section 2, as sketched in the previous paragraph. It suffices to show that the factors $\mathcal{F}_1, \ldots, \mathcal{F}_s$ in their setting satisfy (2.5), and that all the added pseudo factors are uniform. Since the partition of the treatment combinations into disjoint classes at each stage, as done in Ranjan et al (2009), is based on parallel flats in a finite Euclidean geometry, it can be seen that

the resulting unit factors $\mathcal{F}_1, \ldots, \mathcal{F}_s$ for the s stages are uniform and pairwise orthogonal. Therefore (2.5) is satisfied. It is also a consequence of the geometric construction that the disjoint classes induced by $\vee_{j=1}^k \mathcal{F}_{i_j}$ are parallel flats, and hence are of the same size, i.e., $\vee_{j=1}^k \mathcal{F}_{i_j}$ is uniform.

The approach presented here applies to more general situations, including the case where nesting exists among $\mathcal{F}_1, \ldots, \mathcal{F}_s$, as well as the case where the $n_{\mathcal{F}_i}$'s are not necessarily prime powers. We have also related the extra stratum variances to the presence of pseudo factors. This provides a better understanding in the combinatorial context and is useful for design construction.

For the block structure in Figure 3.1, the two pseudo blocks of size 16 create an extra stratum with larger variance than both $\xi_{\mathcal{F}_1}$ and $\xi_{\mathcal{F}_2}$. Formally the block structure, with the two-level pseudo factor included, is the same as the block structure $2/(4 \times 4)$ in Figure 2.1. In general, if there are 2^m units that are to be divided into 2^u groups of size 2^{m-u} at the first stage and 2^v groups of size 2^{m-v} at the second stage, where $u + v > m$, and there are 2^f pseudo blocks of equal size, then we must have $f = u + v - m$. With the pseudo factor included, the block structure is the same as $2^{u+v-m}/(2^{m-v} \times 2^{m-u})$.

Vivacqua and Bisgaard (2009) considered two-stage experiments with a 2^{k-p} row design at the first stage and a 2^{q-r} column design at the second stage, but instead of complete crossing of the row and column designs, to save experimental effort, only a $\frac{1}{2^f}$-fraction is observed. The fraction is defined by f independent generators, called post-fraction generators. It is equivalent to a design in the Bingham et al (2008) setting in which $2^{k-p+q-r-f}$ units are to be divided into 2^{k-p} groups at the first stage and 2^{q-r} groups at the second stage. Therefore there is an extra stratum with larger variance resulting from a 2^f-level pseudo factor. Each of the f post-fraction generators aliases an effect involving only the first-stage treatment factors with an effect involving only the second-stage factors. These aliased effects are confounded with the pseudo blocks, and the f effects involving only the first-stage treatment factors (or the f effects involving only the second-stage factors) together define the 2^f pseudo blocks. Again, with this pseudo factor included, the structure is the same as $2^f/(2^{k-p-f} \times 2^{q-r-f})$. Indeed Vivacqua and Bisgaard (2009) pointed out that the design can be executed as a set of 2^f blocks.

Therefore the construction and selection of two-stage designs in the settings of Vivacqua and Bisgaard (2009), Bingham et al (2008), and Miller (1997) are equivalent. In Section 5, we apply the optimality criterion formulated in the next section to search for good designs and compare them with those tabulated in Vivacqua and Bisgaard (2009).

Remark 3.1. *Under a simple block structure $\mathfrak{F}$, for any two crossing factors $\mathcal{F}_1$ and $\mathcal{F}_2$, $\mathcal{F}_1 \wedge \mathcal{F}_2$ is also included in $\mathfrak{F}$, where $\mathcal{F}_1 \wedge \mathcal{F}_2$ is the coarsest factor that is nested in both $\mathcal{F}_1$ and $\mathcal{F}_2$. This was also done in Paniagua-Quiñones and Box's (2008, 2009) model for three processing-stage experiments. Discussions in this section still apply with simple modifications.*

4 Optimality criterion

For regular designs with unstructured units, Cheng, Steinberg and Sun (1999) showed that minimum aberration is a good surrogate for the model-robustness criterion of maximizing the number of estimable models containing all the main effects and a given number of two-factor interactions. In the multistratum situation, different estimable models may have different efficiencies. In the spirit of Sun (1993) and Tsai, Gilmour, and Mead (2000), we use the average efficiency over a set of candidate models to measure the performance of a multistratum design, as Cheng and Tsai (2009) did for block and split-plot designs. We take the candidate models to be those which contain all the main effects and k two-factor interactions. If the main effect of a factor must be estimated in a designated stratum (for example, in a split-plot experiment, the main effect of a whole-plot factor must be estimated in the whole-plot stratum), then the efficiency of its estimator is fixed. When a main effect is not required to be estimated in a specific stratum, due to the hierarchical principle, we prefer that it be estimated in the bottom stratum to attain the smallest possible variance; then the efficiency of its estimator is also fixed. Thus it is enough to consider the efficiencies of the estimators of two-factor interactions, measured by $D^{1/k}$, where D is the determinant of the information matrix for the two-factor interactions. We normalize the interaction contrasts so that their estimators are of the form $\boldsymbol{c}^T \boldsymbol{y}$ with $\boldsymbol{c}^T \boldsymbol{c} = 1$. For a given design d we define the information capacity $I_k(d)$ as the average of

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$D^{1/k}$ over the candidate models. The objective is to find a design with large $I_k(d)$.

For simplicity only two-level designs are considered here. Extension to the case where the number of levels is a prime number or power of a prime number is straightforward. Suppose $N = 2^{n-p}$, where n is the number of treatment factors. Then there are a total of $2^{n-p} - 1$ alias sets, $2^{n-p} - 1 - n$ of which do not contain main effects. Suppose the effects in $h_{\mathcal{F}}$ of these alias sets are estimated in stratum $W_{\mathcal{F}}$. For each $1 \leq i \leq h_{\mathcal{F}}$, let $m_i^{\mathcal{F}}$ be the number of two-factor interactions contained in the ith of these alias sets. Then an argument similar to that in Cheng and Tsai (2009) shows that $I_k(d)$ is a Schur concave function of the vector whose components consist of $\{\xi_{\mathcal{F}}^{-1/k} m_i^{\mathcal{F}} : 1 \leq i \leq h_{\mathcal{F}}, \mathcal{F} \in \mathfrak{F}, \mathcal{F} \neq \mathcal{U}\}$. Since $I_k(d)$ is increasing in each of these components, a good surrogate for maximizing $I_k(d)$ is to

$$\text{maximize} \sum_{\mathcal{F} \in \mathfrak{F}, \mathcal{F} \neq \mathcal{U}} \xi_{\mathcal{F}}^{-1/k} \sum_{i=1}^{h_{\mathcal{F}}} m_i^{\mathcal{F}}, \tag{4.1}$$

and

$$\text{minimize} \sum_{\mathcal{F} \in \mathfrak{F}, \mathcal{F} \neq \mathcal{U}} \xi_{\mathcal{F}}^{-2/k} \sum_{i=1}^{h_{\mathcal{F}}} [m_i^{\mathcal{F}}]^2 \text{ among those which maximize} \sum_{\mathcal{F} \in \mathfrak{F}, \mathcal{F} \neq \mathcal{U}} \xi_{\mathcal{F}}^{-1/k} \sum_{i=1}^{h_{\mathcal{F}}} m_i^{\mathcal{F}}.$$
$$\tag{4.2}$$

We use the two-stage criterion (4.1) and (4.2) in lieu of the maximization of $I_k(d)$. The study carried out in Cheng and Tsai (2009) for blocked fractional factorial designs showed that the rankings given by the information capacity criterion and its surrogate such as (4.1) and (4.2) are quite consistent.

The following theorem provides a useful sufficient condition for a design to be optimal with respect to (4.1) and (4.2) for all possible stratum variances.

Theorem 4.1. *A sufficient condition for a design d to be optimal with respect to (4.1) and (4.2) for all k and all $\{\xi_{\mathcal{F}}\}_{\mathcal{F} \in \mathfrak{F}, \mathcal{F} \neq \mathcal{U}}$ satisfying (2.10) is that for all subsets $\mathfrak{G}$ of $\mathfrak{F} \backslash \{\mathcal{U}\}$ such that*

$$\mathcal{F} \in \mathfrak{G} \text{ and } \mathcal{F}' \prec \mathcal{F} \Rightarrow \mathcal{F}' \in \mathfrak{G}, \tag{4.3}$$

(i) d maximizes $\sum_{\mathcal{F} \in \mathfrak{G}} \sum_{i=1}^{h_{\mathcal{F}}} m_i^{\mathcal{F}}$, and
(ii) d minimizes $\sum_{\mathcal{F} \in \mathfrak{G}} \sum_{i=1}^{h_{\mathcal{F}}} [m_i^{\mathcal{F}}]^2$ among those which maximize $\sum_{\mathcal{F} \in \mathfrak{G}} \sum_{i=1}^{h_{\mathcal{F}}} m_i^{\mathcal{F}}$.

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Theorem 4.1 can be used to verify the strong optimality with respect to all possible values of the stratum variances. It can also be used to eliminate inferior designs. We say that d_1 dominates another design d_2 if d_1 is at least as good as d_2 with respect to (4.1) and (4.2) for all possible stratum variances, and is better than d_2 for some stratum variances. A design that is dominated by another design is said to be inadmissible. If d_1 is better than d_2 with respect to (i) and (ii) in Theorem 4.1, then d_1 dominates d_2, and d_2 can be ruled out from consideration. This provides a way of eliminating inadmissible designs without knowing the stratum variances. Then the admissible designs can be compared by using available knowledge about stratum variances or other relevant information. In the next section we illustrate applications of this tool by using it to study optimal designs for experiments with two processing stages in the settings of Vivacqua and Bisgaard (2009), Bingham et al (2008), and Miller (1997). The proof of Theorem 4.1 is presented in the appendix.

We say that two designs are equivalent if they have the same $m_i^{\mathcal{F}}$ values for each $\mathcal{F} \in \mathfrak{F}$, $\mathcal{F} \neq \mathcal{U}$. Equivalent designs have the same performance under the information capacity criterion with respect to all possible stratum variances.

5   Optimal designs for experiments with two processing stages

Denote the five unit factors in a blocked strip-plot experiment with block structure $b/(r * c)$ by $\mathcal{U}, \mathcal{R}', \mathcal{C}', \mathcal{B}$, and $\mathcal{E}$. Then by (2.9), there are four strata other than the mean stratum:

$$W_{\mathcal{B}} = V_{\mathcal{B}} \ominus V_{\mathcal{U}}, \ W_{\mathcal{R}'} = V_{\mathcal{R}'} \ominus V_{\mathcal{B}}, \ W_{\mathcal{C}'} = V_{\mathcal{C}'} \ominus V_{\mathcal{B}}, \ W_{\mathcal{E}} = (V_{\mathcal{R}'} + V_{\mathcal{C}'})^{\perp},$$

with variances

$$\xi_{\mathcal{B}} = rc\sigma_{\mathcal{B}}^2 + c\sigma_{\mathcal{R}'}^2 + r\sigma_{\mathcal{C}'}^2 + \sigma_{\mathcal{E}}^2, \ \xi_{\mathcal{R}'} = c\sigma_{\mathcal{R}'}^2 + \sigma_{\mathcal{E}}^2, \ \xi_{\mathcal{C}'} = r\sigma_{\mathcal{C}'}^2 + \sigma_{\mathcal{E}}^2, \ \xi_{\mathcal{E}} = \sigma_{\mathcal{E}}^2.$$

The main effects of the first-stage (row) treatment factors are estimated in the row stratum $W_{\mathcal{R}'}$ and those of the second-stage (column) factors are estimated in the column stratum $W_{\mathcal{C}'}$. To apply Theorem 4.1, we need to verify conditions (i) and (ii) in the theorem for $\mathfrak{G} = \{\mathcal{B}, \mathcal{R}', \mathcal{C}', \mathcal{E}\}, \{\mathcal{R}', \mathcal{C}', \mathcal{E}\}, \{\mathcal{R}', \mathcal{E}\},$

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$\{\mathcal{C}', \mathcal{E}\}$ and $\{\mathcal{E}\}$. Maximizing $\sum_{i=1}^{h_{\mathcal{B}}} m_i^{\mathcal{B}} + \sum_{i=1}^{h_{\mathcal{R}'}} m_i^{\mathcal{R}'} + \sum_{i=1}^{h_{\mathcal{C}'}} m_i^{\mathcal{C}'} + \sum_{i=1}^{h_{\mathcal{E}}} m_i^{\mathcal{E}}$ is the same as maximizing the number of two-factor interactions that are not aliased with main effects; following this by minimizing $\sum_{i=1}^{h_{\mathcal{B}}} [m_i^{\mathcal{B}}]^2 + \sum_{i=1}^{h_{\mathcal{R}'}} [m_i^{\mathcal{R}'}]^2 + \sum_{i=1}^{h_{\mathcal{C}'}} [m_i^{\mathcal{C}'}]^2 + \sum_{i=1}^{h_{\mathcal{E}}} [m_i^{\mathcal{E}}]^2$ assures that these two-factor interactions are distributed over the alias sets as uniformly as possible. These two steps are expected to produce a good overall design (for unstructured units); see Cheng et al (1999). Doing the same thing for $\mathfrak{G} = \{\mathcal{R}', \mathcal{C}', \mathcal{E}\}$ produces a design that maximizes the number of two-factor interactions that are neither aliased with main effects nor confounded with blocks, and these interactions are also distributed over the alias sets as uniformly as possible. This is expected to produce a good blocked design. Similarly, maximizing $\sum_{i=1}^{h_{\mathcal{R}'}} m_i^{\mathcal{R}'} + \sum_{i=1}^{h_{\mathcal{E}}} m_i^{\mathcal{E}}$ (respectively, $\sum_{i=1}^{h_{\mathcal{C}'}} m_i^{\mathcal{C}'} + \sum_{i=1}^{h_{\mathcal{E}}} m_i^{\mathcal{E}}$) is the same as maximizing the number of two-factor interactions that are not aliased with main effects and are confounded with neither blocks nor columns (respectively, rows). Following with the minimization of sum of squares of the m_i values produces a good blocked column (respectively, row) designs. Applying the procedure to $\{\mathcal{E}\}$ assures a good design in the bottom stratum. This has a similar flavor to the Trinca and Gilmour (2001) stratum-to-stratum construction.

As discussed in Section 3, the procedure described in the previous paragraph also produces good designs under our criterion for the problems considered in Bingham et al (2008) and Vivacqua and Bisgaard (2009), with $\mathcal{F}_1$, $\mathcal{F}_2$, and the added pseudo factors corresponding to $\mathcal{R}', \mathcal{C}'$, and $\mathcal{B}$, respectively, provided that we impose the requirement that main effects of all the treatment factors are not confounded with pseudo blocks. In the rest of the paper, we compare our findings with the designs tabulated in Vivacqua and Bisgaard (2009).

Following the notation in Vivacqua and Bisgaard (2009), we use (k, q, p, r, f) to label the case with a 2^{k-p} row design at the first stage, a 2^{q-r} column design at the second stage, and a $\frac{1}{2^f}$-fraction of their product to be observed. The corresponding blocked strip-plot design has $b = 2^f$, $r' = 2^{k-p-f}$ and $c' = 2^{q-r-f}$.

For example, for $k = 2$, $q = 7$, $p = 0$, $r = 3$, and $f = 1$, the first-stage design is a 2^2 complete factorial, the second-stage design is a 2^{7-3} factorial, and the corresponding blocked strip-plot design has 2 rows and 8 columns nested in each of two blocks. In this case, $W_{\mathcal{B}}$, $W_{\mathcal{R}'}$, $W_{\mathcal{C}'}$, and $W_{\mathcal{E}}$ have 1, 2, 14, and 14 degrees of freedom, respectively. Let the first-stage treatment factors be A and

B, and the second-stage factors be N, O, P, Q, R, S, and T. For the design in Table 8 of Vivacqua and Bisgaard (2009), the second-stage design is defined by $R = NOP$, $S = OPQ$, and $T = NPQ$, and the post-fraction generator is $AB = NOPQ$. Let this design be d_1. Consider a design d_2 with the second-stage design also defined by $R = NOP$, $S = OPQ$, $T = NPQ$, but with a different post-fraction generator $AB = NOQ$. Then the following are the m_i values of the two designs in the four strata $W_{\mathcal{B}}$, $W_{\mathcal{R}'}$, $W_{\mathcal{C}'}$ and $W_{\mathcal{E}}$:

	d_1	d_2
$W_{\mathcal{B}}$	4	1
$W_{\mathcal{R}'}$		
$W_{\mathcal{C}'}$	3 3 3 3 3 3 0	3 3 3 3 3 3 3
$W_{\mathcal{E}}$	2 2 2 2 2 2 1 1 0 0 0 0 0 0	1 1 1 1 1 1 1 1 1 1 1 1 1 1

Note that for both designs there is no entry in the row stratum since the two-factor interaction AB is confounded with the pseudo blocks. The values of $\sum_{i=1}^{h_{\mathcal{B}}} m_i^{\mathcal{B}} + \sum_{i=1}^{h_{\mathcal{R}'}} m_i^{\mathcal{R}'} + \sum_{i=1}^{h_{\mathcal{C}'}} m_i^{\mathcal{C}'} + \sum_{i=1}^{h_{\mathcal{E}}} m_i^{\mathcal{E}}$, $\sum_{i=1}^{h_{\mathcal{R}'}} m_i^{\mathcal{R}'} + \sum_{i=1}^{h_{\mathcal{C}'}} m_i^{\mathcal{C}'} + \sum_{i=1}^{h_{\mathcal{E}}} m_i^{\mathcal{E}}$, $\sum_{i=1}^{h_{\mathcal{R}'}} m_i^{\mathcal{R}'} + \sum_{i=1}^{h_{\mathcal{E}}} m_i^{\mathcal{E}}$, $\sum_{i=1}^{h_{\mathcal{C}'}} m_i^{\mathcal{C}'} + \sum_{i=1}^{h_{\mathcal{E}}} m_i^{\mathcal{E}}$, and $\sum_{i=1}^{h_{\mathcal{E}}} m_i^{\mathcal{E}}$ are 36, 32, 14, 32 and 14, respectively for d_1 and 36, 35, 14, 35, 14, respectively, for d_2. We see that d_2 beats d_1 for $\sum_{i=1}^{h_{\mathcal{R}'}} m_i^{\mathcal{R}'} + \sum_{i=1}^{h_{\mathcal{C}'}} m_i^{\mathcal{C}'} + \sum_{i=1}^{h_{\mathcal{E}}} m_i^{\mathcal{E}}$ and $\sum_{i=1}^{h_{\mathcal{C}'}} m_i^{\mathcal{C}'} + \sum_{i=1}^{h_{\mathcal{E}}} m_i^{\mathcal{E}}$, and the two designs are tied for the other three sums. For these three sums, the corresponding sums of squares of the m_i's are 96, 26, 26, respectively, for d_1 and 78, 14, 14, respectively, for d_2. In all three cases, d_2 beats d_1. Therefore d_2 is a better design and d_1 is inadmissible.

Both designs are of resolution IV. The m_i values show that both of them have 14 two-factor interactions estimated in the bottom stratum, but unlike d_1, under d_2 these 14 two-factor interactions are distributed uniformly over the 14 alias sets. Among the other 22 two-factor interactions, four are confounded with the pseudo blocks under d_1, whereas there is only one such two-factor interaction under d_2. In view of (2.10), d_2 is clearly a better design. We also point out that there are 15 clear two-factor interactions under d_2, and there are only two under d_1. Our criterion successfully identifies d_2 as a better design. In fact, d_2 is the unique admissible design (up to equivalence), and therefore is optimal for $k = 2$,

$q = 7$, $p = 0$, $r = 3$ and $f = 1$.

Miller (1997) constructed a blocked strip-plot design with 6 first-stage and 4 second-stage treatment factors for the block structure $2/(4 \times 4)$. It is the same as a design with $(k, q, p, r, f) = (6, 4, 3, 1, 1)$ in Vivacqua and Bisgaard (2009)'s setting. If we switch rows and columns, then it is the design with $(k, q, p, r, f) = (4, 6, 1, 3, 1)$ in Table 8 of Vivacqua and Bisgaard (2009). Let the treatment factors for the row design be A, B, C, D, and the factors for the column design be N, O, P, Q, R, S, respectively. Call this design d_3, which is defined by $D = ABC$, $Q = NO$, $R = NP$, and $S = NOP$, with the post-fraction generator $AB = OP$. In this case, we found that d_3 is one of two admissible designs (up to equivalence). The other admissible design, denoted by d_4, is defined by $D = AC$, $Q = NO$, $R = NP$, and $S = NOP$, with the post-fraction generator $AB = OP$. Neither design dominates the other in all situations.

The following are the m_i values for d_3 and d_4:

	d_3	d_4
$W_{\mathcal{B}}$	5	4
$W_{\mathcal{R}'}$	2 2	1 1
$W_{\mathcal{C}'}$		
$W_{\mathcal{E}}$	2 2 2 2 2 2 2 2 2 2 2 2 0 0 0 0 0 0	2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1

Note that for both designs there is no entry in the column stratum since each two-factor interaction in the column stratum is aliased with a main effect. Design d_3 has more (33) two-factor interactions that are not aliased with main effects than d_4 (30). The two designs have the same number (24) of two-factor interactions estimated in the bottom stratum, but the aliasing of these interactions is less severe under d_4. Thus d_3 is a better overall design for unstructured units, as well as when $\xi_{\mathcal{E}}$ is not too small compared with $\xi_{\mathcal{R}'}$. We expect d_4 to be better if $\xi_{\mathcal{E}}$ is substantially smaller than $\xi_{\mathcal{R}'}$. We also point out that there are 14 clear two-factor interactions under d_4 while there are none under d_3.

Among the 43 cases of 32-run designs in Table 8 of Vivacqua and Bisgaard (2009), 17 have some main effects confounded with pseudo blocks. In five of these cases, we found the following designs that do not confound main effects with

pseudo blocks, and are unique admissible (optimal) designs up to equivalence.

k	q	p	r	f	fractional factorial generators	post-fraction generators
3	4	0	0	2		$AB = NO, AC = NPQ$
3	5	0	2	1	$Q = NOP, R = NP$	$ABC = NO$
4	4	0	0	3		$AB = NO, AC = NP, AD = OQ$
4	5	0	1	3	$R = OPQ$	$AB = NO, AC = NP, AD = OQ$
5	5	1	1	3	$E = ABC, R = NOQ$	$AB = NO, AC = NP, AD = NQ$

There is one case, $(k, q, p, r, f) = (3, 4, 1, 0, 1)$, where the design in the table has a 2^{4-1} column design instead of a 2^4 complete factorial as suggested by the parameter values. In this case one cannot avoid confounding a main effect with pseudo blocks without fractionating the column design. Among the other 25 cases, with equivalent designs considered as the same, there is only one case with four admissible designs and one case with three admissible designs; all the others have at most two admissible designs, in twelve of which the designs in the table are the unique admissible designs. In addition to the case $k = 2$, $q = 7$, $p = 0$, $r = 3$ and $f = 1$ discussed earlier, we found four other cases where the designs in the table are inadmissible: $(k, q, p, r, f) = (2, 5, 0, 1, 1)$, $(3, 5, 0, 1, 2)$, $(4, 6, 0, 2, 3)$, $(5, 6, 1, 2, 3)$. In each of these cases, there is a unique admissible (optimal) design up to equivalence. The following are the defining relations and post-fraction generators for the optimal designs we found for $(k, q, p, r, f) = (2, 5, 0, 1, 1)$, $(3, 5, 0, 1, 2)$, $(2, 7, 0, 3, 1)$, $(4, 6, 0, 2, 3)$, $(5, 6, 1, 2, 3)$:

k	q	p	r	f	fractional factorial generators	post-fraction generators
2	5	0	1	1	$R = OPQ$	$AB = NOP$
3	5	0	1	2	$R = NOP$	$AB = NO, AC = NPQ$
2	7	0	3	1	$R = NOP, S = OPQ, T = NPQ$	$AB = NOQ$
4	6	0	2	3	$R = OPQ, S = NPQ,$	$AB = NO, AC = NP, AD = OQ$
5	6	1	2	3	$E = ACD, R = NOQ, S = NPQ$	$AB = NQ, AC = NO, AD = NP$

Butler (2004) constructed some designs for two, three, and four processing stages. All the 21 designs for two processing stages in his Table 1 are admissible. Except for one 32-run case and two 64-run cases, the designs he

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constructed are optimal. There are three admissible designs (up to equivalence) for $(b, r, c, n_R, n_C) = (2, 4, 4, 4, 4)$, and two admissible designs each for $(b, r, c, n_R, n_C) = (2, 8, 4, 5, 2)$ and $(2, 8, 4, 5, 3)$, where n_R is the number of row (first-stage) treatment factors, and n_C is the number of column (second-stage) treatment factors. Note that in Butler (2004)'s notations, $m_1 = n_R$, $m_2 = n_C$, $n_1 = br$, and $n_2 = bc$.

6 Concluding remarks

We have shown how the theory of randomized experiments with orthogonal block structures developed by Nelder, Speed, Bailey, and Tjur can be used to study design of experiments with multiple processing stages. In particular, it provides a clear understanding of the issue of extra error strata arising from pseudo unit factors. We also propose an optimality criterion for selecting multistratum fractional factorial designs. Using our criterion, which takes stratum variances into account, we are able to find better designs than those available in the literatures in some cases, and verify the optimality of the available designs or identify additional admissible designs as possible alternatives in the others.

Appendix. Proof of Theorem 4.1

We prove the more general result that for any subset $\mathfrak{F}'$ of $\mathfrak{F} \backslash \{\mathcal{U}\}$, if for all subsets $\mathfrak{G}$ of $\mathfrak{F}'$ such that

$$\mathcal{F} \in \mathfrak{G}, \mathcal{F}' \in \mathfrak{F}' \text{ and } \mathcal{F}' \prec \mathcal{F} \Rightarrow \mathcal{F}' \in \mathfrak{G}, \tag{A.1}$$

$$d \text{ maximizes } \sum_{\mathcal{F} \in \mathfrak{G}} \sum_{i=1}^{h_{\mathcal{F}}} m_i^{\mathcal{F}}, \tag{A.2}$$

$$d \text{ minimizes } \sum_{\mathcal{F} \in \mathfrak{G}} \sum_{i=1}^{h_{\mathcal{F}}} [m_i^{\mathcal{F}}]^2 \text{ among those which maximize } \sum_{\mathcal{F} \in \mathfrak{G}} \sum_{i=1}^{h_{\mathcal{F}}} m_i^{\mathcal{F}}, \tag{A.3}$$

then

$$d \text{ maximizes } \sum_{\mathcal{F} \in \mathfrak{F}'} u_{\mathcal{F}} \sum_{i=1}^{h_{\mathcal{F}}} m_i^{\mathcal{F}}, \tag{A.4}$$

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and

$$d \text{ minimizes } \sum_{\mathcal{F} \in \mathfrak{F}'} v_\mathcal{F} \sum_{i=1}^{h_\mathcal{F}} [m_i^\mathcal{F}]^2 \text{ among those maximizing } \sum_{\mathcal{F} \in \mathfrak{F}'} u_\mathcal{F} \sum_{i=1}^{h_\mathcal{F}} m_i^\mathcal{F},$$
(A.5)

for all $\{u_\mathcal{F}\}_{\mathcal{F} \in \mathfrak{F}'}$ and $\{v_\mathcal{F}\}_{\mathcal{F} \in \mathfrak{F}'}$ satisfying

$$u_\mathcal{F} > 0, v_\mathcal{F} > 0,$$
(A.6)

$$\mathcal{F}' \preceq \mathcal{F} \Rightarrow u_\mathcal{F} \leq u_{\mathcal{F}'},$$
(A.7)

$$u_\mathcal{F} < u_{\mathcal{F}'} \Leftrightarrow v_\mathcal{F} < v_{\mathcal{F}'}.$$
(A.8)

Theorem 4.1 then follows by choosing $\mathfrak{F}' = \mathfrak{F} \backslash \{\mathcal{U}\}$, $u_\mathcal{F} = \xi_\mathcal{F}^{-1/k}$, and $v_\mathcal{F} = \xi_\mathcal{F}^{-2/k}$.

We prove this assertion by induction on the number of factors in $\mathfrak{F}'$. The claim is obviously true if $\mathfrak{F}'$ consists of one single factor. Under the hypothesis that it holds whenever $\mathfrak{F}'$ has fewer than s factors, $s \geq 2$, we prove that it is true if $\mathfrak{F}'$ contains s factors.

Suppose (A.2) and (A.3) hold for all the subsets $\mathfrak{G}$ of $\mathfrak{F}'$ that satisfy (A.1). Given $\{u_\mathcal{F}\}_{\mathcal{F} \in \mathfrak{F}'}$ and $\{v_\mathcal{F}\}_{\mathcal{F} \in \mathfrak{F}'}$ for which (A.6), (A.7), and (A.8) hold, we need to show (A.4) and (A.5). Let $\mathcal{F}_0$ have the smallest $u_\mathcal{F}$ among all the $\mathcal{F}$'s in $\mathfrak{F}'$. Then by (A.8), it also has the smallest $v_\mathcal{F}$. Let $u_0 = u_{\mathcal{F}_0}$, $v_0 = v_{\mathcal{F}_0}$, and $\mathfrak{F}^* = \{\mathcal{F} \in \mathfrak{F}' : u_\mathcal{F} > u_0\}$.

If $\mathfrak{F}^*$ is empty, then all the $\mathcal{F}$'s in $\mathfrak{F}'$ have the same $u_\mathcal{F}$ and $v_\mathcal{F}$. In this case (A.4) and (A.5) are obtained by applying (A.2) and (A.3) to $\mathfrak{G} = \mathfrak{F}'$.

On the other hand, if $\mathfrak{F}^*$ is nonempty, then it contains fewer than s factors. For each $\mathcal{F} \in \mathfrak{F}^*$, let $u_\mathcal{F}^* = u_\mathcal{F} - u_0$ and $v_\mathcal{F}^* = v_\mathcal{F} - v_0$. Then (A.6), (A.7), and (A.8) hold for $\{u_\mathcal{F}^*\}_{\mathcal{F} \in \mathfrak{F}^*}$ and $\{v_\mathcal{F}^*\}_{\mathcal{F} \in \mathfrak{F}^*}$. Furthermore,

$$\sum_{\mathcal{F} \in \mathfrak{F}'} u_\mathcal{F} \sum_{i=1}^{h_\mathcal{F}} m_i^\mathcal{F} = \sum_{\mathcal{F} \in \mathfrak{F}^*} u_\mathcal{F}^* \sum_{i=1}^{h_\mathcal{F}} m_i^\mathcal{F} + \sum_{\mathcal{F} \in \mathfrak{F}'} u_0 \sum_{i=1}^{h_\mathcal{F}} m_i^\mathcal{F},$$
(A.9)

$$\sum_{\mathcal{F} \in \mathfrak{F}'} v_\mathcal{F} \sum_{i=1}^{h_\mathcal{F}} [m_i^\mathcal{F}]^2 = \sum_{\mathcal{F} \in \mathfrak{F}^*} v_\mathcal{F}^* \sum_{i=1}^{h_\mathcal{F}} [m_i^\mathcal{F}]^2 + \sum_{\mathcal{F} \in \mathfrak{F}'} v_0 \sum_{i=1}^{h_\mathcal{F}} [m_i^\mathcal{F}]^2.$$
(A.10)

We claim that for all subsets $\mathfrak{G}$ of $\mathfrak{F}^*$, the following condition implies (A.1):

$$\mathcal{F} \in \mathfrak{G}, \mathcal{F}' \in \mathfrak{F}^* \text{ and } \mathcal{F}' \prec \mathcal{F} \Rightarrow \mathcal{F}' \in \mathfrak{G}.$$
(A.11)

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If this claim is true, then by assumption, (A.2) and (A.3) hold for all subsets $\mathfrak{G}$ of $\mathfrak{F}^*$ that satisfy (A.11). Since $\mathfrak{F}^*$ contains fewer than s factors by the induction hypothesis,

$$d \text{ maximizes } \sum_{\mathcal{F} \in \mathfrak{F}^*} u_{\mathcal{F}}^* \sum_{i=1}^{h_{\mathcal{F}}} m_i^{\mathcal{F}}, \tag{A.12}$$

and

$$d \text{ minimizes } \sum_{\mathcal{F} \in \mathfrak{F}^*} v_{\mathcal{F}}^* \sum_{i=1}^{h_{\mathcal{F}}} [m_i^{\mathcal{F}}]^2 \text{ among those maximizing } \sum_{\mathcal{F} \in \mathfrak{F}^*} u_{\mathcal{F}}^* \sum_{i=1}^{h_{\mathcal{F}}} m_i^{\mathcal{F}}. \tag{A.13}$$

Also, by taking $\mathfrak{G} = \mathfrak{F}'$ in (A,1), (A.2), and (A.3), we have

$$d \text{ maximizes } \sum_{\mathcal{F} \in \mathfrak{F}'} \sum_{i=1}^{h_{\mathcal{F}}} m_i^{\mathcal{F}}, \tag{A.14}$$

and

$$d \text{ minimizes } \sum_{\mathcal{F} \in \mathfrak{F}'} \sum_{i=1}^{h_{\mathcal{F}}} [m_i^{\mathcal{F}}]^2 \text{ among those maximizing } \sum_{\mathcal{F} \in \mathfrak{F}'} \sum_{i=1}^{h_{\mathcal{F}}} m_i^{\mathcal{F}}. \tag{A.15}$$

By (A.9), (A.12) and (A.14), d maximizes $\sum_{\mathcal{F} \in \mathfrak{F}'} u_{\mathcal{F}} \sum_{i=1}^{h_{\mathcal{F}}} m_i^{\mathcal{F}}$. If another design d' also maximizes $\sum_{\mathcal{F} \in \mathfrak{F}'} u_{\mathcal{F}} \sum_{i=1}^{h_{\mathcal{F}}} m_i^{\mathcal{F}}$, then it must maximize both $\sum_{\mathcal{F} \in \mathfrak{F}^*} u_{\mathcal{F}}^* \sum_{i=1}^{h_{\mathcal{F}}} m_i^{\mathcal{F}}$ and $\sum_{\mathcal{F} \in \mathfrak{F}'} \sum_{i=1}^{h_{\mathcal{F}}} m_i^{\mathcal{F}}$ as well. Then, by (A.13) and (A.15), its $\sum_{\mathcal{F} \in \mathfrak{F}^*} v_{\mathcal{F}}^* \sum_{i=1}^{h_{\mathcal{F}}} [m_i^{\mathcal{F}}]^2$ and $\sum_{\mathcal{F} \in \mathfrak{F}'} v_0 \sum_{i=1}^{h_{\mathcal{F}}} [m_i^{\mathcal{F}}]^2$ values cannot be smaller than those of d. By (A.10), this implies that the value of $\sum_{\mathcal{F} \in \mathfrak{F}'} v_{\mathcal{F}} \sum_{i=1}^{h_{\mathcal{F}}} [m_i^{\mathcal{F}}]$ for d is at least as small as that for d'.

Therefore it remains to show that for all subsets $\mathfrak{G}$ of $\mathfrak{F}^*$, (A.11) implies (A.1). Suppose (A.11) holds. Given $\mathcal{F} \in \mathfrak{G}$, $\mathcal{F}' \in \mathfrak{F}'$, and $\mathcal{F}' \prec \mathcal{F}$, we need to show that $\mathcal{F}' \in \mathfrak{G}$. By the definition of $\mathfrak{F}^*$, if $\mathcal{F} \in \mathfrak{G}$, $\mathcal{F}' \in \mathfrak{F}'$, and $\mathcal{F}' \prec \mathcal{F}$, then $u_{\mathcal{F}} > u_0$ (since $\mathcal{F} \in \mathfrak{G} \subseteq \mathfrak{F}^*$) and $u_{\mathcal{F}'} \geq u_{\mathcal{F}}$ (since $\mathcal{F}' \prec \mathcal{F}$). It follows that $u_{\mathcal{F}'} > u_0$, i.e., $\mathcal{F}' \in \mathfrak{F}^*$. Then by (A.11), $\mathcal{F}' \in \mathfrak{G}$. $\square$

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